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Accurate simple approximation for the solitary wave

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Abstract

This Note investigates the effect of a renormalization technique on high-order shallow water approximations of gravity waves. The method is illustrated for the solitary surface wave. Applied to the solution of a generalized KdV equation, it is shown that the renormalization significantly increases the accuracy. ***To cite this article: D. Clamond, D. Fructus, C. R. Mecanique 331 (2003).***

Résumé

Approximation simple et précise de l'onde solitaire. On traite de l'effet d'une technique de renormalisation sur les approximations d'ordres supérieurs de type eau peu profonde pour les ondes de gravité. La méthode est illustrée pour l'onde solitaire. En renormalisant la solution d'une équation KdV généralisée, il est montré que la précision s'en trouve grandement augmentée. ***Pour citer cet article : D. Clamond, D. Fructus, C. R. Mecanique 331 (2003).***

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Mots-clés : Ondes ; Eau peu profonde ; Onde solitaire ; Renormalisation

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La recherche de solutions approchées au problème des ondes de gravité (vagues) a conduit à deux types d'approximations [1] : la théorie de Stokes pour les ondes courtes et la théorie de l'eau peu profonde pour les ondes longues. Toutefois, ces théories ont des domaines de validité disjoints. Une solution unifiée, valide sur tout le domaine de variation des paramètres, est souhaitable. Cela a été obtenu par renormalisation de la théorie de l'eau peu profonde du premier ordre [2–4]. Cette solution est valide tant pour les faibles profondeurs qu'en profondeur infinie. Il a aussi été montré que la renormalisation améliore la précision, particulièrement pour les vagues de grandes amplitudes. Alors se pose la question : qu'advient il si la renormalisation est appliquée à des approximations d'ordres plus élevés ? Cette Note est une tentative de réponse. Pour ce faire, on se consacre à l'onde solitaire qui a l'avantage de la simplicité mathématique et suffit à illustrer le propos.

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La théorie de l'eau peu profonde classique, basée sur un schéma de perturbation singulier, à été developpée à différents ordres par plusieurs auteurs (e.g. [5]). Toutefois, les améliorations sont limités car la série diverge pour toute amplitude [7]. Sa renormalisation, pour l'onde solitaire, génère une série en puissances impaires de fonctions tangentes hyperboliques. Ce développement à été proposé par McCowan [8]. McCowan tronquât sa série après le premier terme seulement, et obtint ainsi une approximation toute à la fois mathématiquement simple et remarquablement précise. Il est donc tentant de considérer les ordres supérieurs. Il se trouve que la série de McCowan est aussi divergente [10]. On ne peut donc trop espérer à en retenir plus de termes.

La théorie de l'eau peu profonde au premier ordre correspond à la solution permanente de l'équation KdV [9]. Une alternative à la théorie classique de l'eau peu profonde renormalisée consiste à renormaliser les solutions d'équations KdV généralisées. Gwyther [11] à obtenu une approximation de l'onde solitaire qui est solution d'une equation KdV étendue, i.e. KdV plus un terme non linéaire cubique plus d'autres non linéarités quadratiques plus un terme dispersif du cinquième ordre. Cet article illustre l'effet de la renormalisation sur cette approximation.

La méthode de renormalisation est d'abord brièvement décrite en Section 2 (voir [2–4] pour plus de détails et généralisations). On l'applique ensuite à la solution de Gwyther en Section 3. La nouvelle approximation ainsi obtenue (8) est d'une simplicité comparable à celle de McCowan (6). En revanche, elle est bien plus précise (Section 4). Cette exemple démontre que la renormalisation est un outil simple et efficace pour améliorer les approximations de type *eau peu profonde* généralisées. Les améliorations apportées aux solutions KdV [2–4] ne sont donc pas le fruit d'un « heureux hasard », comme d'aucun pourrait légitimement le penser (discussions privées).

1. Introduction

The search for approximate solutions of surface waves is generally based on some physical considerations, depending on the wave characteristics [1]. This has led to the development of two basic theories: the Stokes theory and the shallow water theory. However, it is well known that these theories are not uniformly valid in the complete range of water depths, from shallow to deep water. An unified description, which is also tractable, is clearly desirable. This can be achieved via renormalization of the first-order shallow water approximation [2–4]. The improved solution consequently derived is uniformly valid from long to short waves, the wavelength being compared to the mean water depth. It has also been shown that the renormalization significantly increases the accuracy of the approximation. Then arises the question: what would happen if a high-order wave theory were applied? This Note is an attempt to answer this question, at least partially. For this purpose, the solitary wave is considered as a simple example. Indeed, its approximations are analytically simple and are sufficient to illustrate the effect of renormalization.

The classical shallow water expansion has been carried out to high-orders by several authors (see, e.g., [5,6]). However, limited improvements can be expected from this expansion because the series diverges forall amplitudes [7]. It is then tempting to apply renormalization to the series. For the solitary, one thus gets an expansion in odd-power series of tanh-function. This series has long been proposed by McCowan [8]. McCowan truncated his series after the first term only, obtaining the approximation with the highest ratio *accuracy/complexity* ever derived. (McCowan's approximation can be viewed as the renormalized Korteweg and de Vries (KdV) [9] approximation for the solitary wave.) Higher accuracy could reasonably be expected taking more terms in the series. Unfortunately, McCowan's series is divergent too (at least) for steep waves [10] and, therefore, the improvement will be limited if taking more terms in the expansion.

Instead, we shall apply renormalization to a little known approximation due to Gwyther [11]. Gwyther's approximation is solution of a modified KdV equation (mKdV), i.e., KdV plus a cubic nonlinear term. Gwyther [12] shown that his approximation is also solution of an extended KdV equation (eKdV), i.e., mKdV plus a fifth-order dispersive term plus other quadratic nonlinearities.

The renormalization method is first briefly described in Section 2. See [2,3] and [4] for further details and generalizations. The renormalization is then apply to Gwyther's approximation in Section 3. Comparisons between an exact numerical solution and McCowan's approximation show that the accuracy is significantly improved by renormalization (Section 4).

2. Renormalization method

Two-dimensional potential flows due to progressive surface solitary wave in water of constant depth h is considered. The surface tension is neglected and g denotes the acceleration due to gravity. Let be $\{x, y, t, \eta, \phi, \psi\}$ the horizontal coordinate, the upward vertical one, the time, the surface elevation from rest, the velocity potential and the stream function, respectively. For progressive waves, the dependant variables are functions of $\theta = x - Ct$, C being the phase velocity. $y = 0$ and $y = h + \eta(\theta)$ are the equations of the impermeable horizontal bottom and of the impermeable free surface where the pressure is zero. 'Hats' denote quantities at the bottom, e.g., $\hat{\phi}(\theta) = \phi(\theta, y = 0)$, and 'tildes' denote quantities at the surface, e.g., $\tilde{\phi}(\theta) = \phi(\theta, y = h + \eta)$. The bottom being impermeable, it is a stream line defined by $\hat{\psi} = 0$.

The velocity potential and the stream function satisfy the Cauchy–Riemann relations

$$\phi_\theta = \psi_y, \quad \phi_y = -\psi_\theta \quad \text{for } 0 \leqslant y \leqslant h + \eta, \text{ with } \phi_y = 0 \text{ at } y = 0 \tag{1}$$

that have the general solutions

$$\phi(\theta, y) = \frac{1}{2}\hat{\phi}(z) + \frac{1}{2}\hat{\phi}(z^*) = \cos(y\partial_\theta)\hat{\phi}(\theta), \qquad \psi(\theta, y) = \frac{1}{2\mathrm{i}}\hat{\phi}(z) - \frac{1}{2\mathrm{i}}\hat{\phi}(z^*) = \sin(y\partial_\theta)\hat{\phi}(\theta) \tag{2}$$

where $z = \theta + \mathrm{i}y$ and $z^* = \theta - \mathrm{i}y$, with $\mathrm{i}^2 = -1$. Thus, if an approximation of $\hat{\phi}$ is known, the application of (2) provides a holomorphic approximation of the complex velocity potential f

$$f(z) \equiv \phi + \mathrm{i}\psi = \hat{\phi}(z) = \exp(\mathrm{i}y\partial_\theta)\hat{\phi}(\theta) \tag{3}$$

Note that (3) is also valid for unsteady flows. After the transformation (3), the free surface can be obtained from the mass conservation equation $\eta_t + \tilde{\psi}_x = 0$. This relation implies the impermeability of the surface. For a progressive solitary wave, the equation can be integrated as

$$\eta = C^{-1}\,\mathrm{Im}\,\tilde{f} \tag{4}$$

providing an implicit definition of η. The dynamic condition at the surface remains to be considered, in order to derive the relations between the parameters. This condition is given by the Bernoulli equation

$$g\eta - C\,\mathrm{Re}(f_z) + \frac{1}{2}|f_z|^2 = 0 \quad \text{at } y = h + \eta \tag{5}$$

written for a solitary wave in the frame of reference where $\hat{\phi}_\theta(\infty) = 0$.

The renormalization principle is as follow. Suppose that an approximation has been obtained via some singular perturbation techniques, meaning that it has been derived from an approximation of ϕ that is not exact solution of (1). Then, from $\hat{\phi}$ and applying (2), a new approximation is derived, this one being exact solution of (1). For example, the KdV approximation for the solitary wave – i.e., $\hat{\phi} \approx \kappa^{-1}AC\tanh\frac{1}{2}\kappa\theta$, κ and A being constants – using (3) yields

$$f = \kappa^{-1}AC\tanh\frac{1}{2}\kappa z, \qquad f_z = \frac{1}{2}AC\,\mathrm{sech}^2\,\frac{1}{2}\kappa z = AC(1 + \cosh\kappa z)^{-1} \tag{6}$$

that is McCowan's approximation (MCA). Then, a new approximation of the surface is obtained from (4). The dynamic condition at the surface (5) can only be solved approximately. A method of approximation, due to McCowan, consists in rewriting (5) as function of η only and expanding in power series (Taylor's expansion around $\eta = 0$). Setting to zero the coefficients of $\eta, \eta^2, \ldots$ thus gives relations between parameters. McCowan modified his method by considering a mixed expansion in η and $\eta - \eta_0$, η_0 being a free parameter. Taking $\eta_0 = a$ (a the amplitude), the Bernoulli condition at the surface is satisfied identically both at the crest and at infinity. McCowan shown that this second method is more accurate than the first one. It turns out that McCowan's modified method is the collocation method proposed by Clamond [3]. Therefore, we shall use the collocation method for finding the relations between parameters.

3. Renormalized Gwyther's approximation (RGA)

Gwyther [11], improving the method of Rayleigh [13], found a more accurate solution for the solitary wave. This approximation, solution of both the mKdV and eKdV equations, can be conveniently written

$$\hat{\phi} = 2\kappa^{-1} AC \operatorname{cosec} \nu \arctan\left(\tan \frac{1}{2}\nu \tanh \frac{1}{2}\kappa\theta\right) \tag{7}$$

where A, κ and ν are constants. The renormalization formula (3) applied to Gwyther's solution yields

$$f = 2\kappa^{-1} AC \operatorname{cosec} \nu \arctan\left[\tan \frac{1}{2}\nu \tanh \frac{1}{2}\kappa z\right], \qquad f_z = AC(\cos\nu + \cosh\kappa z)^{-1} \tag{8}$$

If $\nu = 0$, the approximation (6) is recovered. The relation (4) gives an explicit definition of $\theta(\eta)$ as

$$\cosh\kappa\theta = \sin\nu \sin\kappa(\eta + h) \coth\varkappa\eta - \cos\nu\cos\kappa(\eta + h), \qquad \varkappa = \kappa A^{-1} \sin\nu \tag{9}$$

and the wave amplitude, i.e., $a \equiv \eta(0)$, is given by

$$\kappa a = 2A \operatorname{cosec}\nu \operatorname{arctanh}\left[\tan \frac{1}{2}\nu \tan \frac{1}{2}\kappa(a + h)\right] \tag{10}$$

Thus, at the surface $y = \eta$, we have

$$\operatorname{Re}(f_z) = AC\big[\cot\nu \sinh\varkappa\eta + \cosh\varkappa\eta \cot\kappa(\eta + h)\big] \operatorname{cosec}\nu \sinh\varkappa\eta \tag{11}$$

$$|f_z|^2 = A^2C^2 \operatorname{cosec}^2\nu \sinh^2\varkappa\eta \operatorname{cosec}^2\kappa(\eta + h) \tag{12}$$

To derive the relations between parameters, the Bernoulli equation at the surface (5) is satisfied exactly at three convenient collocation points: for $\eta = 0$ (i.e., at $\theta = \infty$), for $\eta = a$ (i.e., at $\theta = 0$) and for $\eta = \frac{1}{2}a$. These nodes are chosen because they provide simpler relations and are equally spaced for $\eta \in [0; a]$. Many other choices are of course possible. It is not our goal here to investigate the effect of varying the nodes.

For $\eta = 0$, the Bernoulli equation (5) yields the dispersion relation

$$C^2 = g\kappa^{-1} \tan\kappa h \tag{13}$$

For $\eta = a$, the Eq. (5), using (13) and after some elementary algebra, gives

$$A = \big[\cos\nu + \cos\kappa(a + h)\big]\big[1 - \sqrt{1 - 2\kappa a \cot\kappa h}\,\big] \tag{14}$$

For $\eta = \frac{1}{2}a$, the last relation is obtained using (11) and (12) into (5) (no significant simplifications were found). The renormalization of Gwyther's approximation is now complete.

4. Comparisons with the exact solution

Both the McCowan and the renormalized Gwyther approximations are compared to an exact numerical solution [14]. For small and moderately steep waves (say for $2a \leqslant h$) the two approximations are very accurate and their

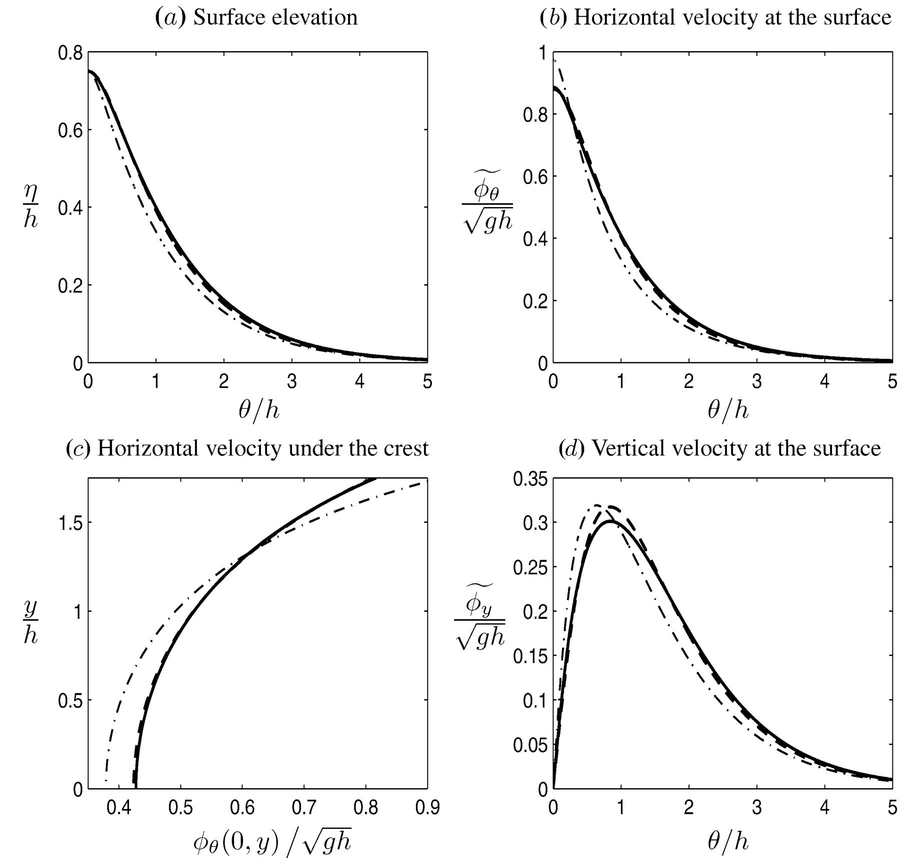

Fig. 1. Example of steep solitary wave ($a/h = 0.75$). — Exact, – – Renormalized Gwyther, – · – McCowan.

Fig. 1. Exemple d'onde solitaire de grande amplitude.

differences are not relevant. For larger amplitudes, on the other hand, RGA is more accurate than MCA, their differences increasing with the amplitude.

For extreme waves, RGA remains quite accurate while MCA is inconsistent. Considering a steep wave with $4a = 3h$, for example, the surface elevation Fig. 1(a) and the horizontal velocity at the surface Fig. 1(b) are well predicted by RGA but MCA shows some important discrepancies. The contrast is even more obvious looking at the horizontal velocity under the crest Fig. 1(c), for which RGA is remarkably precise while MCA is irrelevant. For the vertical velocity at the surface Fig. 1(d) RGA is not as impressive, but is quite accurate and better than MCA. The precision may be improved by choosing other collocation nodes.

5. Conclusion

We have applied the renormalization method to the solitary wave solution of an extended Korteweg and de Vries equation. We have thus derived an analytically simple approximation that is remarkably accurate, even for large waves. This approximation is significantly more accurate than the one obtained renormalizing the solution of the classical Korteweg and de Vries equation (McCowan's first approximation).

This simple example illustrates the power of the renormalization method for improving solutions derived via singular perturbation techniques. The encouraging results obtained in early works [2–4] were thus not the result of a fortunate "accident", as one may have legitimately thought.

Similarly, renormalizing the periodic solutions of the eKdV equation, one should get a very accurate approximation of periodic waves, that is uniformly valid from shallow water to infinite depth.

For the solitary wave, more accurate approximations may be obtained using an odd-power series of the renormalized Gwyther's potential f (8), instead of using McCowan's divergent series.

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