Letters

IOP Publishing | Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. 49 (2016) 31LT01 (11pp) | doi:10.1088/1751-8113/49/31/31LT01

Letter

Multi-symplectic structure of fully nonlinear weakly dispersive internal gravity waves

Didier Clamond[1,3] **and Denys Dutykh**[2]

[1] Laboratoire J. A. Dieudonné, Université de Nice—Sophia Antipolis, Parc Valrose, F-06108 Nice cedex 2, France
[2] LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc, Campus Scientifique, F-73376 Le Bourget-du-Lac Cedex, France

E-mail: diderc@unice.fr and Denys.Dutykh@univ-savoie.fr

Received 1 April 2016, revised 11 June 2016
Accepted for publication 13 June 2016
Published 27 June 2016

Abstract

In this short communication, we present the multi-symplectic structure for the two-layer Serre–Green–Naghdi equations describing the evolution of large amplitude internal gravity water waves when both layers are shallow. We consider only a two-layer stratification with rigid bottom and lid for simplicity, generalisations to several layers being conceivable. This multi-symplectic formulation allows the application of various multi-symplectic integrators (such as Euler or Preissman box schemes) that preserve exactly the multi-symplecticity at the discrete level.

Keywords: internal waves, two-layer fluids, multi-symplectic structure, long waves, Serre–Green–Naghdi equations

(Some figures may appear in colour only in the online journal)

1. Introduction

The density stratification in oceans exists due, mainly, to the dependence of the water density on temperature and salinity [14]. The density stratification supports the so-called 'internal waves'. These are ubiquitous in the ocean and, comparing to surface waves, internal waves may have a huge amplitude of the order of hundreds of meters [13]. These waves play an important role in ocean dynamics and they attract permanent attention of several scientific communities. Compared to surface gravity waves, the physics of internal waves is richer and their modelling leads to more complicated equations, in general (see the reviews [1, 20] for more information).

[3] Author to whom any correspondence should be addressed.

Lagrangian and Hamiltonian formalisms are tools of choice in theoretical physics, in particular for studying nonlinear waves. Quite recently, the multi-symplectic formalism have been proposed as an attractive alternative. This formulation generalises the classical Hamiltonian structure to partial differential equations by treating space and time on the equal footing [5]. Multi-symplectic formulations are gaining popularity, both for mathematical investigations and numerical modelling [7, 19].

Multi-symplectic formulations of various equations modelling surface waves can be found in the literature. However, to our knowledge, no such formulations have been proposed for internal waves. In this note, we show that the multi-symplectic structure of a homogeneous fluid should be easily extended to fluids stratified in several homogeneous layers. For the sake of simplicity, we focus on two-dimensional irrotational motions of internal waves propagating at the interface between two perfect fluids, bounded below by an impermeable horizontal bottom and bounded above by an impermeable rigid lid.

The present article should be considered as a further step in understanding the underlying mathematical structure of an important model of long internal waves—the so-called two-layer Serre equations [21]—where the thicknesses of the fluid layers are small compared to the characteristic wavelength (shallow layers). Serre's equations are approximations for large amplitude long waves. This model is sometimes referred to as *weakly dispersive fully nonlinear* and was first derived by Serre for surface waves [21]. Its generalisation for two layers internal waves was apparently first in the late nineteen eighties [4, 16, 18], both with a rigid lid and with a free surface, and later re-derived using different approaches [2, 3, 9].

The Hamiltonian formulation for the classical Serre equations describing surface waves can be found in [15], for example. However, this structure is non-canonical and quite non-trivial. The two-layer Serre equations also have a non-canonical Hamiltonian structure [2, 11]. In the present study, we propose a multi-symplectic formulation of the two-layer Serre equations with a rigid lid. This work is a direct continuation of [8] where the multi-symplectic structure was proposed for the original Serre equations.

The results obtained in this study can be used to propose new structure-preserving numerical schemes to simulate the dynamics of internal waves. Indeed, now it is straightforward to apply the Euler-box or the Preissman-box schemes [19] to two-layers Serre equations. The advantage of this approach is that such schemes preserve *exactly* the multi-symplectic form at the discrete level [5, 7]. To our knowledge, this direction is essentially open. We are not aware of the existing structure-preserving numerical codes to simulate *internal* waves. This situation may be explained by higher complexity of these models compared to, e.g., surface waves.

The present manuscript is organised as follows. In section 2, we present a simple variational derivation of the governing equations. Their multi-symplectic structure is provided in section 3 and the implied conservation laws are given in section 4. The main conclusions and perspectives of this study are outlined in section 5.

2. Model derivation

We consider a two-dimensional irrotational flow of an incompressible fluid stratified in two homogeneous layers of densities ρ_j, subscripts $j = 1$ and $j = 2$ denoting the lower and upper layers, respectively. The fluid is bounded below by a horizontal impermeable bottom at $y = -d_1$ and above by a rigid lid at $y = d_2$, y being the upward vertical coordinate such that $y = 0$ and $y = \eta(x, t)$ are, respectively, the equations of the still interface and of the wavy interface (see figure 1). Here, x denotes the horizontal coordinate, t is the time, g is the

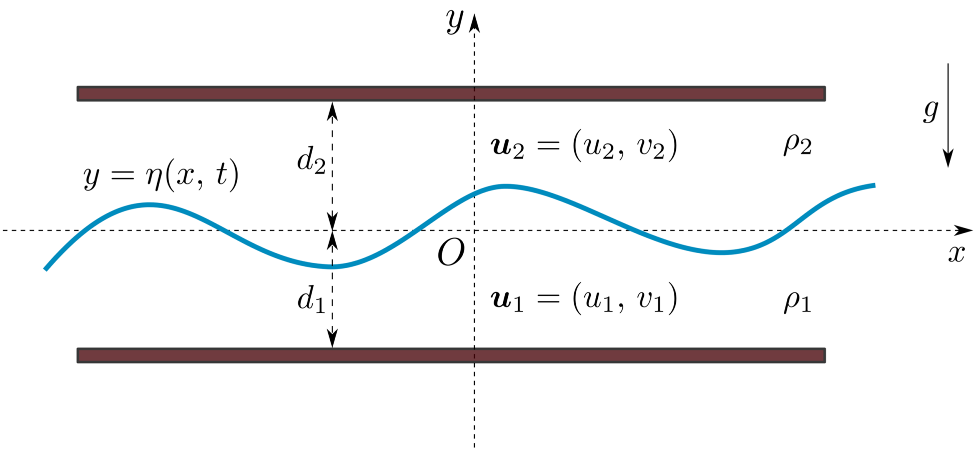

Figure 1. Sketch of the two-fluid domain.

downward (constant) acceleration due to gravity and surface tension is neglected. The lower and upper thicknesses are, respectively, $h_1 = d_1 + \eta$ and $h_2 = d_2 - \eta$, such that $h_1 + h_2 = d_1 + d_2 \stackrel{\text{def}}{=} D$ is a constant. Finally, we denote $\boldsymbol{u}_j = (u_j, v_j)$ the velocity fields in the jth layer (u_j the horizontal velocities, v_j the vertical ones). We derive here the fully nonlinear weakly dispersive long wave approximation Serre-like equations [4, 9] following a variational approach initiated in [22] for the (one-layer) classical Serre equations. Further relations are given in the appendix.

2.1. Ansatz

In order to model long waves in shallow water with rigid horizontal bottom and lid, the velocity fields in each layer is approximated as

$$u_j(x, y, t) \;\approx\; \bar{u}_j(x, t), \qquad v_j(x, y, t) \;\approx\; ((-1)^j d_j - y)\bar{u}_{jx}, \tag{1}$$

where $\bar{u}_j$ is the horizontal velocity averaged over the jth layer, i.e., $\bar{u}_1 \stackrel{\text{def}}{=} h_1^{-1}\int_{-d_1}^{\eta} u_1 \mathrm{d}y$, $\bar{u}_2 \stackrel{\text{def}}{=} h_2^{-1}\int_{\eta}^{d_2} u_2 \mathrm{d}y$. The horizontal velocities u_j are thus (approximately) uniform along the layer column and the vertical velocities v_j are chosen so that the fluid incompressibility is fulfilled together with the bottom and the lid impermeabilities.

With the ansätze (1), the vertical accelerations are

$$D_t v_j \stackrel{\text{def}}{=} v_{jt} + u_j\, v_{jx} + v_j\, v_{jy} \approx \gamma_j\, h_j^{-1}\{d_j - (-1)^j y\},$$

where D_t is the temporal derivative following the motion and γ_j are the vertical accelerations at the interface, i.e.

$$\gamma_j \stackrel{\text{def}}{=} D_t v_j|_{y=\eta} \approx (-1)^j\, h_j\, \{\bar{u}_{jxt} + \bar{u}_j\, \bar{u}_{jxx} \;-\; \bar{u}_{jx}^2\}. \tag{2}$$

The kinetic and potential energies of the liquid column are, respectively

$$\mathscr{K} \stackrel{\text{def}}{=} \int_{-d_1}^{\eta} \rho_1 \frac{u_1^2 + v_1^2}{2}\, \mathrm{d}y + \int_{\eta}^{d_2} \rho_2 \frac{u_2^2 + v_2^2}{2}\, \mathrm{d}y \approx \sum_{j=1}^{2} \rho_j \left(\tfrac{1}{2}\, h_j\, \bar{u}_j^{\,2} + \tfrac{1}{6}\, h_j^{\,3}\, \bar{u}_{jx}^{\,2}\right), \tag{3}$$

$$\mathscr{V} \stackrel{\text{def}}{=} \int_{-d_1}^{\eta} \rho_1\, g\,(y + d_1)\, \mathrm{d}y + \int_{\eta}^{d_2} \rho_2\, g\,(y + d_1)\, \mathrm{d}y = \tfrac{1}{2}\,(\rho_1 - \rho_2)\, g\, h_1^2 + \tfrac{1}{2}\, \rho_2\, g\, D^2. \tag{4}$$

Note that the potential energy is defined relatively to the bottom without loss of generality.

The incompressibility of the fluids and the impermeabilities of the lower and upper boundaries being fulfilled, a Lagrangian density $\mathscr{L}$ is then obtained from the Hamilton principle: the Lagrangian is the kinetic minus potential energies plus constraints for the mass conservation of each layer, i.e.

$$\mathscr{L} \stackrel{\text{def}}{=} \mathscr{K} - \mathscr{V} + \rho_1\, \{h_{1t} + [h_1\, \bar{u}_1]_x\}\, \phi_1 + \rho_2\, \{h_{2t} + [h_2\, \bar{u}_2]_x\}\, \phi_2, \tag{5}$$

where ϕ_j are Lagrange multipliers and with the constraint $h_2 = D - h_1$ being assumed. The latter could be relaxed adding $\lambda\,(h_1 + h_2 - D)$ (λ another Lagrange multiplier) into the right-hand side of (5) [6]. However, we do not do it here in order to handle fewer equations.

2.2. Equations of motion

The Euler–Lagrange equations for the functional $\iint \mathscr{L}\, \mathrm{d}x\, \mathrm{d}t$ yield (together with $h_2 = D - h_1$ and for $j = 1, 2$)

$$\delta\phi_j : 0 = h_{jt} + [h_j\, \bar{u}_j]_x, \tag{6}$$

$$\delta\bar{u}_j : 0 = \phi_j\, h_{jx} - [h_j\, \phi_j]_x - \tfrac{1}{3}\, [h_j^3\, \bar{u}_{jx}]_x + h_j\, \bar{u}_j, \tag{7}$$

$$\delta h_1 : \ 0 = \tfrac{1}{2}(\rho_1\, \bar{u}_1^2 - \rho_2\, \bar{u}_2^2) - (\rho_1 - \rho_2)\, g\, h_1 + \tfrac{1}{2}(\rho_1\, h_1^2\, \bar{u}_{1x}^2 - \rho_2\, h_2^2\, \bar{u}_{2x}^2) - \rho_1\, \phi_{1t} + \rho_2\, \phi_{2t} - \rho_1\, \bar{u}_1\, \phi_{1x} + \rho_2\, \bar{u}_2\, \phi_{2x}. \tag{8}$$

Adding the two equation (6) and integrating the result, one obtains

$$h_1\, \bar{u}_1 + h_2\, \bar{u}_2 = Q(t), \tag{9}$$

Q being an integration 'constant' ($U_{\mathrm{m}} \stackrel{\text{def}}{=} Q/D$ is often called *mix velocity* in the theory of multiphase flows). The relation (7) can be rewritten

$$\phi_{jx} = \bar{u}_j - \tfrac{1}{3}\, h_j^{-1}\, [h_j^{\,3}\, \bar{u}_{jx}]_x = \bar{u}_j - \tfrac{1}{3}\, h_j^2\, \bar{u}_{jxx} - h_j\, h_{jx}\, \bar{u}_{jx} \qquad (j = 1, 2), \tag{10}$$

thence

$$h_1\, \phi_{1x} + h_2\, \phi_{2x} = Q - \tfrac{1}{3}\, [h_1^3\, \bar{u}_{1x} + h_2^3\, \bar{u}_{2x}]_x, \tag{11}$$

$$\rho_1\, \phi_{1x} - \rho_2\, \phi_{2x} = \rho_1\, \bar{u}_1 - \rho_2\, \bar{u}_2 - \tfrac{1}{3}\, \rho_1\, h_1^{-1}\, [h_1^3\, \bar{u}_{1x}]_x + \tfrac{1}{3}\, \rho_2\, h_2^{-1}\, [h_2^3\, \bar{u}_{2x}]_x, \tag{12}$$

$$\rho_1\, u_1\, \phi_{1x} - \rho_2\, u_2\, \phi_{2x} = \rho_1\, \bar{u}_1^{\,2} - \rho_2\, \bar{u}_2^{\,2} \\ - \tfrac{1}{3}\, \rho_1\, u_1\, h_1^{-1}\, [h_1^3\, \bar{u}_{1x}]_x + \tfrac{1}{3}\, \rho_2\, u_2\, h_2^{-1}\, [h_2^3\, \bar{u}_{2x}]_x. \tag{13}$$

The equation (8) then gives

$$\rho_1\, \phi_{1t} - \rho_2\, \phi_{2t} = \tfrac{1}{2}(\rho_1\, h_1^2\, \bar{u}_{1x}^{\,2} - \rho_2\, h_2^2\, \bar{u}_{2x}^{\,2}) - \tfrac{1}{2}(\rho_1\, \bar{u}_1^{\,2} - \rho_2\, \bar{u}_2^{\,2}) - (\rho_1 - \rho_2)\, g\, h_1 \\ + \tfrac{1}{3}\, \rho_1\, \bar{u}_1\, h_1^{-1}\, [h_1^3\, \bar{u}_{1x}]_x - \tfrac{1}{3}\, \rho_2\, \bar{u}_2\, h_2^{-1}\, [h_2^3\, \bar{u}_{2x}]_x, \tag{14}$$

and eliminating ϕ_j between (12) and (14), one obtains

$$\partial_t \Big\{ \rho_1\, \bar{u}_1 - \rho_2\, \bar{u}_2 - \tfrac{1}{3}\, \rho_1\, h_1^{-1}\, [h_1^3\, \bar{u}_{1x}]_x + \tfrac{1}{3}\, \rho_2\, h_2^{-1}\, [h_2^3\, \bar{u}_{2x}]_x \Big\} \\ + \partial_x \Big\{ \tfrac{1}{2}\, \rho_1\, \bar{u}_1^{\,2} - \tfrac{1}{2}\, \rho_2\, \bar{u}_2^{\,2} + (\rho_1 - \rho_2)\, g\, h_1 - \tfrac{1}{2}\, \rho_1\, h_1^2\, \bar{u}_{1x}^{\,2} + \tfrac{1}{2}\, \rho_2\, h_2^2\, \bar{u}_{2x}^{\,2} \\ - \tfrac{1}{3}\, \rho_1\, \bar{u}_1\, h_1^{-1}\, [h_1^3\, \bar{u}_{1x}]_x + \tfrac{1}{3}\, \rho_2\, \bar{u}_2\, h_2^{-1}\, [h_2^3\, \bar{u}_{2x}]_x \Big\} = 0, \tag{15}$$

that, physically, is an equation for the conservation of the difference between the tangential momenta at the interface. One can also easily derive a non-conservative equation for the horizontal momentum

$$\rho_1\, (\bar{u}_{1t} + \bar{u}_1\, \bar{u}_{1x}) - \rho_2\, (\bar{u}_{2t} + \bar{u}_2\, \bar{u}_{2x}) + (\rho_1 - \rho_2)\, g\, h_{1x} \\ + \tfrac{1}{3}\, \rho_1\, h_1^{-1}\, [h_1^2\, \gamma_1]_x + \tfrac{1}{3}\, \rho_2\, h_2^{-1}\, [h_2^2\, \gamma_2]_x = 0. \tag{16}$$

On the other hand, equations for the momentum and energy fluxes are not easily derived from these equations. This is where a multi-symplectic formulation comes to help.

3. Multi-symplectic structure

A system of partial differential equations has a multi-symplectic structure if it can written as a system of first-order equations [5, 17]

$$\mathbb{M} \cdot \boldsymbol{z}_t + \mathbb{K} \cdot \boldsymbol{z}_x = \boldsymbol{\nabla}_{\boldsymbol{z}}\, S(\boldsymbol{z}), \tag{17}$$

where a dot denotes the contracted (inner) product, $\boldsymbol{z} \in \mathbb{R}^n$ is a rank-one tensor (vector) of state variables, $\mathbb{M} \in \mathbb{R}^{n\times n}$ and $\mathbb{K} \in \mathbb{R}^{n\times n}$ are skew-symmetric rank-two tensors (matrices) and S is a smooth rank-zero tensor (scalar) function depending on $\boldsymbol{z}$. The function S is sometimes called the '*Hamiltonian*', though it is generally not a classical Hamiltonian.

The multi-symplectic structure for the one-layer Serre equations is already known [8]. This structure can be easily extended to two (and more) layers. The multi-symplectic formulation for one layer involves 8-by-8 matrices. For two layers, we then expect *a priori* a multi-symplectic formulation with 16-by-16 matrices. However, since we consider a rigid lid, one variable can be eliminated, thus reducing the formulation to 15-by-15. Thus, introducing $h_1 = h$ and $h_2 = D - h$ for brevity, we seek for a multi-symplectic structure with

$$\boldsymbol{z} = h\, \boldsymbol{e}_1 + \varphi_1\, \boldsymbol{e}_2 + \bar{u}_1\, \boldsymbol{e}_3 + \tilde{v}_1\, \boldsymbol{e}_4 + p_1\, \boldsymbol{e}_5 + q_1\, \boldsymbol{e}_6 + r_1\, \boldsymbol{e}_7 + s_1\, \boldsymbol{e}_8 \\ + \varphi_2\, \boldsymbol{e}_9 + \bar{u}_2\, \boldsymbol{e}_{10} + \tilde{v}_2\, \boldsymbol{e}_{11} + p_2\, \boldsymbol{e}_{12} + q_2\, \boldsymbol{e}_{13} + r_2\, \boldsymbol{e}_{14} + s_2\, \boldsymbol{e}_{15}, \tag{18}$$

($\boldsymbol{e}_i$ standard unitary basis vectors) and

$$\mathbb{M} = \rho_1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + \tfrac{1}{3}\,\rho_1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_5 - \boldsymbol{e}_5 \otimes \boldsymbol{e}_1) \\ - \rho_2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_9 - \boldsymbol{e}_9 \otimes \boldsymbol{e}_1) - \tfrac{1}{3}\,\rho_2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_{12} - \boldsymbol{e}_{12} \otimes \boldsymbol{e}_1), \tag{19}$$

$$\mathbb{K} = \tfrac{1}{3}\,\rho_1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_7 - \boldsymbol{e}_7 \otimes \boldsymbol{e}_1) - \rho_1(\boldsymbol{e}_2 \otimes \boldsymbol{e}_6 - \boldsymbol{e}_6 \otimes \boldsymbol{e}_2) \\ - \tfrac{1}{3}\,\rho_2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_{14} - \boldsymbol{e}_{14} \otimes \boldsymbol{e}_1) + \rho_2(\boldsymbol{e}_9 \otimes \boldsymbol{e}_{13} - \boldsymbol{e}_{13} \otimes \boldsymbol{e}_9), \tag{20}$$

$$S = \rho_1\Big(\tfrac{1}{6}\tilde{v}_1{}^2 - \tfrac{1}{2}\bar{u}_1{}^2 - \tfrac{1}{3}s_1\bar{u}_1\tilde{v}_1\Big)h + \rho_2\Big(\tfrac{1}{6}\tilde{v}_2{}^2 - \tfrac{1}{2}\bar{u}_2{}^2 - \tfrac{1}{3}s_2\bar{u}_2\tilde{v}_2\Big)(D-h) \\ - \tfrac{1}{2}\,(\rho_1 - \rho_2)\,g\,h^2 + \tfrac{1}{3}\,\rho_1\,p_1\,(\bar{u}_1 s_1 - \tilde{v}_1) - \tfrac{1}{3}\,\rho_2\,p_2\,(\bar{u}_2 s_2 - \tilde{v}_2) \\ + \rho_1\,q_1\Big(\bar{u}_1 + \tfrac{1}{3}s_1\tilde{v}_1\Big) - \rho_2\,q_2\Big(\bar{u}_2 + \tfrac{1}{3}s_2\tilde{v}_2\Big) - \tfrac{1}{3}\,\rho_1\,r_1\,s_1 + \tfrac{1}{3}\,\rho_2\,r_2\,s_2. \tag{21}$$

These two-layer expressions for $\boldsymbol{z}$, $\mathbb{M}$, $\mathbb{K}$ and S are simple duplication of the corresponding expression for one layer [8]. We show below that they indeed lead to the two-layer Serre-like equations derived in the previous section.

Physically, φ_j are the velocity potentials written at the interface, while ϕ_j are related to the velocity potentials integrated over the fluid layers. Using the velocity potentials at the interface, we obtained rather easily the multi-symplectic structure of the Serre-like equations. Conversely, with the equivalent formulation involving ϕ_j it is difficult, and likely impossible, to obtain a multi-symplectic structure of the Serre-like equations. It is reminiscent of the classical Hamiltonian formulation for finite-depth two-layer flow with a rigid lid, where the weighted difference between the velocity potentials at the interface turns out to be the correct canonical variable [10].

The substitution of (18)–(21) into (17) yields the fifteen equations

$$\rho_1\Big\{\varphi_{1t} + \tfrac{1}{3}\,p_{1t} + \tfrac{1}{3}\,r_{1x}\Big\} - \rho_2\Big\{\varphi_{2t} + \tfrac{1}{3}\,p_{2t} + \tfrac{1}{3}\,r_{2x}\Big\} = -(\rho_1 - \rho_2)\,g\,h \\ + \rho_1\Big\{\tfrac{1}{6}\,\tilde{v}_1^2 - \tfrac{1}{2}\,\bar{u}_1{}^2 - \tfrac{1}{3}\,\bar{u}_1\,\tilde{v}_1\,s_1\Big\} - \rho_2\Big\{\tfrac{1}{6}\,\tilde{v}_2^2 - \tfrac{1}{2}\,\bar{u}_2{}^2 - \tfrac{1}{3}\,\bar{u}_2\,\tilde{v}_2\,s_2\Big\}, \tag{22}$$

$$-\rho_1\,\{h_t + q_{1x}\} = 0, \tag{23}$$

$$0 = \rho_1\Big\{q_1 - h\Big(\bar{u}_1 + \tfrac{1}{3}\,\tilde{v}_1\,s_1\Big) + \tfrac{1}{3}\,p_1\,s_1\Big\}, \tag{24}$$

$$0 = -\tfrac{1}{3}\,\rho_1\{p_1 - h(\tilde{v}_1 - \bar{u}_1\,s_1) - q_1\,s_1\}, \tag{25}$$

$$-\tfrac{1}{3}\,\rho_1\,h_t = -\tfrac{1}{3}\,\rho_1\{\tilde{v}_1 - s_1\,\bar{u}_1\}, \tag{26}$$

$$\rho_1\,\varphi_{1x} = \rho_1\Big\{\bar{u}_1 + \tfrac{1}{3}\,s_1\,\tilde{v}_1\Big\}, \tag{27}$$

$$-\tfrac{1}{3}\,\rho_1\,h_x = -\tfrac{1}{3}\,\rho_1\,s_1, \tag{28}$$

$$0 = -\tfrac{1}{3}\,\rho_1\{r_1 + h\,\bar{u}_1\,\tilde{v}_1 - p_1\,\bar{u}_1 - q_1\,\tilde{v}_1\}, \tag{29}$$

$$\rho_2\,\{h_t + q_{2x}\} = 0, \tag{30}$$

$$0 = -\rho_2 \left\{ q_2 + (D-h)\left(\bar{u}_2 + \tfrac{1}{3}\,\tilde{v}_2\, s_2\right) + \tfrac{1}{3}\, p_2\, s_2 \right\}, \tag{31}$$

$$0 = \tfrac{1}{3}\,\rho_2 \{p_2 + (D-h)(\tilde{v}_2 - \bar{u}_2\, s_2) - q_2\, s_2\}, \tag{32}$$

$$\tfrac{1}{3}\,\rho_2\, h_t = \tfrac{1}{3}\,\rho_2 \{\tilde{v}_2 - s_2\, \bar{u}_2\}, \tag{33}$$

$$-\rho_2\, \varphi_{2x} = -\rho_2 \left\{ \bar{u}_2 + \tfrac{1}{3}\, s_2\, \tilde{v}_2 \right\}, \tag{34}$$

$$\tfrac{1}{3}\,\rho_2\, h_x = \tfrac{1}{3}\,\rho_2\, s_2, \tag{35}$$

$$0 = \tfrac{1}{3}\,\rho_2 \{r_2 - (D-h)\,\bar{u}_2\,\tilde{v}_2 - p_2\,\bar{u}_2 - q_2\,\tilde{v}_2\}. \tag{36}$$

Twelve of these equations are trivial and can be simplified as (with $j = 1, 2$)

$$q_j = (-1)^{j-1}\, h_j\, \bar{u}_j, \qquad p_j = (-1)^{j-1}\, h_j\, \tilde{v}_j, \qquad s_j = (-1)^{j-1}\, h_{jx},$$
$$r_j = \bar{u}_j\, p_j, \qquad \tilde{v}_j = (-1)^{j-1}(h_{jt} + \bar{u}_j\, h_{jx}), \qquad \varphi_{jx} = \bar{u}_j + \tfrac{1}{3}\, s_j\, \tilde{v}_j,$$

the remaining three giving the mass conservation equations (together with $h_1 + h_2 = D$)

$$h_{1t} + [h_1\,\bar{u}_1]_x = 0, \qquad h_{2t} + [h_2\,\bar{u}_2]_x = 0, \tag{37}$$

and, exploiting the relations (37), the equation for the tangential momenta at the interface

$$\begin{aligned} \rho_1\,\varphi_{1t} - \rho_2\,\varphi_{2t} = {} & \tfrac{1}{2}(\rho_1\, h_1^2\, \bar{u}_{1x}^2 - \rho_2\, h_2^2\, \bar{u}_{2x}^2) - \tfrac{1}{2}(\rho_1\, \bar{u}_1^2 - \rho_2\, \bar{u}_2^2) \\ & + \tfrac{1}{3}\,\rho_1\,\bar{u}_1\, h_1^{-1}\,[h_1^3\,\bar{u}_{1x}]_x - \tfrac{1}{3}\,\rho_2\,\bar{u}_2\, h_2^{-1}\,[h_2^3\,\bar{u}_{2x}]_x \\ & + \tfrac{1}{3}\,\rho_1\,[h_1^2\,\bar{u}_{1x}]_t - \tfrac{1}{3}\,\rho_2\,[h_2^2\,\bar{u}_{2x}]_t - (\rho_1 - \rho_2)\, g\, h_1. \end{aligned} \tag{38}$$

One can verify that these equations are equivalent to the ones obtained above.

The multi-symplectic structure described above involves the potentials φ_j that are different from the potentials ϕ_j used to derive Serre-like equations of the section 2. After elimination of ϕ_j and φ_j, the two systems of equations are identical. Indeed, the differences between these two velocity potentials are—from (10), (33) and (37)—given by

$$\begin{aligned} \varphi_{jx} - \phi_{jx} &= \tfrac{1}{3}\, h_{jx}\,(h_{jt} + u_j\, h_{jx}) + \tfrac{1}{3}\, h_j^2\, \bar{u}_{jxx} + h_j\, h_{jx}\, \bar{u}_{jx} \\ &= \tfrac{1}{3}\, h_j^2\, \bar{u}_{jxx} + \tfrac{2}{3}\, h_j\, h_{jx}\, \bar{u}_{jx} = \tfrac{1}{3}[h_j^2\, \bar{u}_{jx}]_x, \end{aligned} \tag{39}$$

thence with $\varphi_j = \phi_j + \frac{1}{3}{h_j}^2 \bar{u}_{jx}$ substituted into (38), the equation (14) is recovered.

4. Conservation laws

From the multi-symplectic structure, one obtains local conservation laws for the energy and the momentum

$$E_t + F_x = 0, \qquad I_t + G_x = 0, \tag{40}$$

where $E(z) = S(z) + \frac{1}{2} z_x \cdot \mathbb{K} \cdot z$, $F(z) = -\frac{1}{2} z_t \cdot \mathbb{K} \cdot z$, $G(z) = S(z) + \frac{1}{2} z_t \cdot \mathbb{M} \cdot z$ and $I(z) = -\frac{1}{2} z_x \cdot \mathbb{M} \cdot z$. For the Serre-like equations, exploiting the results of the previous section and after some algebra, one obtains

$$\begin{aligned} E = {} & \rho_1 \Big[\tfrac{1}{2}\, \varphi_1\, h_1\, \bar{u}_1 + \tfrac{1}{6}\, h_1^3\, \bar{u}_1\, \bar{u}_{1x} \Big]_x - \tfrac{1}{2}\, \rho_1\, h_1\, \bar{u}_1^2 - \tfrac{1}{6}\, \rho_1\, h_1^3\, \bar{u}_{1x}^2 \\ & + \rho_2 \Big[\tfrac{1}{2}\, \varphi_2\, h_2\, \bar{u}_2 + \tfrac{1}{6}\, h_2^3\, \bar{u}_2\, \bar{u}_{2x} \Big]_x - \tfrac{1}{2}\, \rho_2\, h_2\, \bar{u}_2^2 - \tfrac{1}{6}\, \rho_2\, h_2^3\, \bar{u}_{2x}^2 \\ & - \tfrac{1}{6}\, \rho_2\, D\, [h_2^2\, \bar{u}_2\, \bar{u}_{2x}]_x - \tfrac{1}{2}\, (\rho_1 - \rho_2)\, g\, h_1^2, \end{aligned} \tag{41}$$

$$\begin{aligned} F = {} & -\rho_1 \Big[\tfrac{1}{2}\, \varphi_1\, h_1\, \bar{u}_1 + \tfrac{1}{6}\, h_1^3\, \bar{u}_1\, \bar{u}_{1x} \Big]_t - \rho_1\, h_1\, \bar{u}_1 \Big(\tfrac{1}{2}\, \bar{u}_1^2 + \tfrac{1}{6}\, h_1^2\, \bar{u}_{1x}^2 + \tfrac{1}{3}\, h_1\, \gamma_1 \Big) \\ & - \rho_2 \Big[\tfrac{1}{2}\, \varphi_2\, h_2\, \bar{u}_2 + \tfrac{1}{6}\, h_2^3\, \bar{u}_2\, \bar{u}_{2x} \Big]_t - \rho_2\, h_2\, \bar{u}_2 \Big(\tfrac{1}{2}\, \bar{u}_2^2 + \tfrac{1}{6}\, h_2^2\, \bar{u}_{2x}^2 - \tfrac{1}{3}\, h_2\, \gamma_2 \Big) \\ & + \tfrac{1}{6}\, \rho_2\, D\, [h_2^2\, \bar{u}_2\, \bar{u}_{2x}]_t - (\rho_1 - \rho_2)\, g\, h_1^2\, \bar{u}_1 + \rho_2\, Q\, \phi_{2t} + \tfrac{1}{2}\, \rho_2\, Q\, \bar{u}_2^2 \\ & - \rho_2\, Q \Big[\tfrac{1}{2}\, h_2^2\, \bar{u}_2\, \bar{u}_{2x} \Big]_x + \tfrac{1}{6}\, \rho_2\, Q\, h_2^2\, \bar{u}_2\, \bar{u}_{2xx}, \end{aligned} \tag{42}$$

$$\begin{aligned} G = {} & \rho_1\, h_1\, \bar{u}_1^2 + \tfrac{1}{2}\, (\rho_1 - \rho_2)\, g\, h_1^2 + \tfrac{1}{3}\, \rho_1\, h_1^2\, \gamma_1 + \rho_1 \Big[\tfrac{1}{2}\, \varphi_1\, h_1 + \tfrac{1}{6}\, h_1^3\, \bar{u}_{1x} \Big]_t \\ & - \tfrac{1}{2}\, \rho_2\, (h_1 - h_2)\, \bar{u}_2^2 - \tfrac{1}{3}\, \rho_2\, h_2^2\, \gamma_2 + \rho_2 \Big[\tfrac{1}{2}\, \varphi_2\, h_2 + \tfrac{1}{6}\, h_2^3\, \bar{u}_{2x} \Big]_t \\ & - \tfrac{1}{2}\, \rho_2\, D\, \phi_{2t} + \tfrac{1}{2}\, \rho_2\, D\, [h_2^2\, \bar{u}_2\, \bar{u}_{2x}]_x - \tfrac{1}{6}\, \rho_2\, D\, h_2^2\, \bar{u}_2\, \bar{u}_{2xx}, \end{aligned} \tag{43}$$

$$\begin{aligned} I = {} & \rho_1\, h_1\, \bar{u}_1 - \rho_1 \Big[\tfrac{1}{2}\, \varphi_1\, h_1 + \tfrac{1}{6}\, h_1^3\, \bar{u}_{1x} \Big]_x - \tfrac{1}{2}\, \rho_2\, D\, \phi_{2x} \\ & + \rho_2\, h_2\, \bar{u}_2 - \rho_2 \Big[\tfrac{1}{2}\, \varphi_2\, h_2 + \tfrac{1}{6}\, h_2^3\, \bar{u}_{2x} \Big]_x. \end{aligned} \tag{44}$$

Note that these relations involve both φ_j and ϕ_j in order to handle more compact expressions.

4.1. Momentum flux

From the relations (43) and (44), after simplifications and some elementary algebra, one obtains the equation for the conservation of the momentum

$$\begin{aligned} & \sum_{j=1}^{2} \partial_t [\rho_j\, h_j\, \bar{u}_j] + \partial_x \Big[\rho_j\, h_j \Big\{ \bar{u}_j^2 - (-1)^j\, \tfrac{1}{2}\, g\, h_j - (-1)^j\, \tfrac{1}{3}\, h_j\, \gamma_j \Big\} \Big] \\ & \quad = D\, \rho_2\, \partial_x \Big[\phi_{2t} + \tfrac{1}{2}\, \bar{u}_2^2 - h_2\, h_{2x}\, \bar{u}_2\, \bar{u}_{2x} - \tfrac{1}{2}\, h_2^2\, \bar{u}_{2x}^2 - \tfrac{1}{3}\, h_2^2\, \bar{u}_2\, \bar{u}_{2xx} - g\, h_2 \Big], \end{aligned} \tag{45}$$

and comparison with the equation (58) gives an expression for the pressure at the interface

$$\tilde{P}/\rho_2 = K_2(t) - \phi_{2t} - \tfrac{1}{2}\, \bar{u}_2^2 + h_2\, h_{2x}\, \bar{u}_2\, \bar{u}_{2x} + \tfrac{1}{2}\, h_2^2\, \bar{u}_{2x}^2 + \tfrac{1}{3}\, h_2^2\, \bar{u}_2\, \bar{u}_{2xx} + g\, h_2, \tag{46}$$

where $K_2(t)$ is an arbitrary function (integration 'constant'). The horizontal derivative of this relation, with (6) and (10), yields after some algebra

$$\tilde{P}_x/\rho_2 = g\, h_{2x} - \bar{u}_{2t} - \bar{u}_2\, \bar{u}_{2x} + \tfrac{1}{3}\, h_2^{-1} [h_2^2\, \gamma_2]_x, \tag{47}$$

that is the upper-layer averaged horizontal momentum equation (59), as it should be. We have obtained the Cauchy–Lagrange equation (46) because we used $h = h_1$ as main variable for the interface in the multi-symplectic formalism and because we eliminated ϕ_1 from the equations. Had we instead used h_2 and eliminated ϕ_2, we would have obtained a Cauchy–Lagrange equation for the lower layer. The latter can be easily derived in the form

$$\tilde{P}/\rho_1 = K_1(t) - \phi_{1t} - \tfrac{1}{2}\, \bar{u}_1^2 + h_1\, h_{1x}\, \bar{u}_1\, \bar{u}_{1x} + \tfrac{1}{2}\, h_1^2\, \bar{u}_{1x}^2 + \tfrac{1}{3}\, h_1^2\, \bar{u}_1\, \bar{u}_{1xx} - g\, h_1. \tag{48}$$

The arbitrary functions K_j are Bernoulli 'constants'. Their determination requires gauge conditions on ϕ_j and P_j (in order to have unequivocal definitions of these quantities) and a precise definition of the mean interface level and of the frame of reference.

4.2. Energy flux

From the relations (41) and (42), after simplifications and some elementary algebra, one obtains the equation for the conservation of the energy flux

$$\sum_{j=1}^{2} \partial_t\Big[\tfrac{1}{2}\,\rho_j\,h_j\,\Big\{\bar{u}_j^{\,2} + \tfrac{1}{3}\,h_j^{\,2}\,\bar{u}_{jx}^{\,2} - (-1)^j\,g\,h_j\Big\}\Big]$$
$$+\,\partial_x\Big[\rho_j\,h_j\,\bar{u}_j\,\Big\{\tfrac{1}{2}\,\bar{u}_j^{\,2} + \tfrac{1}{6}\,h_j^{\,2}\,\bar{u}_{jx}^{\,2} - (-1)^j\,\tfrac{1}{3}\,h_j\,\gamma_j - (-1)^j\,g\,h_j\Big\}\Big]$$
$$= Q\,\rho_2\,\partial_x\Big[\phi_{2t} + \tfrac{1}{2}\,\bar{u}_2^{\,2} - h_2\,h_{2x}\,\bar{u}_2\,\bar{u}_{2x} - \tfrac{1}{2}\,h_2^{\,2}\,\bar{u}_{2x}^{\,2} - \tfrac{1}{3}\,h_2^2\,\bar{u}_2\,\bar{u}_{2xx} - g\,h_2\Big]. \tag{49}$$

The right-hand sides of equations (45) and (49) both involve ϕ_2. The latter can be eliminated computing $D \times (49) - Q \times (45)$, yielding after one integration by parts

$$\sum_{j=1}^{2} \partial_t\Big[\tfrac{1}{2}\,D\,\rho_j\,h_j\,\Big\{\bar{u}_j^{\,2} + \tfrac{1}{3}\,h_j^{\,2}\,\bar{u}_{jx}^{\,2} - (-1)^j\,g\,h_j\Big\} - Q\,\rho_j\,h_j\,\bar{u}_j\Big]$$
$$+\,\partial_x\Big[D\,\rho_j\,h_j\,\bar{u}_j\,\Big\{\tfrac{1}{2}\,\bar{u}_j^{\,2} + \tfrac{1}{6}\,h_j^{\,2}\,\bar{u}_{jx}^{\,2} - (-1)^j\,\tfrac{1}{3}\,h_j\,\gamma_j - (-1)^j\,g\,h_j\Big\}$$
$$-\,Q\,\rho_j\,h_j\,\Big\{\bar{u}_j^{\,2} - (-1)^j\,\tfrac{1}{2}\,g\,h_j - (-1)^j\,\tfrac{1}{3}\,h_j\,\gamma_j\Big\}\Big]$$
$$= -\frac{\mathrm{d}\,Q}{\mathrm{d}t}\{\rho_1\,h_1\,\bar{u}_1 + \rho_2\,h_2\,\bar{u}_2\} \;=\; -\frac{\rho_2}{2}\,\frac{\mathrm{d}\,Q^2}{\mathrm{d}t} - \frac{\mathrm{d}\,Q}{\mathrm{d}t}\,(\rho_1 - \rho_2)\,h_1\,\bar{u}_1. \tag{50}$$

5. Conclusions and perspectives

We have derived the shallow two-layer Serre-type equations from a variational framework. The main contribution of this study is that we presented their multi-symplectic structure. The rather complicated nonlinear dispersion of the Serre-like equations makes non-trivial the derivation of the multi-symplectic structure, *a priori*. However, we have shown here that this structure for fluids stratified in several homogeneous layers can be rather easily obtained from the one layer case. For the sake of simplicity, we focused here on two layers in two-dimension with horizontal bottom and lid. Generalisations for three-dimensional several-layer stratifications should be straightforward from the multi-symplectic structure of the one-layer three-dimensional Serre equations with a varying bottom. This will be the subject of future investigations.

The size of the multi-symplectic structure increases rapidly with the number of layers and the number of spatial dimensions. However, the matrices involved are sparse and most of the equations are algebraically elementary. Thus, the calculus can be achieved easily and straightforwardly with any computer algebra system capable of performing symbolic computations.

Finding a multi-symplectic structure opens up new directions in the analysis and numerics of equations. In this paper, we illustrate another advantage of the multi-symplectic formalism: it also provides an efficient tool of calculus. Thanks to the multi-symplectic formalism, the conservation laws are obtained automatically, and thus the conserved quantities and their fluxes are obtained as well. The 'automatic' derivation of these conservation laws is an advantage of the multi-symplectic structure.

As the main perspective, we would like to mention the structure-preserving numerical simulations of nonlinear internal waves. The proposed multi-symplectic structure can be transposed to the discrete level if one employs a multi-symplectic integrator [7, 19]. These schemes were already rested in complex KdV simulations [12], but this direction seems to be essentially open for internal wave models.

Acknowledgments

The authors would like to acknowledge the support from CNRS under the PEPS 2015 Inphyniti programme and exploratory project FARA. The authors are grateful to the anonymous referees whose remarks enhanced our manuscript.

Appendix. Complementary equations

Multiplying by $(y + d_1)$ and $(d_2 - y)$ the vertical momentum (full Euler) equation for the lower and upper layers, respectively, and integrating over the layer thicknesses, we have

$$\int_{-d_1}^{\eta} \rho_1\,(y + d_1)\left[\frac{D\,v_1}{Dt} + g\right]\mathrm{d}y = -\int_{-d_1}^{\eta} (y + d_1)\,\frac{\partial\,P_1}{\partial y}\,\mathrm{d}y = h_1(\bar{P}_1 - \tilde{P}), \tag{51}$$

$$\int_{\eta}^{d_2} \rho_2\,(d_2 - y)\left[\frac{D\,v_2}{Dt} + g\right]\mathrm{d}y = -\int_{\eta}^{d_2} (d_2 - y)\,\frac{\partial\,P_2}{\partial y}\,\mathrm{d}y = h_2(\tilde{P} - \bar{P}_2), \tag{52}$$

where $\tilde{P}$ is the (unknown) pressure at the interface and $\bar{P}_j$ is the layer-averaged pressure. With the ansätze (1) one obtains

$$\bar{P}_j = \tilde{P} + (-1)^{j-1}\,\tfrac{1}{2}\,\rho_j\,g\,h_j + (-1)^{j-1}\,\tfrac{1}{3}\,\rho_j\,h_j\,\gamma_j, \qquad (j = 1, 2). \tag{53}$$

Integrating over the layer thicknesses the horizontal momenta, one obtains

$$\int_{-d_1}^{\eta} \rho_1\,\frac{D\,u_1}{Dt}\,\mathrm{d}y = -\int_{-d_1}^{\eta} \frac{\partial\,P_1}{\partial x}\,\mathrm{d}y \;=\; [h_1(\tilde{P} - \bar{P}_1)]_x - h_1\,\tilde{P}_x, \tag{54}$$

$$\int_{\eta}^{d_2} \rho_2\,\frac{D\,u_2}{Dt}\,\mathrm{d}y = -\int_{\eta}^{d_2} \frac{\partial\,P_2}{\partial x}\,\mathrm{d}y \;=\; [h_2(\tilde{P} - \bar{P}_2)]_x - h_2\,\tilde{P}_x, \tag{55}$$

thence with the ansätze (1) (with $j = 1, 2$)

$$[h_j\,\bar{u}_j]_t + \left[h_j\,\bar{u}_j^{\,2} - (-1)^j\,\tfrac{1}{2}\,g\,h_j^{\,2} - (-1)^j\,\tfrac{1}{3}\,h_j^{\,2}\,\gamma_j\right]_x = -\rho_j^{\,-1}\,h_j\,\tilde{P}_x. \tag{56}$$

From these relations, we obtain at once (with $\dot{Q} = \mathrm{d}Q/\mathrm{d}t$)

$$\dot{Q} + \left[h_1\,\bar{u}_1^2 + h_2\,\bar{u}_2^2 + g\,D\,h_1 + \tfrac{1}{3}\,h_1^2\,\gamma_1 - \tfrac{1}{3}\,h_2^2\,\gamma_2\right]_x = -(h_1\,\rho_1^{\,-1} + h_2\,\rho_2^{\,-1})\tilde{P}_x, \tag{57}$$

$$\sum_{j=1}^{2}\rho_j\left\{[h_j\,\bar{u}_j]_t + \left[h_j\,\bar{u}_j^{\,2} - (-1)^j\,\tfrac{1}{2}\,g\,h_j^{\,2} - (-1)^j\,\tfrac{1}{3}\,h_j^{\,2}\,\gamma_j\right]_x\right\} = -D\,\tilde{P}_x, \tag{58}$$

$$\bar{u}_{jt} + \bar{u}_j\,\bar{u}_{jx} - (-1)^j\,g\,h_{jx} - (-1)^j\,\tfrac{1}{3}\,h_j^{\,-1}[h_j^{\,2}\,\gamma_j]_x = -\rho_j^{\,-1}\,\tilde{P}_x. \tag{59}$$

The elimination of $\tilde{P}_x$ between the two equations (59) yields (16).

References

[1] Apel J R, Ostrovsky L A, Stepanyants Y A and Lynch J F 2007 Internal solitons in the ocean and their effect on underwater sound *J. Acoust. Soc. Am.* **121** 695
[2] Barros R and Gavrilyuk S 2007 Dispersive nonlinear waves in two-layer flows with free surface: II. Large amplitude solitary waves embedded into the continuous spectrum *Stud. Appl. Math.* **119** 213–51
[3] Barros R, Gavrilyuk S L and Teshukov V M 2007 Dispersive nonlinear waves in two-layer flows with free surface: I. model derivation and general properties *Stud. Appl. Math.* **119** 191–211
[4] Barthélémy E 1989 Etude des ondes longues internes et de leur dynamique dans les zones côtières *PhD Thesis* University Joseph Fourier
[5] Bridges T J 1997 Multi-symplectic structures and wave propagation *Math. Proc. Camb. Phil. Soc.* **121** 147–90
[6] Bridges T J and Donaldson N M 2007 Reappraisal of criticality for two-layer flows and its role in the generation of internal solitary waves *Phys. Fluids.* **19** 072111
[7] Bridges T J and Reich S 2001 Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity *Phys. Lett.* A **284** 184–93
[8] Chhay M, Dutykh D and Clamond D 2016 On the multi-symplectic structure of the Serre–Green–Naghdi equations *J. Phys. A: Math. Theor.* **49** 03LT01
[9] Choi W and Camassa R 1999 Fully nonlinear internal waves in a two-fluid system *J. Fluid Mech.* **396** 1–36
[10] Christodoulides P and Dias F 1995 Stability of capillary-gravity interfacial waves between two bounded fluids *Phys. Fluids* **7** 3013
[11] Duchêne V, Israwi S and Talhouk R 2016 A new class of two-layer Green–Naghdi systems with improved frequency dispersion *Stud. Appl. Math.* (in press) (doi:10.1111/sapm.12125)
[12] Dutykh D, Chhay M and Fedele F 2013 Geometric numerical schemes for the KdV equation *Comput. Math. Math. Phys.* **53** 221–36
[13] Garrett C and Munk W 1975 Space–time scales of internal waves: a progress report *J. Geophys. Res.* **80** 291–7
[14] Garrett C and Munk W 1979 Internal waves in the ocean *Ann. Rev. Fluid Mech.* **11** 339–69
[15] Johnson R S 2002 Camassa–Holm, Korteweg–de Vries and related models for water waves *J. Fluid Mech.* **455** 63–82
[16] Mal'tseva Z L 1989 Unsteady long waves in a two-layer fluid *Dinamika Sploshn. Sredy* **93–94** 96–110
[17] Marsden J E, Patrick G W and Shkoller S 1998 Multisymplectic geometry, variational integrators, and nonlinear PDEs *Commun. Math. Phys.* **199** 52
[18] Miyata M 1987 Long internal waves of large amplitude *Nonlinear Water Waves: IUTAM Symp. (Tokyo, Japan)* (Berlin: Springer) pp 399–405
[19] Moore B and Reich S 2003 Multi-symplectic integration methods for Hamiltonian PDEs *Future Gener. Comput. Syst.* **19** 395–402
[20] Ostrovsky L, Pelinovsky E, Shrira V and Stepanyants Y 2015 Beyond the KdV: post-explosion development *Chaos* **25** 097620
[21] Serre F 1953 Contribution à l'étude des écoulements permanents et variables dans les canaux *Houille Blanche* **8** 374–88
[22] Staub R 1972 Contribution à l'étude des ondes de gravité en eau peu profonde *PhD Thesis* University of Grenoble