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Response of a noncohesive packing of grains to a localized force: Deviation from continuum elasticity

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In order to characterize the mechanical behavior of grain piles, we investigate the response of a noncohesive two-dimensional packing of cylinders submitted to a localized force. By means of image processing, we obtain an accurate measurement of the individual grain displacements in the reversible regime of deformation. The measured displacement field deviates unambiguously from the predictions of linear elasticity and of other theoretical descriptions commonly used to model the behavior of cohesionless soils in civil engineering or in soil mechanics. Surprisingly, the analysis of the deformation field reveals a tendency to localization in the reversible regime.

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Noncohesiveness is an essential feature distinguishing grain packings from continuous media, because the former cannot resist tensile stresses. In such packings, grains interact via elastic contact forces and by solid friction. The relationship between contact forces and individual grain deformations is essentially nonlinear, as exemplified by the Hertz-Mindlin law valid for spherical grains [1–3]. It is worth noting that the number of contact forces between neighboring grains is generally larger than the number of equilibrium mechanical equations: this leads to multivalued equilibrium solutions [4–6]. These equilibrium equations have to be supplemented by a set of inequalities, stating that the contact forces exist in the Coulomb cone. This set of inequalities precludes any variational formulation to determine the contact forces, unlike the case of elasticity [7]. Moreover, for a loose packing, new contacts are created as the confining pressure is increased. All these considerations cast doubt on the applicability of the linear elasticity theory for modeling the behavior of dry grain packings. Nevertheless, according to classical civil engineering textbooks [8], grain piles or soils submitted to a purely compressive external load are considered to obey linear elasticity below the Mohr-Coulomb plastic yield criterion. In the elastic equilibrium state, the stress field obeys the Beltrami biharmonic equation, which is of elliptic nature. This description implies therefore the uniqueness of the solution for the internal stress field and for the deformation field, for a given set of boundary conditions. The existence of multiple equilibrium states for the ensemble of contact forces in the microscopic description calls into question the uniqueness expected from the elastic modeling in the continuous limit. Note, moreover, that for a grain pile there is a memory effect, that is, a dependence of the internal state of stress on the preparation process. This memory effect was first recognized by Darwin more than one century ago [9], and then reassessed by Geng *et al.* [10]. The memory effect implies the existence of multiple equilibrium states for identical boundary conditions, which obviously opposes the elastic description.

In order to gain insight into the nature of the stress equilibrium equations, we performed a point-punch test on a two-dimensional packing. The point-punch test represents the Green function of the mechanical response for the discrete medium, provided it is linear (note that this last assumption is also questionable [11]). The sample studied is prepared by cutting a 10-mm-thick elastomeric plate into grains that are identical cylinders (8 mm diameter, 10 mm long). The two-dimensional collection of grains is then packed into contact according to a triangular compact lattice (i.e., centered hexagonal) and bounded by a rigid hexagonal metal frame. The frame ensures zero normal displacement boundary conditions. The Young's modulus and the Poisson ratio of the polyurethane elastomer are respectively $E = 4 \times 10^6$ Pa and $\nu = 0.46$. The punch consists of a steel blade of 10 mm long, 3 mm thick, and a Young's modulus of 2×10^{11} Pa. The elastomeric material has been chosen because the small Young's modulus allows a high relative precision in the measurement of the deformations induced by a gentle point load. Note that the two-dimensional (2D) triangular packing (in-plane strain) can be considered as isotropic from the viewpoint of linear elasticity [12].

Previous experimental studies probing the response of a granular piling submitted to a point load have led to controversial analyses. Using a photoelastic visualization method and a piling consisting of square tiles, Da Silva and Rajchenbach [13] concluded that their observations cannot be interpreted in the framework of linear elastic modeling, and rather supported a hierarchical process for the stress transmission, consistently with the models of Harr [14] and of Coppersmith *et al.* [15]. Indeed, in two dimensions, the elastic response of a semi-infinite medium to a point normal force can be described as follows [16]. If the origin (of polar coordinates r,θ) is taken as the point of application of the load P (defined as a force per unit of length), the stress is everywhere radial, and its magnitude is given by $\sigma(r,\theta) = (2P/\pi r)\cos\theta$. Hence the contours of constant stress magnitude are a set of circles passing through the point of application of the force. This result holds for a semi-infinite elastic medium, and the isostress contours are slightly modified in the presence of rigid boundary conditions, as discussed below. By means of the same photoelastic method, but using pentagonal grains, Geng *et al.* [17] observed a wedge-shaped brightened domain (with a wedge angle close to 60°), which

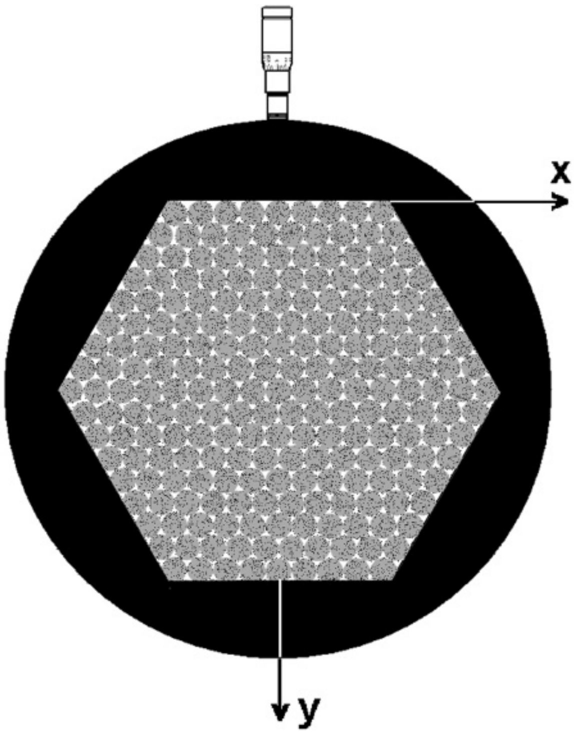

FIG. 1. Sketch of the load cell. The collection of cylinders is packed into a triangular (centered hexagonal) compact lattice, and confined into a rigid hexagonal frame. Cylinders are 8 mm in diameter. Grains are seeded with tracer particles. The displacements of the tracers are tracked as the punch is progressively moved inward.

does not correspond to the circular isostress contours expected from isotropic linear elasticity. Another interesting study is that of Goldenberg *et al.* [18], who probed the response of a 2D triangular packing by means of numerical simulations. Unilateral (i.e., noncohesive) normal contact forces were modeled by one-sided linear springs, and the tangential friction between grains was disregarded. In the limit of large systems, this model was shown to obey linear elasticity, which was consistent with the fact that the elastic solution as a response to a compressive load involves no tensile stress.

In our experimental procedure, a sequence of pictures is taken as the external load is increased. Then successive pictures are processed in order to access the displacement field. For the sake of accuracy, the bulk of the grains has been seeded with fine tracer particles (0.2 mm in diameter) which allow a precise tracking of local displacements. The measurement accuracy is of the order of 2 pixels, which corresponds to 2/100 particle diameter. The experimental load cell is sketched in Fig. 1.

Figure 2 depicts a 3D representation of the local displacement amplitude as a function of the position (x,y), resulting from a punch indentation of 2 mm. Note that the observed fluctuations should not be attributed to limitations on the measurement accuracy, but to the inhomogeneities in the displacements on the scale of one grain. Indeed, the tracers located in the vicinity of contacts experience a displacement much larger than that undergone by the tracers located near the center of the same grain. It is of interest to compare the displacement field to that obtained in the case of a continuous elastic material. With that aim, we submitted a continuous

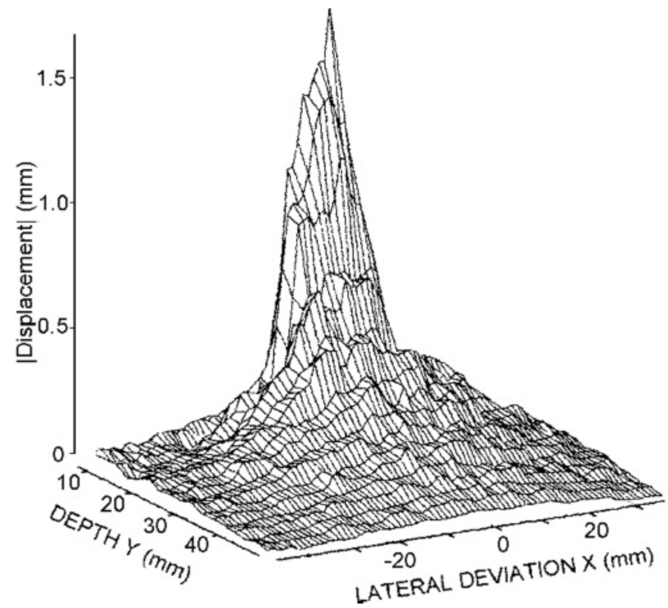

FIG. 2. 3D representation of the local displacement field obtained in the 2D packing of cylinders for a punch indentation of 2 mm.

plate of the same elastomeric material, of identical thickness (10 mm), positioned in the same hexagonal frame, to the same punching test. In Fig. 3, we show the displacements plotted as a function of the distance from the punch, measured along the direction of loading Oy, both for the discrete packing and for the continuous plate. Moreover, in order to compare the actual measured response and the theoretical elastic predictions, we show in the same figure the displacement curves obtained by solving analytically the Navier-Lamé equation (with Poisson ratios $\nu = -0.9$ and 0.5) with the same boundary conditions, and in the planar strain configuration. Note that a variation in the Young's modulus would be ineffective on displacements along the punching axis, since the present boundaries impose conditions on displacements, not on stresses. The experimental points obtained for the continuous plate compare well with

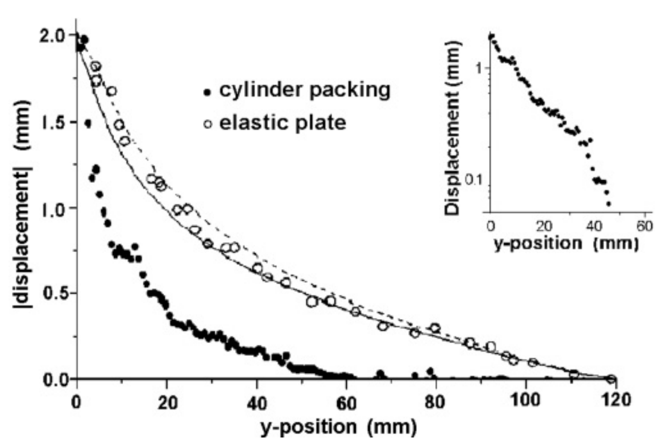

FIG. 3. Displacement amplitude, measured along the load axis Oy (•, discrete piling; ○, continuous plate) and elastic predictions for various Poisson moduli in a plane strain modeling (continuous line, $\nu = -0.9$; dashed line, $\nu = 0.5$). In inset, semilogarithmic plot of the displacement amplitude as a function of y.

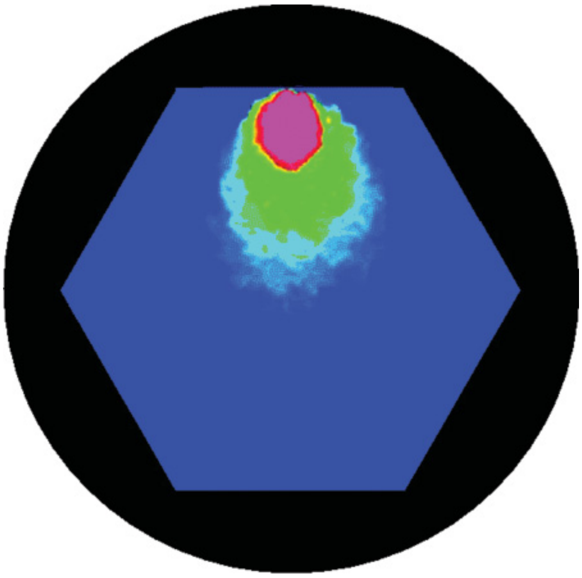

FIG. 4. (Color online) Same data as Fig. 2. Displacement amplitude as a function of the position (x,y), coded in color levels. The contours corresponding to displacements of equal amplitude are approximately elliptic.

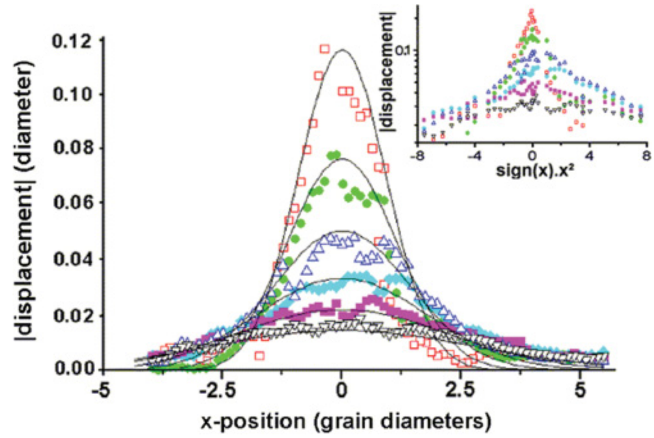

FIG. 5. (Color online) Displacement amplitude as a function of the deviation x from the axis of punching, plotted for various ordinates y (□, $y = 3.1$ mm; •, $y = 10.0$ mm; △, $y = 17.0$ mm; ⋄, $y = 21.0$ mm; ■, $y = 30.0$ mm; ∇, $y = 39.0$ mm. Continuous lines correspond to Gaussian fits according to Eq. (1) (see text). The lengths a and b appearing in Eq. (1) are of comparable magnitude, typically 3 or 4 grain diameters. In inset is shown a semilogarithmic plot of the displacement amplitude as a function of $x^2\mathrm{sgn}(x)$ (where x is here expressed in units of the grain diameter).

the theoretical elastic description, and thus clearly indicate an elastic response of the polymeric material. On the other hand, the displacement field corresponding to the 2D discrete packing deviates unambiguously from the predictions of the linear elasticity of continuous media. Instead, it follows roughly an exponential-like decay with the distance along the direction of loading (see inset, Fig. 3). The exponential-like decay along Oy suggests a localization process.

To sharpen our comparison of the mechanical response of a 2D granular packing to a punch with that of a continuous elastic medium, it is interesting to examine the displacement field in locations off the axis of loading. In Fig. 4, we show the map of the displacement amplitude as a function of the position (x,y), coded in gray levels. Apart from the fluctuations originating in the position of the tracers, according to their distance relative to grain centers or to contacts (as indicated above), it is clear that the isodisplacement contours closely resemble a family of ellipses.

From an assumed exponential-like decay of the displacements according to the punching axis direction, and of the ellipticlike shape of the isodisplacement contours in the (x,y) plane, elementary geometrical considerations lead to the following form for the displacement magnitude in the discrete medium:

$$|\,\mathrm{displacement}\,| \propto \exp(-y/a)\ \exp(-x^2/by), \qquad (1)$$

where the lengths a and b are here of comparable magnitude, typically 3 to 4 grain diameters. To confirm the validity of relation (1), we have plotted in Fig. 5 the displacement amplitude as a function of the transverse position x for various ordinates y. It is clear that the set of experimental data can reasonably be fitted by a family of Gaussian curves, with standard deviation varying as $\sqrt{y}$. Note that the data obtained by Da Silva *et al.* [13] can be accounted for by the same fitting function (1), with $a \simeq 4$ and $b \simeq 0.5$ grain diameters. The Gaussian widening along Ox is reminiscent of the diffusive models previously proposed by Harr [14] and by Coppersmith *et al.* [15]. However, note that the latter models predict a $1/\sqrt{y}$ decay along the direction of loading, rather than the exponential-like one, as observed.

At this point, it is worth addressing the issue of sample size, i.e., the cell size compared to the grain diameter. The packing here comprises about 220 grains. As shown above, the localization length is typically 4 grain diameters, and the tracer displacements attain a zero value (within the experimental accuracy) at positions far from the cell boundaries. We conclude that using larger cells (with rigid boundaries imposing a zero normal displacement) would not lead to any significant change in either the localization phenomena or the reported localization lengths.

As emphasized above, our data are incompatible with the predictions of linear elasticity commonly used in civil engineering to model the behavior of noncohesive soils below plastic yielding. It is worth noting that our experimental results also contradict some alternative theoretical approaches aimed at modeling the equilibrium behavior of granular assemblies, such as micropolar elasticity [19], micropolar elastoplasticity [20], hypoplastic theory [21], and the hyperbolic model of Bouchaud *et al.* [22].

Indeed, micropolar elasticity does not predict any localization effects in the response to a point load. Taking into account micropolar effects and couple stresses brings only negligible corrections (acting as $1/r^3$ with the distance to the punch) to the stress field obtained within the frame of continuum elasticity (which decreases as $1/r$) (e.g., see [23]). On the other hand, micropolar elastoplasticity predicts strain localization effects and the formation of shear bands at the plastic yielding. Thus, the localization process is predicted as irreversible. Irreversibility is of course also obtained within the frame of hypoplasticity. Concerning the hyperbolic model of

Bouchaud *et al.* [22], the observed response is fully incompatible with the picture of internal stresses confined in two "rays" starting from the loading point.

In summary, we have performed accurate measurements of the displacement field in a two-dimensional packing of elastic grains, as a response to a point load in the reversible regime of deformation. The coarse-grained displacement field deviates unambiguously from the predictions of continuum linear elasticity. We have found that the deformation amplitude, measured along the punching axis Oy, follows an exponential-like decay as a function of the distance y to the punch, which indicates a localization process. On the other hand, the isodisplacement contours resemble a family of ellipses, and the transverse width of the strained region varies approximately as $\sqrt{y}$. We emphasize that, although each individual grain behaves elastically, the collective response is *reversible*, so that the coarse-grained strain field does not map onto linear elasticity predictions. Moreover, current alternative models devised to overcome some known limitations of the standard elastoplastic model seem to fail in accounting for our data.

[1] H. Hertz, J. Reine Angew. Math. **92**, 156 (1881).

[2] R. D. Mindlin, J. Appl. Mech. **16**, 259 (1949).

[3] The elastic displacement δ of two equal spheres submitted to a compressive normal force F reads as $\delta = [9(1-\nu^2)^2/16E^2R]^{1/3}\ F^{2/3}$ (see Ref. [1]). For two cylinders of length l maintained in contact along a generatrix, the displacement is $\delta = [2F(1-\nu^2)/\pi El]\ \{\ln[4\pi ERl/(1-\nu^2)F]-1\}$ in the plane strain approximation (see K. L. Johnson, *Contact Mechanics*, (Cambridge University Press, Cambridge, UK, 1985)].

[4] S. Alexander, Phys. Rep **296**, 65 (1998).

[5] J. J. Moreau, in *Actes du 6ème Colloque en Calcul des Structures*, edited by M. Pottier-Ferry *et al.* (Teknea, Toulouse, 2003), Vol. 3, p. 465.

[6] T. Unger, J. Kertész, and D. E. Wolf, Phys. Rev. Lett. **94**, 178001 (2005).

[7] Note that there are attempts to describe the Coulombian friction within a bipotential formalism. This description succeeds in accounting for the plurality of solutions [see M. Buliga, G. De Saxcé, and C. Vallée, J. Convex Anal. **17**, 081 (2010)].

[8] K. Terzaghi, *Theoretical Soil Mechanics* (John Wiley & Son, New York, 1943).

[9] G. H. Darwin, *Proc. Inst. Civ. Eng*. **71**, 350 (1883).

[10] J. Geng, E. Longhi, R. P. Behringer, and D. W. Howell, Phys. Rev. E **64**, R060301 (2001).

[11] J. J. Moreau, Oral communication, G.D.R. MIDI Meeting, Carry le Rouet (2004, unpublished).

[12] M. H. Sadd and Qingli Dai, Mech. Mater. **37**, 641 (2005).

[13] M. Da Silva and J. Rajchenbach, Nature (London) **406**, 708 (2000).

[14] M. E. Harr, *Foundations of Theoretical Soil Mechanics* (McGraw-Hill, New York, 1966).

[15] S. N. Coppersmith, C.-h. Liu, S. Majumdar, O. Narayan, and T. A. Witten, Phys. Rev. E **53**, 4673 (1996).

[16] M. Flamant, Comptes Rendus Hebdomadaires de Séances de l'Académie des Sciences (Paris) **114**, 1465 (1892).

[17] Junfei Geng, D. Howell, E. Longhi, R. P. Behringer, G. Reydellet, L. Vanel, E. Clément, and S. Luding, Phys. Rev. Lett. **87**, 035506 (2001).

[18] C. Goldenberg and I. Goldhirsch, Phys. Rev. Lett. **89**, 084302 (2002).

[19] A. C. Eringen, in *Fracture II*, edited by H. Leibovitz (Academic, New York, 1968).

[20] H.-B. Mulhaus and I. Vardoulakis, Géotechnique **37**, 271 (1987).

[21] D. Kolymbas, *Introduction to Hypoplasticity*, (Balkema, Rotterdam, 2000).

[22] J. P. Bouchaud, M. E. Cates, and P. Claudin, J. Phys. I **5**, 639 (1995).

[23] W. Nowacki, in *Prípevsky k Teórii Stavebných Konstrukcií*, (Veda, Vydavatelstvo Slovenskej Akadémie Vied , Bratislava, 1974), p. 76.