Random Graph Coloring

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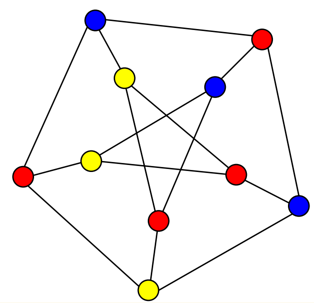
A simple question...



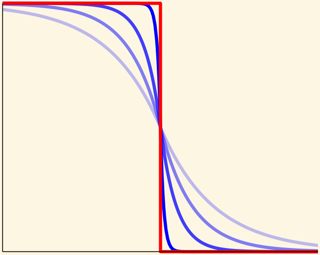
"What is the chromatic number of $\mathbf{G}(n, m)$?" [ER 60]

- n vertices
- $m = dn/2$ random edges

…that lacks a simple answer

- Early work: a factor two approximation
  - the greedy algorithm [GMcD ’75; BE ’76]
  - sparse case [SU ’84]
- Getting the asymptotics right
  - factor $\frac{4}{3}$ approximation [M’87]
  - factor $1 + o(1)$ for $d \gg n^{2/3}$… [B’88]
  - …and indeed for $d \gg 1$ [Ł’91]
- Concentration results
  - Concentration within $O(\sqrt{n})$. [SS 1987]
  - Two-point concentration for $d \ll n^{1/6}$… [Ł 1991]
  - …and in fact for $d \ll n^{1/2}$. [AK 1997]

Two moments do not suffice



The k-colorability threshold [ER'60]

- consider $\mathbf{G} = \mathbf{G}(n,m)$ with $2m/n \sim d$
- let $Z_k(\mathbf{G}) = \#k$-colorings
- 1st moment $d_{k-\mathrm{col}} \leq (2k-1)\ln k$
- 2nd moment $d_{k-\mathrm{col}} \geq (2k-2)\ln k$ [AN'05]
- improved bound $d_{k-\mathrm{col}} \leq (2k-1)\ln k - 1 + o(1)$ [CO'13]

The “cavity method”

The “cavity method”

- A generic but “recipe”. [“Belief/Survey Propagation”]
- A precise prediction as to the k-colorability threshold.
- A variety of “predictions” in
  - mathematical physics,
  - information theory,
  - probabilistic combinatorics,
  - compressive sensing.

The “cavity method”

Conjectures [KMRTSZ’07]

- the k-colorability threshold is

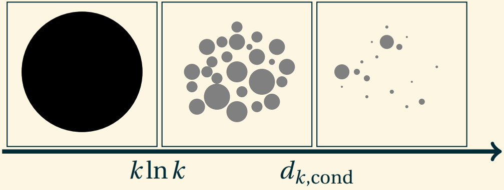
$$d_{k-\mathrm{col}} = (2k-1)\ln k - 1 + o(1)$$

- there occurs a condensation phase transition at

$$d_{k,\mathrm{cond}} = (2k-1)\ln k - 2\ln 2 + o(1)$$

- non-rigorous calculations based on Belief Propagation

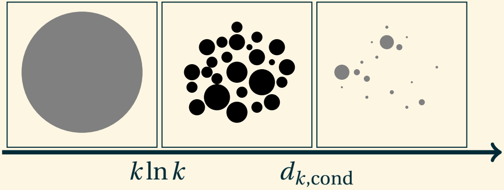
"Replica symmetry breaking"



"Replica symmetry"

- Can walk from one coloring to any another.
- Only *short-range* effects matter.
- Simple coloring algorithms succeed.

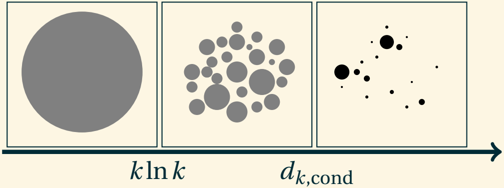
“Replica symmetry breaking”



“Dynamic replica symmetry breaking”

- The set of k-colorings shatters into tiny clusters.[ACO’08, M’12]
- *Long-range* effects emerge, stalling algorithms.
- Yet pairs of solutions “look uncorrelated”.

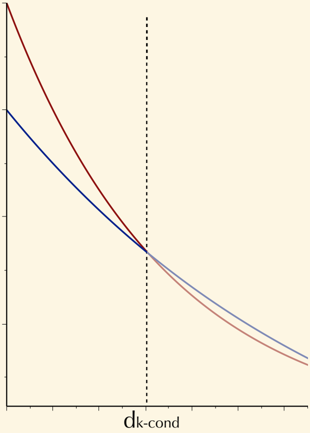
“Replica symmetry breaking”



“Condensation”

- A *bounded* number of clusters dominate.
- Pairs of solutions are *heavily correlated*.
- A second “phase transition”.

The “entropy crisis”



- as $d \to d_{k,\mathrm{cond}}$, both $\mathbb{E}\sqrt[n]{Z_k(\mathbf{G})}$ and the cluster size drop
- at $d_{k,\mathrm{cond}}$ they equalise

Chasing the k-colorability threshold

Theorem [BCOHRV '13]

We have $d_{k-\mathrm{col}} \geq d_{k,\mathrm{cond}}$.

- $d_{k,\mathrm{cond}} = (2k-1)\ln k - 2\ln 2 + \varepsilon_k$.
- Within $2\ln 2 + o_k(1) \approx 1.39$ of the first moment.

The condensation phase transition

Theorem [BCOHRV'14]

Assume $k > k_0$ and $d > (2k-1)\ln k - 2$. Define

$$\mathrm{BP} : \mathscr{P}([k])^{\gamma} \to \mathscr{P}([k]), \qquad \mathrm{BP}[\mu_1, \ldots, \mu_\gamma](\cdot) \propto \prod_{h \in [\gamma]} 1 - \mu_h(\cdot)$$

$$\mathscr{T} : \mathscr{P}^2([k]) \to \mathscr{P}^2([k]),$$

$$\pi \mapsto \sum_{\gamma=0}^{\infty} \frac{d^{\gamma} \exp(-d)}{\gamma! Z_\gamma(\pi)} \int \left[\sum_{h \in [k]} \prod_{i \in [\gamma]} 1 - \mu_i(h) \right] \delta_{\mathrm{BP}[\mu_1, \ldots, \mu_\gamma]} \mathrm{d} \bigotimes_{j \in [\gamma]} \pi(\mu_j)$$

Then $\mathscr{T}$ has a unique frozen fixed point $\pi^*_{d,k}$.

The condensation phase transition

$$\mathscr{B}(\pi) = \mathscr{B}^e(\pi) + \frac{1}{k}\sum_{i\in[k]}\sum_{\gamma_1,\dots,\gamma_k=0}^{\infty}\mathscr{B}^v(\pi;i;\gamma)\prod_{h\in[k]}\left(\frac{d}{k-1}\right)^{\gamma_h}\frac{\exp(-d/(k-1))}{\gamma_h!}$$

$$\mathscr{B}^e(\pi) = -\frac{d}{2k(k-1)}\sum_{h_1=1}^{k}\sum_{h_2\in[k]\setminus\{h_1\}}\int \ln\left[1-\sum_{h\in[k]}\mu_1(h)\mu_2(h)\right]\mathrm{d}\bigotimes_{i=1}^{2}\pi_{h_i}(\mu_i)$$

$$\mathscr{B}^v(\pi;i;\gamma) = \int \ln\left[\sum_{h=1}^{k}\prod_{h'\in[k]\setminus\{i\}}\prod_{j=1}^{\gamma_{h'}}1-\mu_{h'}^{(j)}(h)\right]\mathrm{d}\bigotimes_{h'\in[k]}\bigotimes_{j=1}^{\gamma_{h'}}\pi_{h'}(\mu_{h'}^{(j)})$$

Theorem (ctd.) [BCOHRV'14]

Further,

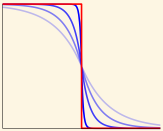
$$d \mapsto k(1-1/k)^{d/2} - \exp(\mathscr{B}(\pi^*_{d,k}))$$

has a unique zero $d_{k,\mathrm{cond}}$.

- $d < d_{k,\mathrm{cond}} \Rightarrow \lim \mathbb{E}\sqrt[n]{Z_k(\mathbf{G})} = k(1-1/k)^{d/2}$
- $d > d_{k,\mathrm{cond}} \Rightarrow \limsup \mathbb{E}\sqrt[n]{Z_k(\mathbf{G})} < k(1-1/k)^{d/2}$

Implies that $d_{k-\mathrm{col}} \geq d_{k,\mathrm{cond}} \approx (2k-1)\ln k - 2\ln 2$

Random regular graphs



Theorem [COEH'13]

For large k there is $d_{k-\mathrm{reg}}$ s.t. the random regular graph

- is k-colorable w.h.p. if $d < d_{k-\mathrm{reg}}$
- fails to be k-colorable w.h.p. if $d > d_{k-\mathrm{reg}}$

- about 61% of the time $d_{k-\mathrm{reg}}$ is not an integer
- "small subgraph conditioning" [KPGW'10]

The second moment method

- Let $Z(\mathbf{G}) \geq 0$ and $Z(G) > 0$ only if G is k-colorable.
- Suppose

$$0 < \mathbb{E}[Z^2] \leq C \cdot \mathbb{E}[Z]^2 \quad \text{with } C = C(k) > 0.$$

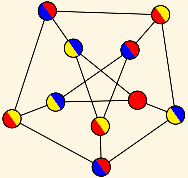
- By the Paley-Zygmund inequality,

$$\mathbb{P}[\mathbf{G} \text{ is } k\text{-col}] \geq \mathbb{P}\Big[Z > 0\Big] \geq \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} > 0.$$

Lemma [AF '99]

If $\liminf \mathbb{P}[\mathbf{G} \text{ is } k\text{-col}] > 0$ for some d, then $d_{k-\mathrm{col}} \geq d - o(1)$.

The Birkhoff polytope



- Call $\sigma : [n] \to [k]$ *balanced* if $|\sigma^{-1}(i)| = \frac{n}{k}$ for all i.
- Let $Z_{k,\text{bal}} = \#$balanced k-colorings of $\mathbf{G}$.
- Then
$$\frac{1}{n} \ln \mathbb{E}[Z_{k,\text{bal}}] \sim \ln k + \frac{d}{2} \ln(1 - 1/k).$$
- Define the $k \times k$ overlap matrix $\varrho(\sigma, \tau)$ by
$$\varrho_{ij}(\sigma, \tau) = \frac{k}{n} \cdot \left|\sigma^{-1}(i) \cap \tau^{-1}(j)\right|.$$
- Doubly-stochastic because σ, τ are balanced.

Balanced colorings

- Let $\mathscr{R} = \{\text{all possible overlap matrices}\}$ and

$$Z_{\varrho,\text{bal}} = \#\left\{(\sigma,\tau) \text{ balanced } k\text{-colorings with overlap } \varrho\right\}.$$

- Then

$$\mathbb{E}[Z_{k,\text{bal}}^2] = \sum_{\varrho\in\mathscr{R}} \mathbb{E}[Z_{\varrho,\text{bal}}]$$

  and thus

$$\ln \mathbb{E}[Z_{k,\text{bal}}^2] \sim \max_{\varrho\in\mathscr{R}} \ln \mathbb{E}[Z_{\varrho,\text{bal}}]$$

Balanced colorings

- We have

$$\ln \mathbb{E}[Z_{k,\mathrm{bal}}^2] \sim \max_{\varrho \in \mathscr{R}} \ln \mathbb{E}[Z_{\varrho,\mathrm{bal}}].$$

- Furthermore,

$$\begin{aligned} \frac{1}{n} \ln \mathbb{E}[Z_{\varrho,\mathrm{bal}}] &\sim f(\varrho) = H(\varrho) + E(\varrho), \quad \text{where} \\ H(\varrho) &= \ln k - \frac{1}{k} \sum_{i,j=1}^{k} \varrho_{ij} \ln(\varrho_{ij}) \qquad [\text{“entropy”}] \\ E(\varrho) &= \frac{d}{2} \ln\left[1 - \frac{2}{k} + \frac{1}{k^2} \sum_{i,j=1}^{k} \varrho_{ij}^2\right] \quad [\text{“probability”}] \end{aligned}$$

Balanced colorings

- As $n \to \infty$, $\mathscr{R}$ is dense in the Birkhoff polytope

$$\mathscr{D} = \left\{\text{doubly-stochastic } k \times k \text{ matrices}\right\}.$$

- Hence,

$$\frac{1}{n} \ln \mathbb{E}[Z_{k,\mathrm{bal}}^2] \sim \max_{\rho \in \mathscr{D}} f(\rho).$$

- At the barycenter $\bar{\rho} = \frac{1}{k}\mathbf{1}$ we have

$$f(\bar{\rho}) \sim \frac{2}{n} \ln \mathbb{E}[Z_{k,\mathrm{bal}}].$$

- $\mathbb{E}[Z_{k,\mathrm{bal}}^2] \leq C \cdot \mathbb{E}[Z_{k,\mathrm{bal}}]^2 \Leftrightarrow \max_{\rho \in \mathscr{D}} f(\rho)$ is attained at $\bar{\rho}$.

The singly-stochastic bound

Theorem [AN'05]

Let

$$\mathscr{S} = \{\text{singly-stochastic } k \times k \text{ matrices}\}.$$

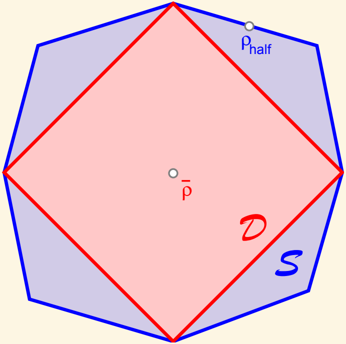
For $d \le d_{k,\mathrm{AN}} = 2k\ln k - 2\ln k - 2$ we have

$$\max_{\varrho \in \mathscr{D}} f(\varrho) \le \max_{\varrho \in \mathscr{S}} f(\varrho) \le f(\bar{\varrho}).$$

Proof

- Optimisation over a product of simplices.
- Going to the 6th derivative…

Singly vs. doubly-stochastic

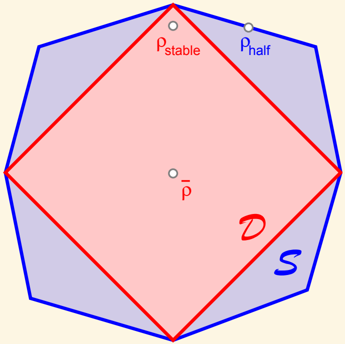


- For $d > d_{k,\mathrm{AN}}$, $\max_{\varrho \in \mathscr{S}} f(\varrho)$ is attained near ϱ_{half} with

$$\varrho_{\mathrm{half},ij} = \begin{cases} \mathbf{1}_{i=j} & \text{if } i \leq k/2, \\ \frac{1}{k} & \text{if } i > k/2. \end{cases}$$

- ϱ_{half} fails to be doubly-stochastic.

Singly vs. doubly-stochastic

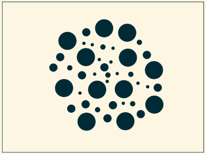


- For $d > d_{k,\mathrm{cond}} - (1 + \ln 2)$

$$\varrho_{\mathrm{stable}} = (1 - 1/k)\mathrm{id} + k^{-2}\mathbf{1} \quad \text{satisfies} \quad f(\varrho_{\mathrm{stable}}) > f(\bar{\varrho}).$$

- Thus, $\max_{\varrho \in \mathscr{D}} f(\varrho) > f(\bar{\varrho})$.

Clustering

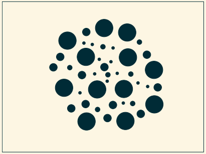


- Assume $k \ln k < d < d_{k,\text{cond}}$.
- Clusters $\mathscr{C}_1, \ldots, \mathscr{C}_N$.
- $\max_{i \le N} |\mathscr{C}_i| \le \exp(-\Omega(n)) \cdot Z_{k,\text{bal}}$.
- Clusters are well-separated.

Key idea

Add constraints to the maximisation problem to reflect clustering.

Tame colorings



Definition

A balanced k-coloring σ is *tame* if

- its *cluster*

$$\mathscr{C}(\sigma) = \left\{\tau : \rho_{ii}(\sigma,\tau) > 0.51 \text{ for all } i = 1,\ldots,k\right\}$$

  has size $|\mathscr{C}(\sigma)| \leq \mathbb{E}[Z_{k,\mathrm{bal}}]$,
- for any balanced k-coloring τ and any $1 \leq i, j \leq k$ we have

$$\rho_{ij}(\sigma,\tau) > 0.51 \;\Rightarrow\; \rho_{ij}(\sigma,\tau) \geq 1 - \frac{\ln^2 k}{k}.$$

The first moment

Proposition

Let $Z_{k,\text{tame}} = \#\text{good } k$-colorings. Then for $d < d_{k,\text{cond}}$,

$$\mathbb{E}[Z_{k,\text{tame}}] \sim \mathbb{E}[Z_{k,\text{bal}}].$$

Proof

- Consider the planted model.
- Exhibit a frozen core.
- *Cluster size???*

The second moment

Proposition ["second moment"]

For $d < d_{k,\text{cond}}$, $\mathbb{E}[Z_{k,\text{tame}}^2] \le C \cdot \mathbb{E}[Z_{k,\text{tame}}]^2$.

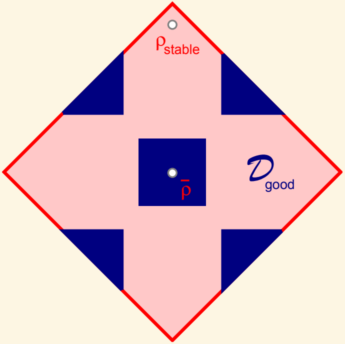
- Call a doubly-stochastic ϱ *separable* if

$$\varrho_{ij} > 0.51 \;\Rightarrow\; \varrho_{ij} \ge 1 - \frac{\ln^2 k}{k} \qquad \text{for all } i, j.$$

- Call ϱ *s-stable* if $s = \#\left\{(i,j) : \varrho_{ij} > 0.51\right\}$.
- Let

$$\mathcal{D}_{s,\text{tame}} = \left\{\text{all } s\text{-stable separable } \varrho\right\} \text{ and } \mathcal{D}_{\text{tame}} = \bigcup_{s=0}^{k-1} \mathcal{D}_{s,\text{tame}}.$$

The second moment

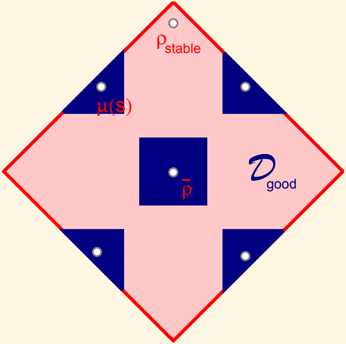


- We have

$$\frac{1}{n}\ln\mathbb{E}[Z_{k,\mathrm{tame}}^2] \sim \max_{\varrho\in\mathscr{D}_{\mathrm{tame}}} f(\varrho).$$

- Note that $\varrho_{\mathrm{stable}} \notin \mathscr{D}_{\mathrm{tame}}$.

The second moment



Key insight

Let $d \leq d_{k,\text{cond}}$ and $0 \leq s < k$. On $\mathscr{D}_{s,\text{tame}}$ the maximiser has the form

$$\mu_{ij}(s) = \begin{cases} 1-\alpha & \text{if } i = j \leq s, \\ \beta & \text{if } i \neq j,\ i, j \leq s \\ \gamma & \text{if } i, j > s, \\ \zeta & \text{otherwise.} \end{cases}$$

The cluster size

The planted model

- choose a random map $\hat{\boldsymbol{\sigma}} : [n] \to [k]$
- choose a random graph $\hat{\mathbf{G}}$ given that $\hat{\boldsymbol{\sigma}}$ is a k-coloring

The cluster size

The cluster

- assuming $d > (1+\varepsilon)k \ln k$, define

$$\mathscr{C}(\hat{\mathbf{G}}, \hat{\boldsymbol{\sigma}}) = \left\{ \tau : \min_{j \in [k]} \frac{|\tau^{-1}(j) \cap \hat{\boldsymbol{\sigma}}^{-1}(j)|}{|\hat{\boldsymbol{\sigma}}^{-1}(j)|} \geq 0.99 \right\}$$

- equivalent to other natural definitions [M'12]

The cluster size

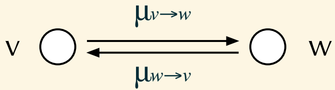
Lemma [COV'13]

We have $\mathbb{E}\sqrt[n]{Z_k(\mathbf{G})} \sim \sqrt[n]{\mathbb{E}[Z_k(\mathbf{G})]}$ iff with high probability

$$\sqrt[n]{\mathscr{C}(\hat{\mathbf{G}}, \hat{\boldsymbol{\sigma}})} \leq \sqrt[n]{\mathbb{E}[Z]} \sim k(1-1/k)^{d/2}$$

- Hence, we need to calculate $\sqrt[n]{\mathscr{C}(\hat{\mathbf{G}}, \hat{\boldsymbol{\sigma}})}$

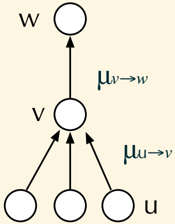
Belief Propagation



Messages

- messages $\mu_{v\to w}^{(t)}(\cdot) \in \mathscr{P}([k])$ for any adjacent pair (v, w)
- initialise $\mu_{v\to w}^{(0)}(j) = \mathbf{1}\{\hat{\boldsymbol{\sigma}}(v) = j\}$

Belief Propagation



The update rule

- define for $t \geq 0$ and adjacent (v, w)

$$\mu_{v\to w}^{(t+1)}(j) \propto \prod_{u \in \partial v \setminus w} 1 - \mu_{u\to v}^{(t)}(j)$$

- let

$$\mu_{v\to w}^{*}(\cdot) = \lim_{t\to\infty} \mu_{v\to w}^{(t)}(\cdot)$$

Belief Propagation

The Bethe free energy

- define

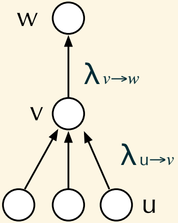
$$\mu_v^*(j) \propto \prod_{u \in \partial v} 1 - \mu_{u \to v}^*(j),$$

$$\mu_{(v,w)}^*(i,j) \propto \mathbf{1}\{i \neq j\} \mu_{v \to w}^*(i) \mu_{w \to v}^*(j),$$

$$\mathscr{B}(\mu^*) = \sum_v (1 - d(v)) H(\mu_v^*) + \frac{1}{2} \sum_{(v,w)} H(\mu_{(v,w)}^*)$$

- *Physics prediction:* $\sqrt[n]{\mathscr{C}(\hat{\mathbf{G}}, \hat{\boldsymbol{\sigma}})} \sim \exp(\mathscr{B}(\mu^*))$

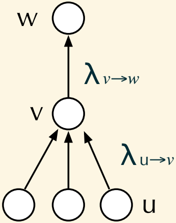
Warning Propagation



Discrete messages

- $\lambda^{(t)}_{v\to w}(j) \in \{0,1\}$ for any (v,w), $j \in [k]$
- initialise $\lambda^{(0)}_{v\to w}(j) = \mathbf{1}\{\hat{\boldsymbol{\sigma}}(v) = j\}$

Warning Propagation



Discrete updates

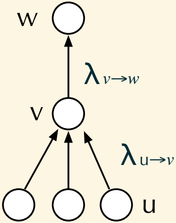
- define for $t \geq 0$ and adjacent (v, w)

$$\lambda_{v \to w}^{(t+1)}(j) = \min_{h \neq j} \max_{u \in \partial v \setminus w} \lambda_{u \to v}^{(t)}(h)$$

- let

$$\lambda_{v \to w}^{*}(j) = \lim_{t \to \infty} \lambda_{v \to w}^{(t)}(j)$$

Warning Propagation



Frozen vertices

- define

$$\lambda_v^*(j) = 1 - \max_{u \in \partial v \setminus w} \lambda_{u \to v}^*(j)$$

- then

$$\lambda_v^*(j) = 0 \Leftrightarrow \mu_v^*(j) = 0$$

List coloring

Lemma

W.h.p. $\mathscr{C}(\hat{\mathbf{G}}, \hat{\boldsymbol{\sigma}})$ is the set of all colorings such that for all v,

$$\tau(v) \in \Lambda^*(v) = \left\{ j \in [k] : \lambda_v^*(j) = 1 \right\}$$

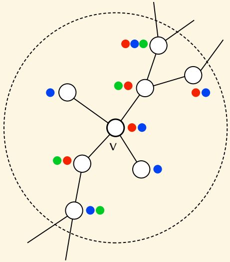
Lemma

Obtain $\tilde{\mathbf{G}} \subset \hat{\mathbf{G}}$ by deleting all edges $\{v, w\}$ such that

$$\Lambda^*(v) \cap \Lambda^*(w) = \emptyset.$$

Then τ is a list coloring of $\hat{\mathbf{G}}$ iff τ is a list coloring of $\tilde{\mathbf{G}}$.

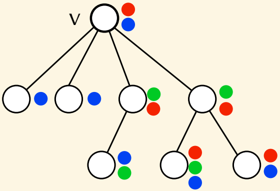
A branching process



The local structure of $\tilde{\mathbf{G}}$

- pick a random vertex $\boldsymbol{v}$ and consider $\partial^{\omega}_{\tilde{\mathbf{G}}}\boldsymbol{v}$…
- …including the color lists
- most likely acyclic

A branching process



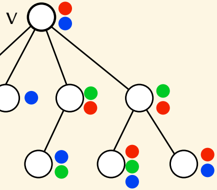
A random tree

- set of types $\mathscr{L} = \{(i,\ell) : i \in \ell \subset [k]\}$
- define a suitable distribution $q = (q_{i,\ell})$ on $\mathscr{L}$
- specifically,

$$q_{(i,\ell)} = \frac{(1-\exp(-\varrho d/(k-1)))^{k-|\ell|}}{k\exp(\varrho d/(k-1))^{|\ell|-1}}, \quad \text{where}$$

$$\varrho = (1-\exp(-\varrho d/(k-1)))^{k-1}$$

A branching process



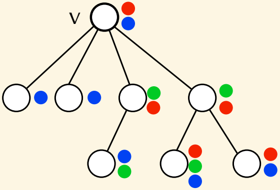
A random tree

- set of types $\mathscr{L} = \{(i, \ell) : i \in \ell \subset [k]\}$
- choose the type of the root vertex from q
- a vertex of type (i, ℓ) spawns

$$\mathrm{Po}(d q_{i',\ell'})$$

  children of type (i', ℓ'), provided $i \neq i'$, $\ell \cap \ell' \neq \emptyset$
- $\boldsymbol{T}$ = resulting tree

A branching process



Lemma

- $\boldsymbol{T}$ captures the local structure of $\tilde{\mathbf{G}}$
- $\boldsymbol{T}$ is *finite* almost surely
- $\mathbb{E}\sqrt[|\boldsymbol{T}|]{Z(\boldsymbol{T})} = \exp(\mathscr{B}(\pi^*_{d,k}))$

Corollary

With high probability we have $\sqrt[n]{\mathscr{C}(\hat{\mathbf{G}}, \hat{\boldsymbol{\sigma}})} \sim \exp(\mathscr{B}(\pi^*_{d,k}))$

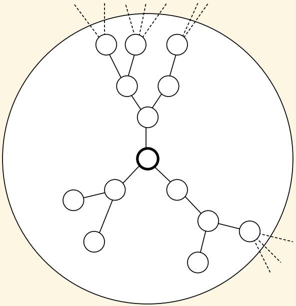
Upper bounding the k-colorability threshold

Theorem [CO'13]

We have $d_{k-\mathrm{col}} \le (2k-1)\ln k - 1 - o(1)$

- first moment over Warning Propagation fixed points
- vanilla first moment $d_{k-\mathrm{col}} \le (2k-1)\ln k$

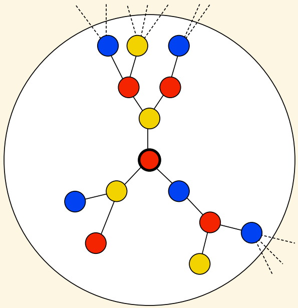
Long-range vs short-range



Local structure

- fix $t > 0$ and choose v randomly $\Rightarrow \partial^t(\mathbf{G}, v)$ is a tree w.h.p.
- the local structure converges to a Galton-Watson tree

Long-range vs short-range



Questions

- A random probability measure on $[k]^n$
- Are there “forbidden” local colorings?

The Boltzmann distribution

- let $\mathscr{S}_k(G) = \{k\text{-colorings of } G\}$ and $Z_k(G) = |\mathscr{S}_k(G)|$
- define a probability measure

$$\mu_{k,G} : [k]^{V(G)} \to [0,1], \quad \sigma \mapsto \frac{\mathbf{1}\{\sigma \in \mathscr{S}_k(G)\}}{Z_k(G)}$$

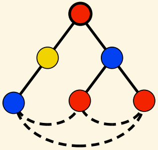
- for $U \subset V(G)$ define a distribution on $[k]^U$ by

$$\mu_{k,G|U}(\sigma_0) = \mu_{k,G}\left\{\sigma \in [k]^{V(G)} : \forall x \in U : \sigma(x) = \sigma_0(x)\right\}$$

- letting $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots$ be independet samples from $\mu_{k,G}$, write

$$\langle X(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_l)\rangle = \frac{1}{Z_k(G)^l} \sum_{\sigma_1, \ldots, \sigma_l \in \mathscr{S}_k(G)} X(\sigma_1, \ldots, \sigma_l)$$

Correlation decay



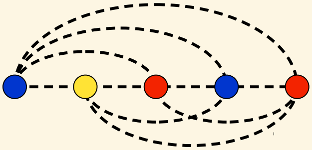
Theorem [COEJ'14]

Let $d < d_{k,\mathrm{cond}}$ and fix $t > 0$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v\in[n]} \mathbb{E}\left\| \mu_{k,\mathbf{G}|\partial^t(\mathbf{G},v)} - \mu_{k,\partial^t(\mathbf{G},v)} \right\| = 0.$$

- "The coloring induced on the depth-t neighborhood of v is asymptotically uniform."

Correlation decay



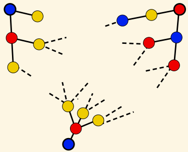
Theorem [COEJ'14]

Let $d < d_{k,\text{cond}}$ and fix $l > 0$. Then

$$\lim_{n\to\infty} \frac{1}{n^l} \sum_{v_1,\ldots,v_l} \mathbb{E} \left\| \mu_{k,\mathbf{G}|\{v_1,\ldots,v_l\}} - \bigotimes_{i=1}^{l} \mu_{k,\mathbf{G}|\{v_i\}} \right\| = 0.$$

- "asymptotic l-wise independence"
- earlier work: $d < 2(k-1)\ln(k-1)$ [MRT'11]

Correlation decay



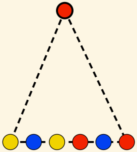
Theorem [COEJ'14]

Let $d < d_{k,\mathrm{cond}}$ and fix $l, t > 0$. Then

$$\lim_{n\to\infty} \frac{1}{n^l} \sum_{v_1,\dots,v_l} \mathbb{E} \left\| \mu_{k,\mathbf{G}|\partial^t(\mathbf{G},v_1)\cup\cdots\cup\partial^t(\mathbf{G},v_l)} - \bigotimes_{i=1}^{l} \mu_{k,\partial^t(\mathbf{G},v_i)} \right\| = 0.$$

- "asymptotic l-wise independence and uniformity"

Reconstruction



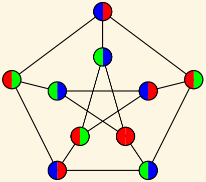
Corollary [COEJ'14]

Assume that $d < d_{k,\mathrm{cond}}$. Then

$$\text{non-reconstruction in } \mathbf{G} \Leftrightarrow \text{non-reconstruction in } \mathbf{T}(d,k).$$

- previously known for $d < 2(k-1)\ln(k-1)$ [MRT'11]
- reconstruction threshold in $\mathbf{T}(d,k)$ is $\sim k \ln k$ [E'14]

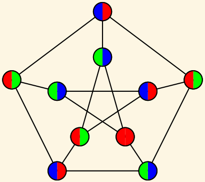
Bicolored graphs



The random replica model

- Generate a random graph $\mathbf{G}$.
- Sample two k-colorings $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ uniformly and independently.

Bicolored graphs

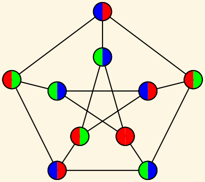


The planted replica model

- Choose $\boldsymbol{\sigma}'_1, \boldsymbol{\sigma}'_2 : [n] \to [k]$ uniformly and independently.
- $\mathbf{G}'$ = random graph given that $\boldsymbol{\sigma}'_1, \boldsymbol{\sigma}'_2$ are k-colorings.

- *Easy to analyse:* local structure converges to branching process

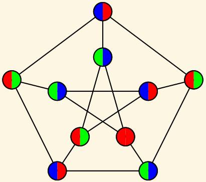
Bicolored graphs



Key lemma [COEJ'15]

If $d < d_{k,\mathrm{cond}}$, then *random replica* $\triangleleft$ *planted replica*.

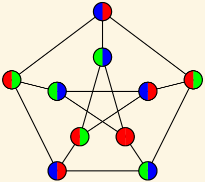
Bicolored graphs



Completing the proof

- study the statistics of bicolored trees in the *planted model*
- the result carries over to the *random replica model*
- the theorem follows from an averaging argument

Bicolored graphs



Averaging replicas [GM'07]

To show

$$\mathbb{E}\left\|\mu_{k,\mathbf{G}|\{v_1,v_2\}} - \mu_{k,\mathbf{G}|\{v_1\}} \otimes \mu_{k,\mathbf{G}|\{v_2\}}\right\| \to 0,$$

use

$$\mathbb{E}\left[\left\langle \mathbf{1}\{\boldsymbol{\sigma}(v_1) = c_1\}\mathbf{1}\{\boldsymbol{\sigma}(v_2) = c_2\} - k^{-2}\right\rangle^2\right]$$

$$= \mathbb{E}\left\langle \prod_{j=1}^{2}\left(\mathbf{1}\left\{\boldsymbol{\sigma}_j(v_1) = c_1\right\}\mathbf{1}\left\{\boldsymbol{\sigma}_j(v_2) = c_2\right\} - k^{-2}\right)\right\rangle$$

Summary

- physics-inspired rigorous proofs
- thorough understanding for $d < d_{k,\mathrm{cond}}$
- techniques generalise to other problems
- open problem: $d_{k-\mathrm{col}}$