Modelling self-organizing networks

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Cargese Fall School on Random Graphs
(September 2015)
Introduction

Spatial Preferred Attachment (SPA) Model

Future work
Multidisciplinary research

**Pure Mathematics:**
- Graph Theory
- Random Structures and Algorithms
- Modelling

**Applied Computer Science:**
- ... 

**Social Science:** for example,
- *Homophily, contagion and the decay of community structure in self-organizing networks* (PNAS paper!)
- *Social learning in a large, evolving network* (BlackBerry)
Multidisciplinary research

Applied Computer Science:

- *Utilizing big data for business-to-business matching and recommendation system* (ComLinked Corp., 2014-15)
- *A self-organizing dynamic network model increasing the efficiency of outdoor digital billboards* (KPM, 2014)
- *Personalized Mobile Recommender System* (Blackberry, 2013-14)
- *Intelligent Rating System* (Mako, 2012-13)
- *Dynamic clustering and prediction of taxi service demand* (Winston, 2012)
Multidisciplinary research

Applied Computer Science (currently):

- *Hypergraphs and their applications* (Tutte Institute for Mathematics and Computing)
- *Relationship Mapping Analytics for Fundraising and Sales Prospect Research* (Charter Press Ltd.)

Applied Computer Science (near future):

- *Network Modeling of Trust in Online Scientific Information Sources* (Bell Labs)

...
Outline

1. Introduction
2. Spatial Preferred Attachment (SPA) Model
3. Future work
Every human-technology interaction, or sensor network, generates new data points that can be viewed, based on the type of interaction, as a self-organizing network.
The web graph

nodes: web pages  edges: hyperlinks
Social networks

nodes: people  edges: social interaction
(e.g. Facebook friendship)
nodes: *scientists*  
edges: *co-authorship*

An induced subgraph of the collaboration graph with authors of Erdős number ≤ 2.
Are these networks similar?
Are these networks similar?

Answer: Yes!

- large scale
- ‘small world’ property (e.g. low diameter of $O(\log n)$, high clustering coefficient)
- degree distribution (power-law, the number of nodes of degree $k$ is proportional to $k^{-\gamma}$)
- bad expansion
- etc.
Why model self-organizing networks?

- uncover the generative mechanisms underlying self-organizing networks,
- models are a predictive tool,
- community detection,
- improving search engines (the web graph),
- spam and worm defense,
- nice mathematical challenges.
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(For example, PA model justifies “rich get richer” principle.)
A good graph model should...

...reproduce experimentally observed graph properties:

- degree distribution follows a power law,
- small average distance between nodes, (“small world”),
- locally dense, globally sparse,
- expansion properties (conductance),...

...include a credible model for agent behaviour guiding the formation of the link structure,

...agents should not need global knowledge of the network to determine their link environment.
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Common assumptions in the study of real-life networks

- Communities in a social network can be recognized as densely linked subgraphs.

- Web pages with many common neighbours contain related topics.

- Co-authors usually have similar research interests, etc.
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Such assumptions, commonly used in experimental and heuristic treatments of real-life networks, imply that there is an a priori “community structure” or “relatedness measure” of the nodes, which is reflected by the link structure of the graph.

**The network is a visible manifestation of an underlying hidden reality.**
Spatial graph models

- Nodes correspond to points in a (high-dimensional) feature space.
- The metric distance between nodes is a measure of “closeness.”
- The edge generation is influenced by the position and relative distance of the nodes.

This gives a basis for reverse engineering: given a graph, and assuming a spatial model, it is possible to estimate the distribution of nodes in the feature space from information contained in the graph structure.
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Spatial Preferred Attachment (SPA) Model

Nodes are points in *Euclidean space* (randomly and uniformly distributed).

We let $S$ be the unit hypercube in $\mathbb{R}^m$, equipped with the torus metric derived from any of the $L_p$ norms. This means that for any two points $x$ and $y$ in $S$,

$$d(x, y) = \min \left\{ \| x - y + u \|_p : u \in \{-1, 0, 1\}^m \right\}.$$
Nodes are points in *Euclidean space* (randomly and uniformly distributed).

Each node has a “sphere of influence” centered at the node. The size is determined by the *in-degree* of the node.

\[
|S(v, t)| = \frac{A_1 \deg^{-}(v, t) + A_2}{t}
\]
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- If $v$ falls into the sphere of influence $u$, it will link to $u$ with probability $p$. 
Spatial Preferred Attachment (SPA) Model

There are at least three features that distinguish the SPA model from previous models:

- A new node can choose its links purely based on local information.
- Since a new node links to each visible node independently, the out-degree is not a constant nor chosen according to a pre-determined distribution, but arises naturally from the model.
- The varying size of the influence regions allows for the occasional long links, edges between nodes that are spaced far apart. (This implies a certain “small world” property.)
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Spatial Preferred Attachment (SPA) Model

A simulation of the SPA model on the unit square with $t = 5,000$ and $p = 1$
Power law with exponent $x = 1 + \frac{1}{p}$.

**Theorem (Aiello, Bonato, Cooper, Janssen, Prałat)**

A.a.s.

$$N(0, t) = (1 + o(1)) \frac{t}{1 + p},$$

and for all $k$ satisfying $1 \leq k \leq \left( \frac{t}{\log^8 t} \right)^{\frac{p}{4p+2}},$

$$N(k, t) = (1 + o(1)) \frac{p^k}{1 + p + kp} t \prod_{j=0}^{k-1} \frac{j}{1 + p + jp}.$$

(The differential equations method is used.)
A little taste of DEs method

Definition

A **martingale** is a sequence $X_0, X_1, \ldots$ of random variables defined on the random process such that

$$
\mathbb{E}(X_{n+1} \mid X_0, X_1, \ldots, X_n) = X_n.
$$

In most applications, the martingale satisfies the property that

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\mathbb{E}(X_{n+1} \mid X_0, X_1, \ldots, X_n) = \mathbb{E}(X_{n+1} \mid X_n) = X_n.
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Example

Toss a coin $n$ times. Let $S_n$ be the difference between the number of heads and the number of tails after $n$ tosses.
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Theorem (Hoeffding-Azuma inequality)

Let $X_0, X_1, \ldots$ be a martingale. Suppose that there exist constants $c_k > 0$ such that

$$|X_k - X_{k-1}| \leq c_k$$

for each $k \leq n$. Then, for every $t > 0$,

$$\mathbb{P}(X_n \geq \mathbb{E}X_n + t) \leq \exp \left( -\frac{t^2}{2 \sum_{k=1}^{n} c_k^2} \right),$$

$$\mathbb{P}(X_n \leq \mathbb{E}X_n - t) \leq \exp \left( -\frac{t^2}{2 \sum_{k=1}^{n} c_k^2} \right).$$
A little taste of DEs method

\[ \mathbb{E}(N(0, t + 1) - N(0, t) | N(0, t)) = 1 - \frac{N(0, t)pA_2}{t} \]

We first transform \( N(0, t) \) into something close to a martingale. It provides some insight if we define real function \( f(x) \) to model the behaviour of the scaled random variable \( \frac{N(0, x)}{n} \). If we presume that the changes in the function correspond to the expected changes of random variable, we obtain the following differential equation

\[ f'(x) = 1 - f(x) \frac{pA_2}{x} \]

with the initial condition \( f(0) = 0 \).
A little taste of DEs method

The general solution of this equation can be put in the form

\[ f(x)x^{pA_2} - \frac{x^{1+pA_2}}{1 + pA_2} = C. \]

Consider the following real-valued function

\[ H(x, y) = yx^{pA_2} - \frac{x^{1+pA_2}}{1 + pA_2}. \]

(We expect \( H(w_t) = H(t, N(0, t)) \) to be close to zero.)

\[ \mathbb{E}(H(w_{t+1}) - H(w_t) \mid G_t) = O(t^{pA_2-1}) \]

\[ |H(w_{t+1}) - H(w_t)| = O(t^{pA_2} \log^2 n). \]


\[ |H(w_t) - H(w_{t_0})| = O(n^{1/2+pA_2} \log^3 n). \]
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**Out-degree:** An important difference between the SPA model and many other models is that the out-degree is not a parameter of the model, but is the result of a stochastic process.

**Theorem (Aiello, Bonato, Cooper, Janssen, Prałat)**

\[
\max_{0 \leq i \leq t} \deg^+ (v_i, t) \geq (1 + o(1))p \frac{\log t}{\log \log t}.
\]

However, a.a.s. all nodes have out-degree \(O(\log^2 t)\).

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A.a.s. \( \deg^+(v_t, t) = O(\log^2 t) \).
Let us partition the vertex set $V_t$ as follows:

$$V'_t = \left\{ x = (x_1, x_2, \ldots, x_m) \in V_t : x_1 < \frac{1}{2} \right\}$$

and $V''_t = V_t \setminus V'_t$. 
Sparse cuts

Theorem (Cooper, Frieze, Prałat)

A.a.s. the following holds

\[ |V_t'| = (1 + o(1))t/2, \]
\[ |V_t''| = (1 + o(1))t/2, \text{ and} \]
\[ |E(V_t', V_t'')| = O(t^{\max\{1 - 1/m, pA_1\} \log^5 t}) = o(t). \]
Let $l(v_i, v_j)$ denote the length of the shortest directed path from $v_j$ to $v_i$ if such a path exists, and let $l(v_i, v_j) = 0$ otherwise.

The directed diameter of a graph $G_t$ is defined as

$$D(G_t) = \max_{1 \leq i < j \leq t} l(v_i, v_j).$$
Diameter

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Theorem (Cooper, Frieze, Prałat)

There exists absolute constant $c_2$ such that a.a.s.

$$D(G_t) \geq \frac{c_2 \log t}{\log \log t}.$$  

(The lower bound requires the additional assumption that $A_1 < 3A_2$, and it is showed for dimension 2 only. However, it can be easily generalized.)
The distance between \( u \) and \( v \) can be estimated from the graph properties \((cn(u, v, n), \deg^- (u) \text{ and } \deg^- (v))\).

**Theorem (Janssen, Prałat, Wilson)**

\[ \text{Theorem 3.1. Let } \omega = \omega(n) \text{ be any function tending to infinity together with } n. \text{ The following holds a.a.s. Let } v_k \text{ and } v_\ell \text{ be vertices such that} \]
\[ k = \deg(v_k, n) \geq \deg(v_\ell, n) = \ell \geq \omega^2 \log n \]
\[ \text{in a graph generated by the SPA model. Let } d = d(v_k, v_\ell) \text{ be the distance between} \]
\[ v_k \text{ and } v_\ell \text{ in the metric space. Finally, let } T = f^{-1}(\ell/(\omega \log n)). \text{ Then,} \]
\[ \text{Case 1. If } d \geq \varepsilon(\omega \log n/T)^{1/m} \text{ for some } \varepsilon > 0, \text{ then} \]
\[ cn(v_k, v_\ell, n) = O(\omega \log n). \]

\[ \text{Case 2. If } k \geq (1 + \varepsilon) \ell \text{ for some } \varepsilon > 0 \text{ and} \]
\[ d \leq \left( \frac{A_1k + A_2}{c_m n} \right)^{1/m} - \left( \frac{A_1\ell + A_2}{c_m n} \right)^{1/m} = \Theta \left( \frac{k}{n} \right)^{1/m}, \quad (5) \]
\[ \text{then} \]
\[ cn(v_k, v_\ell, n) = (1 + o(1))p \ell. \]
\[ \text{If } k = (1 + o(1)) \ell \text{ and } d \ll (k/n)^{1/m} = (1 + o(1))(\ell/n)^{1/m}, \text{ then} \]
\[ cn(v_k, v_\ell, n) = (1 + o(1))p \ell \text{ as well.} \]

\[ \text{Case 3. If } k \geq (1 + \varepsilon) \ell \text{ for some } \varepsilon > 0 \text{ and} \]
\[ \left( \frac{A_1k + A_2}{c_m n} \right)^{1/m} - \left( \frac{A_1\ell + A_2}{c_m n} \right)^{1/m} < d \ll (\omega \log n/T)^{1/m}, \quad (6) \]
\[ \text{then} \]
\[ cn(v_k, v_\ell, n) = C \left( \frac{i_k}{i_\ell} \right)^{1/m} \text{ where } i_k = f^{-1}(k) \text{ and } i_\ell = f^{-1}(\ell) \text{ and } C = p A_1^{-1} A_2^{-1} c_m^{-1} \text{ for some } \varepsilon > 0, \]
\[ \text{then} \]
\[ cn(v_k, v_\ell, n) = \Theta \left( \frac{i_k}{i_\ell} \right)^{1/m}. \]
The distance between $u$ and $v$ can be estimated from the graph properties ($cn(u, v, n)$, $\text{deg}^-(u)$ and $\text{deg}^-(v)$).

Actual distance vs. estimated distance from simulated data
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**Spatial Preferred Attachment (SPA) Model**

**Future work**

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**Giant component**

**Conjecture (Cooper, Frieze, Prałat)**

\[ p_3 := \left( 2A_1 + 2A_2 \right)^{-1} \] is the threshold for the giant component.

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(a) \( A_1 = 1, A_2 = 1 \)  
(b) \( A_1 = 1, A_2 = 3 \)  
(c) \( A_1 = 3, A_2 = 1 \)

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**Conjecture**

The clustering coefficient of a vertex of degree \( k \) is of order \( 1/k \).
Common directions

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- Find the right parameters for power law exponent etc.
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Spatial Preferred Attachment (SPA) Model

- Generalize the model:
  - Node and edge deletion
  - Adding edges to existing nodes
  - Updating the out-links of a node
  - Shifting coordinates (“learning process”)

- Undirected graphs
- Non-uniform distribution of points

- Use the model to estimate the underlying geometry of the nodes.
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Use the model to estimate the underlying geometry of the nodes.
Consider two homophily hypotheses:

- the likelihood of tie formation between two actors increases with greater similarities in the actors’ tastes
- the likelihood of tie deletion between two actors increases with greater differences in the actors’ tastes

The role of social influence—third main hypothesis:

- actors tend to adopt the tastes of others they share direct connections with
Story 2: GEO-P model and domination number