Almost all 5-regular graphs have a 3-flow

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(joint work with Nick Wormald)

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Outline

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Tutte's conjecture

Definition

A nowhere-zero 3-flow in an undirected graph $G = (V, E)$ is an orientation of its edges and a function f assigning a number $f(e) \in \{1, 2\}$ to any oriented edge e such that for any vertex $v \in V$,

$$\sum_{e \in D^+(v)} f(e) - \sum_{e \in D^-(v)} f(e) = 0,$$

where $D^+(v)$ is the set of all edges emanating from v, and $D^-(v)$ is the set of all edges entering v.

Conjecture (Tutte, 1972)

Any 4-edge connected graph admits a nowhere-zero 3-flow.

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For a long time, it was not even known whether or not there is a finite k so that any k-edge connected graph has a nowhere-zero 3-flow...

Theorem (Lai, Zhang, 1992)

$k = 4\log_2 n$ *works for any n-vertex graph.*

Theorem (Alon, Linial, Meshulam, 1991)

$k = 2\log_2 n$ *works for any n-vertex graph (somewhat implicit but stronger form).*

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…now we know!

Theorem (Thomassen, 2012)

Every 8-edge-connected graph admits a nowhere-zero 3-flow.

Theorem (Lovász, Thomassen, Wu, Zhang, 2013)

Every 6-edge-connected graph admits a nowhere-zero 3-flow.

Tutte's conjecture

Conjecture (Tutte, 1972)

Any 4-edge connected graph admits a nowhere-zero 3-flow.

A graph admits a nowhere-zero 3-flow if and only if it has an edge orientation in which the difference between the outdegree and the indegree of any vertex is divisible by 3 (see, e.g., Seymour).

It is enough to prove the conjecture for 5-regular graphs (see, e.g., da Silva and Dahab).

Conjecture (Tutte, 1972, equivalent form)

Every 4-edge connected 5-regular graph has an edge orientation in which every outdegree is either 4 or 1.

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Conjecture (Jaeger, 1988, equivalent form)

For any fixed integer $p \geq 1$, every $4p$-edge connected, $(4p+1)$-regular graph has a mod $(2p+1)$-orientation, that is, an edge orientation in which every outdegree is either $3p+1$ or p.

Still open and appears to be difficult!

Our goal: prove that its assertion holds for almost all $(4p+1)$-regular graphs.

Fact: typical $(4p+1)$-regular graph is $(4p+1)$-edge connected.

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Jaeger conjecture — result

$\mathcal{G}_{n,d}$ — the probability space of random $d = (4p+1)$-regular n-vertex graphs with uniform probability distribution (d is fixed; n is even since d is odd).

A property holds 'asymptotically almost surely' (a.a.s.) if the probability that a member $G \in \mathcal{G}_{n,d}$ satisfies the property tends to 1 as $n \to \infty$.

Theorem (Alon and Prałat, 2011)

There exists a finite p_0 so that for any fixed integer $p > p_0$, a random $(4p+1)$-regular graph G admits, a.a.s., a mod $(2p+1)$-orientation, that is, an orientation in which every outdegree is either $3p+1$ or p.

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Jaeger conjecture — main observation

Theorem (Lai , Shao, Wu, and Zhou, 2009)

Let G be a $(4p+1)$-regular graph for some $p \in \mathbb{Z}^+$. Then $G=(V,E)$ has a mod $(2p+1)$-orientation iff there is a partition $V = V^+ \cup V^-$ with $|V^+| = |V^-|$ such that for any $S \subseteq V$,

$$|E(S,S^c)| \geq (2p+1)\big||S \cap V^+| - |S \cap V^-|\big|.$$

Theorem (Alon and Prałat, 2011)

There exists $c > 0$ so that the following holds. Let $G=(V,E)$ be a random $d=(4p+1)$-regular graph for some $p \in \mathbb{N}$. Then, a.a.s. V has a partition $V = V^+ \cup V^-$ with $|V^+| = |V^-|$ such that for any $S \subseteq V$,

$$|E(S,S^c)| \geq \left(2p + \frac{1}{2\sqrt{2}}\sqrt{p} - cp^{3/8}\right)\big||S \cap V^+| - |S \cap V^-|\big|.$$

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The lower bound for p_0 was not optimized, but it could not be reduced to $p_0 = 1$. Using the small subgraph conditioning method of Robinson and Wormald we show the following.

Theorem (Prałat and Wormald, 2015+)

A random 5-regular graph G_n on n vertices a.a.s. admits a nowhere-zero flow over $\mathbb{Z}_3$, that is, an edge orientation in which every out-degree is either 1 or 4.

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Outline

Spectral graph theory

The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of a graph are the eigenvalues of its adjacency matrix.

The value of $\lambda = \max(|\lambda_2|, |\lambda_n|)$ for a random d-regular graphs has been studied extensively.

Theorem (Friedman)

For every $\varepsilon > 0$ and $G \in \mathcal{G}_{n,d}$,

$$\mathbb{P}(\lambda(G) \leq 2\sqrt{d-1} + \varepsilon) = 1 - o(1) .$$

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Expander Mixing Lemma

Lemma (Alon, Chung, 1988)

Let G be any d-regular graph with n vertices and set $\lambda = \lambda(G)$. Then for all $S, T \subseteq V$

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \le \lambda \sqrt{|S||T|}.$$

(Note that $S \cap T$ does not have to be empty; $|E(S,T)|$ is defined to be the number of edges between $S \setminus T$ to T plus twice the number of edges that contain only vertices of $S \cap T$.)

Dense bisection

Condition

There exists a partition $V = V^+ \cup V^-$ *with* $|V^+| = |V^-| = |V|/2$ *such that for any* $S \subseteq V$,

$$|E(S, S^c)| \geq \left(2p + \frac{1}{2\sqrt{2}}\sqrt{p} - cp^{3/8}\right) \big||S \cap V^+| - |S \cap V^-|\big|.$$

Observation: For $S = V^+$ (or $S = V^-$) we need

$$|E(V^+, V^-)| \geq (2p + \Omega(\sqrt{p}))|V^+| = \frac{dn}{4} + \Omega(\sqrt{d}n).$$

Therefore, it is natural to start with a proof that there is such a dense bisection.

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Dense bisection

Proof.

- Since a random regular graph has only $O(1)$ triangles in expectation, a.a.s. there are $o(n)$ triangles.
- Delete an arbitrary edge of each triangle.
- Apply the result of Shearer (1992) who showed that a triangle-free graph $G = (V, E)$ with degree sequence $(d_1, d_2, \dots, d_n)$ has a cut of size at least $|E|/2 + \frac{1}{8\sqrt{2}} \sum_{i=1}^{n} \sqrt{d_i}$.
- Therefore, a.a.s. there is a cut (A, A^c) with

$$|E(A, A^c)| \geq \frac{dn}{4} + \frac{1}{8\sqrt{2}} \sqrt{d}n - o(n).$$

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- The Expander Mixing Lemma implies that, a.a.s.

$$\left(\frac{1}{2} - \frac{1}{d^{1/4}}\right) n < |A| < \left(\frac{1}{2} + \frac{1}{d^{1/4}}\right) n.$$

- Now it is enough to modify the cut (A, A^c) by shifting at most $n/d^{1/4}$ vertices, to get a bisection cut (V^+, V^-) so that $|V^+| = |V^-|$ and $|V^+ \setminus A| + |A \setminus V^+| \le n/d^{1/4}$.
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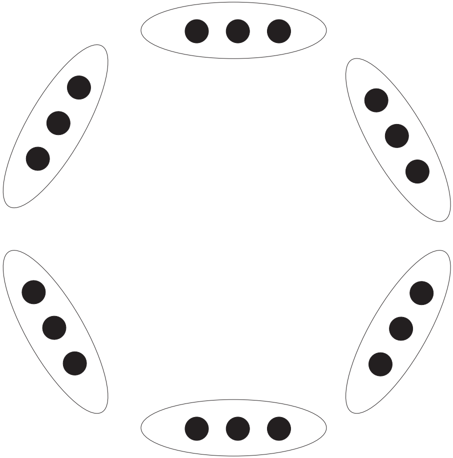

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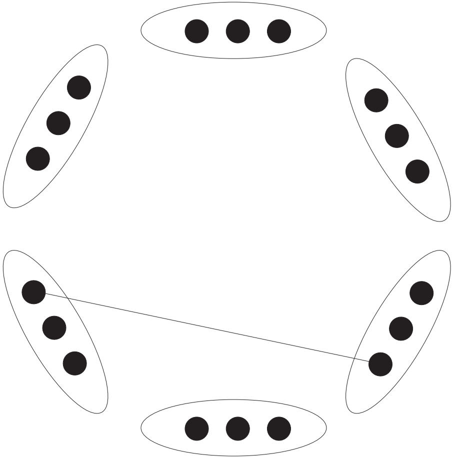

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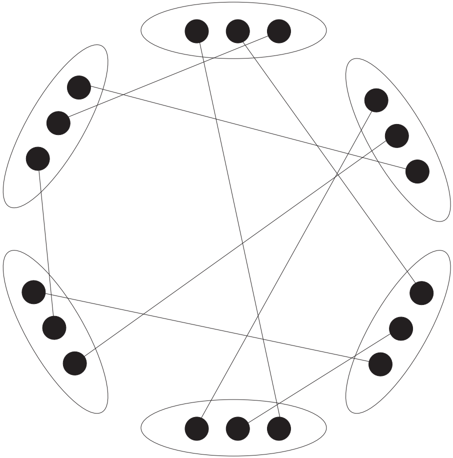

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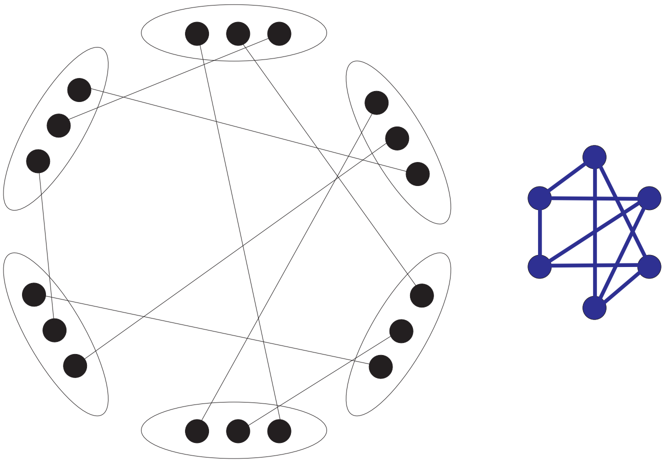

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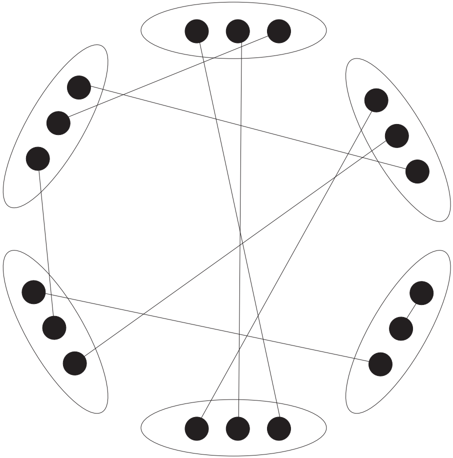

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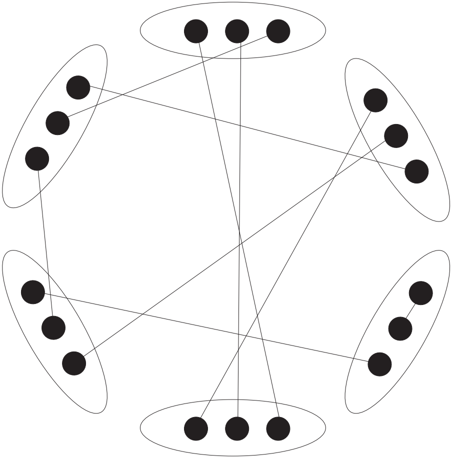

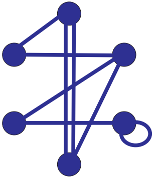

Pairing model $\mathcal{P}_{n,d}$

The probability of a random pairing corresponding to a given simple graph G is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n,d}$.

Moreover, a random pairing generates a simple graph with probability asymptotic to $e^{(1-d^2)/4}$ depending on d.

Therefore, any event holding a.a.s. over the probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n,d}$.

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Consider $\mathcal{P}_{n,5}$. Let Y be the number of valid orientations.

$$\mathbb{E}Y = \frac{\binom{n}{n/2} 5^n (5n/2)!}{M(5n)} \sim \left(\frac{25}{8}\right)^{n/2} \sqrt{5},$$

where

$$M(s) = \frac{s!}{(s/2)! 2^{s/2}}$$

is the number of perfect matchings of s points.

Indeed, there are $\binom{n}{n/2}$ ways to select in-vertices (since exactly half of the vertices must be such), 5^n ways to select one special point in each vertex, which determines each point to be either in or out, $(5n/2)!$ ways to pair up the points so that each “in” is paired with an “out”, and $M(5n)$ pairings in total.

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It can be shown that

$$\mathbb{E} Y(Y-1) \sim \left(\frac{25}{8}\right)^n \frac{25}{\sqrt{21}},$$

and so

$$\frac{\mathbb{E}\, Y^2}{(\mathbb{E}\, Y)^2} \sim \frac{5}{\sqrt{21}}.$$

The second moment method fails, but just barely.

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Solution: Under such circumstances, we can hope to apply the small subgraph conditioning method.

Small subgraph conditioning method

The distribution of Y is affected by the presence of certain small but not too common subgraphs in the random graph—usually the short cycles of given lengths.

Conditioning on the small subgraph counts affects $\mathbb{E}Y$, altering it by some constant factor.

Luckily and yet mysteriously, such conditioning reduces the variance of Y, to the point that conditioning on the numbers of enough small subgraphs reduces the variance to any desired small fraction of $(\mathbb{E}Y)^2$.

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- ...then hope for the best (all constants work out).

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Let X_k ($k \geq 1$) be the number of cycles of length k in $\mathcal{P}_{n,5}$. It is known that for each $k \geq 1$, $X_1, X_2, \ldots, X_k$ are asymptotically independent Poisson random variables with

$$\mathbb{E}X_k = \binom{n}{k} \frac{(k-1)!}{2} 5^k 4^k \frac{M(5n-2k)}{M(5n)} \to \lambda_k := \frac{4^k}{2k}.$$

The next step is to show that for each $k \geq 1$, there is a constant μ_k such that

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \to \mu_k$$

and, more generally, such that the joint factorial moments satisfy

$$\frac{\mathbb{E}(Y[X_1]_{j_1} \cdots [X_k]_{j_k})}{\mathbb{E}Y} \to \prod_{i=1}^{k} \mu_i^{j_i}$$

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for any fixed $j_1, \ldots, j_k$.

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \sum_{0\le i\le k/2} a_i \frac{(5\cdot 4)^k [n]_k \binom{n-2i}{n/2-i} 3^{2i} 5^{n-k} (5n/2-k)!}{2k\binom{n}{n/2} 5^n (5n/2)!}$$

$$\sim \sum_{0\le i\le k/2} \frac{a_i}{2k} \left(\frac{8}{5}\right)^k \left(\frac{3}{2}\right)^{2i},$$

where a_i is the number of orientations of the cycle C of length k with i vertices of in-degree 2.

We need to find the number of triples (P, C, O) where P is a pairing, C a k-cycle of P and O an orientation of P (and then divide by $M(5n)$). In fact, we count the triples (P, C, O) which have i vertices on C with in-degree 2 in C (these are in-vertices).

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \sum_{0\le i\le k/2} a_i \frac{(5\cdot 4)^k [n]_k \binom{n-2i}{n/2-i} 3^{2i} 5^{n-k} (5n/2-k)!}{2k\binom{n}{n/2} 5^n (5n/2)!}$$

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where a_i is the number of orientations of the cycle C of length k with i vertices of in-degree 2.

The number of ways to choose the pairs of (i.e. inducing the edges of) the cycle.

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \sum_{0\le i\le k/2} a_i \frac{(5\cdot 4)^k [n]_k \binom{n-2i}{n/2-i} 3^{2i} 5^{n-k} (5n/2-k)!}{2k\binom{n}{n/2} 5^n (5n/2)!}$$

$$\sim \sum_{0\le i\le k/2} \frac{a_i}{2k} \left(\frac{8}{5}\right)^k \left(\frac{3}{2}\right)^{2i},$$

where a_i is the number of orientations of the cycle C of length k with i vertices of in-degree 2.

The number of ways to select the remaining in- and out-vertices.

Vertices on the cycle: i of in-degree 2 in C (in-vertices), i of out-degree 2 in C (out-vertices), $k-2i$ of in/out degree 1 in C (in- or out- vertices).

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \sum_{0\le i\le k/2} a_i \frac{(5\cdot 4)^k [n]_k \binom{n-2i}{n/2-i} 3^{2i} 5^{n-k} (5n/2-k)!}{2k\binom{n}{n/2} 5^n (5n/2)!}$$

$$\sim \sum_{0\le i\le k/2} \frac{a_i}{2k}\left(\frac{8}{5}\right)^k \left(\frac{3}{2}\right)^{2i},$$

where a_i is the number of orientations of the cycle C of length k with i vertices of in-degree 2.

The number of ways to choose the special points of the vertices of C.

It only needs to be done for vertices of in-degree 0 or 2 in C; vertices of in-degree 1 in the cycle have their special point already determined.

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \sum_{0 \le i \le k/2} a_i \frac{(5 \cdot 4)^k [n]_k \binom{n-2i}{n/2-i} 3^{2i} 5^{n-k} (5n/2-k)!}{2k \binom{n}{n/2} 5^n (5n/2)!}$$

$$\sim \sum_{0 \le i \le k/2} \frac{a_i}{2k} \left(\frac{8}{5}\right)^k \left(\frac{3}{2}\right)^{2i},$$

where a_i is the number of orientations of the cycle C of length k with i vertices of in-degree 2.

The number of ways to choose the special points of vertices outside C.

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \sum_{0 \le i \le k/2} a_i \frac{(5 \cdot 4)^k [n]_k \binom{n-2i}{n/2-i} 3^{2i} 5^{n-k} (5n/2-k)!}{2k \binom{n}{n/2} 5^n (5n/2)!}$$

$$\sim \sum_{0 \le i \le k/2} \frac{a_i}{2k} \left(\frac{8}{5}\right)^k \left(\frac{3}{2}\right)^{2i},$$

where a_i is the number of orientations of the cycle C of length k with i vertices of in-degree 2.

The number of ways to pair up the points of appropriate types.

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \sum_{0\le i\le k/2} a_i \frac{(5\cdot 4)^k [n]_k \binom{n-2i}{n/2-i} 3^{2i} 5^{n-k} (5n/2-k)!}{2k\binom{n}{n/2} 5^n (5n/2)!}$$

$$\sim \sum_{0\le i\le k/2} \frac{a_i}{2k}\left(\frac{8}{5}\right)^k \left(\frac{3}{2}\right)^{2i},$$

where a_i is the number of orientations of the cycle C of length k with i vertices of in-degree 2.

Hence,

$$\mu_k := \frac{1}{2k}\cdot\left(\frac{8}{5}\right)^k \sum_{0\le i\le k/2} a_i \left(\frac{3}{2}\right)^{2i}.$$

To find a_i, one can select the $2i$ vertices of C that are to have in- or out-degree 2 in C. Since there are exactly two ways to orient C, $a_i = 2\binom{k}{2i}$, and this is the coefficient of x^{2i} in $q(x) := 2\,(1+x)^k$. It follows that

$$\sum_{0\le i\le k/2} a_i \left(\frac{3}{2}\right)^{2i} = \frac{1}{2}\Big(q(3/2)+q(-3/2)\Big) = \left(\frac{5}{2}\right)^k + \left(-\frac{1}{2}\right)^k,$$

and thus

$$\mu_k = \frac{1}{2k}\left(4^k + (-4/5)^k\right).$$

The final step is to compute

$$\delta_k = \frac{\mu_k}{\lambda_k} - 1 = \left(-\frac{1}{5}\right)^k$$

and then, using $-\log(1-x) = \sum_{k\geq 1} x^k/k$,

$$\begin{aligned} \exp\left(\sum_{k\geq 1} \lambda_k \delta_k^2\right) &= \exp\left(\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}\left(\frac{4}{25}\right)^k\right) \\ &= \exp\left(-\frac{1}{2}\log\left(1-\frac{4}{25}\right)\right) = \frac{5}{\sqrt{21}}. \end{aligned}$$

The fact that this is coincides with the asymptotic value of $\frac{\mathbb{E}\,Y^2}{(\mathbb{E}\,Y)^2}$ implies that $\mathbb{P}(Y>0) \sim 1$.

Theorem 4.1 ([56], see also [87]) *Let $\lambda_i > 0$ and $\delta_i \geq -1$, $i = 1, 2, \ldots$, be real numbers and suppose that for each n there are random variables $X_i = X_i(n)$, $i = 1, 2, \ldots$, and $Y = Y(n)$ defined on the same probability space $\mathcal{G} = \mathcal{G}(n)$ such that X_i is non-negative integer valued, Y is non-negative and $\mathbf{E}Y > 0$ (for n sufficiently large). Suppose furthermore that*

(a) For each $k \geq 1$ X_i, $i = 1, 2, \ldots, k$ are asymptotically independent Poisson random variables with $\mathbf{E}X_i \to \lambda_i$;

(b)

$$\frac{\mathbf{E}(Y[X_1]_{j_1} \cdots [X_k]_{j_k})}{\mathbf{E}Y} \to \prod_{i=1}^{k} (\lambda_i(1+\delta_i))^{j_i}$$

for every finite sequence $j_1, \ldots, j_k$ of non-negative integers;

(c) $\sum_i \lambda_i \delta_i^2 < \infty$;

(d) $$\frac{\mathbf{E}Y_n^2}{(\mathbf{E}Y_n)^2} \leq \exp\left(\sum_i \lambda_i \delta_i^2\right) + o(1) \qquad \text{as } n \to \infty.$$

Then

$$\mathbf{P}(Y_n > 0) = \exp\left(-\sum_{\delta_i = -1} \lambda_i\right) + o(1),$$

and, provided $\sum_{\delta_i=-1} \lambda_i < \infty$,

$$\overline{\mathcal{G}}^{(Y)} \approx \overline{\mathcal{G}}$$

where $\overline{\mathcal{G}}$ is the probability space obtained from $\mathcal{G}$ by conditioning on the event $\bigwedge_{\delta_i=-1}(X_i = 0)$.