DIAMETER PINCHING IN ALMOST POSITIVE RICCI CURVATURE

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Abstract. In this paper we prove a diameter sphere theorem and its corresponding \( \lambda_1 \) sphere theorem under \( L^p \) control of the curvature. They are generalizations of some results due to S. Ilias [8].

1. Introduction

Let \((M^n, g)\) be a complete manifold with Ricci curvature \( \text{Ric} \geq n-1 \). Then \((M^n, g)\) satisfies the following classical results (the proofs can be found in [13] for instance):

- \( \text{Diam}(M^n, g) \leq \pi \) (S. Myers) with equality iff \((M^n, g) = (S^n, \text{can})\) (S. Cheng),
- \( \lambda_1(M^n, g) \geq n \) (A. Lichnerowicz) with equality iff \((M^n, g) = (S^n, \text{can})\) (M. Obata),

where \( \text{Diam} \) is the diameter and \( \lambda_1 \) is the first positive eigenvalue.

Studying the properties of the sphere kept by manifold with \( \text{Ric} \geq n-1 \) and almost extremal diameter or \( \lambda_1 \), S. Ilias proved in [8] the following results:

Theorem 1.1 (S. Ilias). For any \( A > 0 \), there exists \( \epsilon(A, n) > 0 \) such that any \( n \)-manifolds with \( \text{Ric} \geq n-1 \), sectional curvature \( \sigma \leq A \) and \( \lambda_1 \leq n+\epsilon \) is homeomorphic to \( S^n \).

Theorem 1.2 (S. Ilias). For any \( A > 0 \), there exists \( \epsilon(A, n) > 0 \) such that any \( n \)-manifolds with \( \text{Ric} \geq n-1 \), \( \sigma \leq A \) and \( \text{Diam}(M) \geq \pi - \epsilon \) is homeomorphic to \( S^n \).

Remark 1.3. C. Croke proves in [7] that for \( n \)-manifolds with \( \text{Ric} \geq n-1 \), \( \lambda_1(M) \) close to \( n \) implies \( \text{Diam}(M) \) close to \( \pi \). The converse is proved in [8] (using a spectral inequality due to S. Cheng [6]).

Remark 1.4. For \( n \geq 4 \), M. Anderson [1] and Y. Otsu [10] construct sequences of complete metrics \( g_i \) with \( \text{Ric}(g_i) \geq n-1 \), \( \lambda_1(g_i) \to n \) and \( \text{Diam}(g_i) \to \pi \) on manifolds that are not homeomorphic to \( S^n \) (more precisely, Otsu shows that if \( n \geq 5 \), these manifolds can have infinitely many different fundamental groups).

Remark 1.5. The two results of S. Ilias have been improved by G. Perelman in [11], where the assumption \( \sigma \leq A \) is replaced by \( \sigma \geq -A \) (note that under the Ilias’s assumptions \( \sigma \leq A \) and \( \text{Ric} \geq n-1 \) we have \( |\sigma| \leq (n-2)A \)).

Subsequently, we denote \( \text{Ric}(x) \) the lowest eigenvalue of the Ricci tensor and \( \bar{\sigma}(x) \) the maximal sectional curvature at \( x \). In [4], we prove the following generalization of the Myers and Lichnerowicz theorems:

Theorem 1.6. For any \( p > n/2 \), there exists \( C(p, n) \) such that if \((M^n, g)\) is a complete manifold with \( \int_M (\text{Ric} - (n-1))^p \leq \frac{\text{Vol} M}{C(p, n)} \), then \( M \) is compact, has finite fundamental group and satisfies

\[
\text{Diam}(M) \leq \pi \left[ 1 + C(p, n) \left( \frac{\rho_p}{\text{Vol} M} \right)^{\frac{1}{p}} \right],
\]

\[
\lambda_1(M) \geq n \left[ 1 - C(p, n) \left( \frac{\rho_p}{\text{Vol} M} \right)^{\frac{1}{p}} \right],
\]

where \( \rho_p = \int_M (\text{Ric} - (n-1))^p \) and \( x_- = \max(0, -x) \).

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Remark 1.7. It follows from [4] that the constant $C(p,n)$ is computable, if $\int_M (\text{Ric} - (n - 1))_p^\rho$ is finite (for $p > n/2$) then $\text{Vol} M$ is finite, and that we can not bound the diameter or the first non zero eigenvalue under the assumption $\rho_p \leq \frac{1}{c(p, n)}$ or \( \rho \_{\pi} \) small (see [4]).

In this paper we prove the following extensions of the Ilias's stability results.

Theorem 1.8. Let $n \geq 2$ be an integer, $A > 0$ and $p > n$ be some reals. There exists a positive constant $C(p,n,A)$ such that any complete $n$-manifold which satisfies
\[
\int_M (\text{Ric} - (n - 1))_p^\rho < C(p, n, A) \text{Vol} M, \quad \int_M \sigma^p < A \text{Vol} M
\]
and
\[
\text{Diam}(M) \geq \pi (1 - C(p, n, A))
\]
is homeomorphic to $S^n$ (where $x_+ = \max(0, x)$).

Theorem 1.9. Let $n \geq 2$ be an integer, $A > 0$ and $p > n$ be some reals. There exists a positive constant $C(p,n,A)$ such that any complete $n$-manifold which satisfies
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\]
and
\[
\lambda_1(M) \leq n (1 + C(p, n, A))
\]
is homeomorphic to $S^n$.

Remark 1.10. By the Hölder inequality, the two curvature assumptions of Theorem 1.9 can be replaced by
\[
\int_M (\text{Ric} - (n - 1))_p^\rho < C(p, n, A) \text{Vol} M, \quad \int_M \sigma^p < A \text{Vol} M,
\]
where $\sigma(x)$ is an upper bound for the absolute value of the sectional curvatures at $x$.

2. Comparison results in almost positive Ricci curvature

Subsequently we denote $B(x, r)$ (resp. $S(x, r)$) the geodesic ball (resp. sphere) of center $x$ and radius $r$ and $L_k(r)$ (resp. $A_k(r)$) the volume of a geodesic sphere (resp. ball) of radius $r$ in $(\mathbb{S}^n, \frac{1}{n} g)$. Besides the theorem 1.6, we will need the following comparison results for manifolds of almost positive Ricci curvature (see [4] for a proof).

Proposition 2.1. For any $n \geq 2$ and $p \geq n/2$ ($p \geq 1$ if $n = 2$) there exists a constant $C(p,n)$ such that for any complete Riemannian $n$-manifold $(\mathbb{M}^n, g)$ with $\eta^{10} = \frac{\text{Vol} M}{\text{Vol} M} \leq c_{(p,n)}$, we have
\[
\left( \frac{\text{Vol}_{n-1} S(x, R)}{L_{1-\eta}(R)} \right)^{\eta^{10}} - \left( \frac{\text{Vol}_{n-1} S(x, r)}{L_{1-\eta}(r)} \right)^{\eta^{10}} \leq C(p, n, \eta) \eta^2 (R - r)^{\frac{p}{2p - 1}},
\]
\[
\frac{\text{Vol} B(x, r)}{\text{Vol} B(x, R)} \geq (1 - C(p, n, \eta)) \frac{A_1(r)}{A_1(R)},
\]
\[
\text{Vol}_{n-1} S(x, R) \leq (1 + \eta^2) L_{1-\eta}(R),
\]
\[
\text{Vol} B(x, R) \leq (1 + \eta) A_1(R).
\]
for all $x \in M$ and all radii $0 \leq r \leq R$.

For any $n \geq 2$ and $p > n/2$ there exists a constant $C(p,n)$ such that if $(\mathbb{M}^n, g)$ is a complete $n$-manifold with $\rho_\eta \leq \frac{1}{c_{(p,n)}}$, then $\|u\|_{L^2} \leq \text{Diam}(M) C(p, n) \|u\|_2 + \|u\|_2$, for any $u \in H^{1,2}(M)$. In the case $n = 2$, we have $\|u\|_4 \leq \text{Diam}(M) C \|u\|_2 + \|u\|_2$ if $\|u\|_2 \leq \frac{1}{c_{(p,n)}}$.

Similar estimates are proved in [12] under the assumption $\text{Diam}^{2p} \frac{\rho_\eta}{\text{Vol} M} \leq \frac{1}{c_{(p,n)}}$. 
The main tool to prove this proposition is the following lemma:

**Lemma 3.2.** Let \( n \geq 2 \) and \( p > n/2 \) \((p \geq 1 \text{ if } n = 2)\) and \( x_0 \in \mathbb{S}^n \). There exists a constant \( C(p,n) \) such that if \((M^n, g)\) is a complete \( n \)-manifold with \( \eta^{10} = \overline{p}_p \leq \frac{1}{C(p,n)} \) and \( \text{Diam}(M) \geq \pi - \frac{1}{C(p,n)} \) then there exists \( x_0 \in M \) such that for any \( C^1 \)-function \( u : [0, 2\pi] \to \mathbb{R} \) we have

\[
\left| \frac{1}{\text{Vol} M} \int_M u \circ d_M(x_0, \cdot) \, dv_g - \frac{1}{\text{Vol} \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, \cdot) \, dv_{\mathbb{S}^n} \right| \leq \|u'\|_{\infty} C(p,n) \left[ \eta + (\text{Diam}(M) - \pi) \right].
\]

**Proof.** Let \((x_0, y_0) \in M^2\) such that \( d = \text{Diam}(M) = d(x_0, y_0) \). The functions \( A, L, A_1 \) and \( L_1 \) are defined in Proposition 2.1 and prolonged by 0 to \( \mathbb{R} \) (note that the diameter of \( M \) can be greater than \( \pi \)). The function \( r \to u(r)A(r) \) is continuous and has right differential on \( \mathbb{R} \) equal to \( u'A + uL \). We infer the equalities

\[
\begin{align*}
  u(d) \text{Vol} M &= \int_0^d u(r)L(r) \, dr + \int_0^d u'(r)A(r) \, dr, \\
  u(\pi) \text{Vol} \mathbb{S}^n &= \int_0^\pi u(r)L_1(r) \, dr + \int_0^\pi u'(r)A_1(r) \, dr,
\end{align*}
\]

which imply

\[
\begin{align*}
  &\left| \frac{1}{\text{Vol} M} \int_M u \circ d_M(x_0, x) \, dv_g - \frac{1}{\text{Vol} \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, x) \, dv_{\mathbb{S}^n} \right| \\
  &= \left| \int_0^d \frac{u(r)L(r) \, dr}{\text{Vol} M} - \int_0^\pi \frac{u(r)L_1(r) \, dr}{\text{Vol} \mathbb{S}^n} \right| = \left| u(d) - u(\pi) + \int_0^\pi \frac{u'(r)A_1 \, dr}{\text{Vol} \mathbb{S}^n} - \int_0^d \frac{u'A \, dr}{\text{Vol} M} \right| \\
  &= \left| \int_0^d u' \left( \frac{A_1}{\text{Vol} \mathbb{S}^n} - \frac{A}{\text{Vol} M} \right) \, dr + \int_0^\pi u' \left( \frac{A_1}{\text{Vol} \mathbb{S}^n} - 1 \right) \, dr \right| \\
  &\leq \|u'\|_{\infty} \left( \int_0^d \frac{A_1}{\text{Vol} \mathbb{S}^n} - \frac{A}{\text{Vol} M} \, dr + |\pi - d| \right).
\end{align*}
\]

By Proposition 2.1 we have, for all \( r \leq d \):

\[
(1 - C(p,n)\eta) \frac{A_1(r)}{\text{Vol} \mathbb{S}^n} - \frac{A(r)}{\text{Vol} M} \leq 1 - \frac{\text{Vol} B(y_0, d - r)}{\text{Vol} M} \leq 1 - (1 - C(p,n)\eta) \frac{A_1(d - r)}{\text{Vol} \mathbb{S}^n} \leq \frac{A_1(r + \pi - d)}{\text{Vol} \mathbb{S}^n} + C(p,n)\eta.
\]

Hence

\[
\left| \frac{A(r)}{\text{Vol} M} - \frac{A_1(r)}{\text{Vol} \mathbb{S}^n} \right| \leq C(p,n)\eta + \frac{(A_1(r) - A_1(r + \pi - d))}{\text{Vol} \mathbb{S}^n}. \quad \text{An easy computation}
\]

gives \( \| \frac{1}{\text{Vol} M} \int_M u \circ d_M(x_0, x) \, dv_g - \frac{1}{\text{Vol} \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, x) \, dv_{\mathbb{S}^n} \|_{\infty} \leq C(p,n)\eta + (d - \pi) \). We now finish the proof of Proposition 3.1.
Proof. Lemma 3.2 applied to \( u = \sin^2, u = \cos^2 \) and \( u = \cos \) gives:

\[
\left| \int_M \frac{\sin^2 d_M(x_0, \cdot)}{\text{Vol} M} - \int_{S^n} \frac{\sin^2 d_{E^n}(x_0, \cdot)}{\text{Vol} S^n} \right| \leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\
\left| \int_M \frac{\cos^2 d_M(x_0, \cdot)}{\text{Vol} M} - \int_{S^n} \frac{\cos^2 d_{E^n}(x_0, \cdot)}{\text{Vol} S^n} \right| \leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\
\left| \int_M \frac{\cos d_M(x_0, \cdot)}{\text{Vol} M} - \int_{S^n} \frac{\cos d_{E^n}(x_0, \cdot)}{\text{Vol} S^n} \right| \leq C(p, n)(\eta + (d - \pi)_-) \leq 1.
\]

Hence, if we set \( f = \cos d_M(x_0, \cdot) \), we get

\[
\left| \|\nabla f\|_2^2 - \frac{n}{n+1} \right| \leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\
\left| \|f\|_2^2 - \frac{1}{n+1} \right| \leq C(p, n)(\eta + (d - \pi)_-) \leq 1, \\
\left| \frac{1}{\text{Vol} M} \int_M f \right| \leq C(p, n)(\eta + (d - \pi)_-) \leq 1.
\]

Which readily implies that

\[
\lambda_1(M) \leq \frac{\|\nabla (f - \bar{f})\|_2^2}{\|f - \bar{f}\|_2^2} \leq n(1 + C(p, n)(\eta + (d - \pi)_-)),
\]

where we have set \( \bar{f} = \frac{1}{\text{Vol} M} \int_M f \). \( \square \)

Remark 3.3. The same technique as in [5] can be used to prove that manifolds with almost positive Ricci curvature and \( \lambda_1 \) is close to \( n \) have a diameter close to \( \pi \) (see [12]).

4. Proof of Theorem 1.9

4.1. Fiber Bundle \( E \). Let \( E \) be the fiber bundle \( TM \oplus \mathbb{R} e \to M \) endowed with the following scalar product and linear connection:

\[
<X + fe, Y + he>_{E} = g(X, Y) + fh \\
D_f^E(X + fe) = D^M_f X + fZ + (df(Z) - g(Z, X)).e
\]

Where \( D^M_f \) is the Levi-Civita connection of the metric \( g \) on \( M \). We set \( p \) the orthogonal projection of \( E \) on \( TM \), \( \text{Ric}'(S) = \text{Ric}_M(p(S)) - (n-1)p(S) \) and \( \text{\triangle sph} = \text{\triangle sph}_E + \text{Ric}' \).

The following Lemma is proved in [3]:

Lemma 4.1. If \( f : M \to \mathbb{R} \) satisfies \( \triangle f = \lambda f \) then \( S_f = \nabla f + f.e \) satisfies \( \text{\triangle sph}(S_f) = (\lambda - n)(\nabla f - fe) \) and \( (D^E_f S_f, X) = Ddf(X, X) + fg(X, X) \).

Note also that we have

\[
R^E_{(Z, Y)}(X + fe) = R^M(Z, Y)X - (g(Y, X)Z - g(Z, X)Y).
\]

4.2. Bound on the Hessian of the first eigenfunction. To prove Theorem 1.9 we need a \( L^\infty \) bound on the Hessian of the first eigenfunction. In that purpose, we will modify the proof of Theorem 2.4 in [2] (whose proof would give us only a bound on \( \|DS_f\|_{n+\epsilon}/\|S_f\|_{\infty} \) for a given \( \epsilon = \epsilon(p, n) \)). In our case we really need to perform a Möser iteration.

Proposition 4.2. Let \( n \geq 2 \) and \( \infty \geq p > n/2 \). There exists a constant \( C(p, n) \) such that if \( (M^n, g) \) is any manifold with \( \bar{p}_p \leq \frac{1}{c(p, n)} \) and \( \lambda_1 \leq n + \frac{c(p, n)}{1} \) then for \( f : M \to \mathbb{R} \) such that \( \triangle f = \lambda_1 f \) we have:

\[
\frac{\|D^E_f S_f\|_{\infty}}{\|S_f\|_{\infty}} \leq C(p, n)(\lambda_1 + \|R\|_{2p})^\gamma (|\lambda_1 - n| + \bar{p}_p)^{\frac{1}{2p-n}},
\]

where \( S_f = \nabla f + f.e \) and \( \gamma = \frac{pn}{2p-n} \).
To prove Proposition 4.2 we need a commutation Lemma (see [2]):

**Lemma 4.3.** For any section \( S \in \Gamma(E) \) we have

\[
\frac{1}{2} \Delta (|DS|^2) + |D^2S|^2 \leq \langle D^* R^E S, DS \rangle + \text{Ric}^- |DS|^2 + \langle D\nabla S, DS \rangle + \|R^E\| |DS|^2,
\]

where \( \|R^E\| \) is the norm of the linear map \( R^E : \bigwedge^2 \mathbb{T}_u M \to \bigwedge^2 E_u \) defined by \( R^E(u \wedge v)(T, S) = \langle R^E(u \wedge v) T, S \rangle \).

**Remark 4.4.** This Lemma is valid for any Riemannian fiber bundle \( (E, D, \langle \cdot, \cdot \rangle) \).

We now give the proof of Proposition 4.2.

**Proof.** We set \( u = \sqrt{|DS|^2 + \epsilon^2} \), we have

\[
u \nabla u = \frac{1}{2} \nabla (u^2) + |du|^2 = \frac{1}{2} \nabla (u^2) + \frac{|D^2S, DS|^2}{|DS|^2 + \epsilon^2}
\]

Hence, by Lemma 4.3

\[
\int_M |d(u^k)|^2 \leq \frac{k^2}{2k - 1} \int_M \left( \frac{1}{2} \nabla |DS|^2 + |D^2S|^2 \right) u^{2(k - 1)}
\]

\[
\leq \frac{k^2}{2k - 1} \left( \int_M \text{Ric}^- u^{2k} + \int_M <D\nabla S, DS > u^{2(k - 1)} + \int_M <D^* R^E S, DS > u^{2(k - 1)} + \int_M \|R^E\| |u^{2k}| \right)
\]

We now apply the divergence theorem to the form \( u^{2(k - 1)}(\nabla S, D_S) \), and get for any \( k \geq 1 \):

\[
\int_M \langle D\nabla S, DS \rangle u^{2(k - 1)}
\]

\[
= \int_M \nabla S^2 u^{2(k - 1)} - 2(k - 1) \sum_i \int_M \langle D\nabla S(i), Du(i) \rangle u^{2k - 3}
\]

\[
\leq \int_M \nabla S^2 u^{2(k - 1)} + 2(k - 1) \int_M |\nabla S||du|^2 u^{2(k - 1)}
\]

\[
\leq \frac{k - 1}{2} \int_M |du|^2 u^{2(k - 1)} + (2k - 1) \int_M |\nabla S|^2 u^{2(k - 1)}
\]

We do the same with the form \( u^{2(k - 1)}(\text{tr}_{1, 3}(\langle R^E, S \rangle, D_S)) \) and get

\[
\int_M \langle D^* R^E S, DS \rangle u^{2(k - 1)}
\]

\[
= \int_M \frac{1}{2} |R^E|^2 u^{2(k - 1)} + 2(k - 1) \sum_{i,j} \langle R^E(i, j)S, D_S \rangle du(i)u^{2k - 3}
\]

\[
\leq \frac{k - 1}{2} \int_M |du|^2 u^{2(k - 1)} + (2k - 1) \int_M |R^E|^2 u^{2(k - 1)}
\]

Where we have used \( \sum_{i,j} \langle R^E(i, j), D^2S(i, j) \rangle = \frac{1}{2} |R^E|^2 \).

Since \( \int_M |du|^2 u^{2(k - 1)} = \frac{1}{k} \int_M |d(u^k)|^2 \), the three last inequalities give, for any \( k \geq 1 \):

\[
\|d(u^k)\|_2^2
\]

\[
\leq k(\int_M \text{Ric}^- u^{2k} + \int_M |R^E| |u^{2k}|) + k(2k - 1)(\int_M |R^E|^2 u^{2k - 2} + \int_M |\nabla S|^2 u^{2k - 2})
\]
\[ \leq 4k^2 \left( B_1 \|u\|_{\frac{2k}{p-1}}^{\frac{2k}{p}} + B_2 \|S\|_{\infty}^2 \|u\|_{\frac{2(k-1)}{p-1}}^{\frac{2(k-1)}{p}} \right), \]

where we have set

\[ B_1 = \|\text{Ric}^-\|_p + \|R^E\|_p \leq C(n)(\|R^M\|_2^2 + \lambda_1^2) = B^2, \]

\[ B_2 = \|\Delta_S\|_2^2 + \|R^E S\|_p^2 \leq \|\Delta S_{\text{ph}}\|_2^2 + \|\text{Ric}'\|_2^2 + \|R^E\|_p^2 \leq C(n)(\lambda_1^2 + \|R^M\|_p^2) = B^2. \]

By the Sobolev inequality given by Proposition 2.1, we get

\[ \|D S\|_{\frac{2n}{n+1}} \leq \|D S\|_{2k} + C(p, n) B k \sqrt{\|D S\|_{\frac{2k}{p}} + \|S\|_{\infty}^2 \|D S\|^{\frac{2(k-1)}{p-1}}}, \]

and by \( \|D S\|_{2k} \leq \|D S\|_{\frac{2n}{n+1}} \leq \|D S\|_{\frac{1}{2}} \|D S\|_{\frac{1}{2}} (\|D S\|_{\frac{2}{p}} \leq \|D S\|_{\frac{1}{2}} \|D S\|_{\frac{1}{2}}), \) we have

\[ \left( \|D S\|_{\frac{2n}{n+1}} \right) \left( \|D S\|_{n+1} \right) \leq \left[ 1 + B k C(p, n) \left(1 + \|S\|_{\infty}^2 \right) \frac{2n}{n+1} \right] \left( \|D S\|_{\frac{2n}{n+1}} \right) \left( \|D S\|_{n+1} \right), \]

where \( \nu = \frac{\frac{(p-1)}{2p} + 1}{(n-2)p} \) and \( a_0 = \frac{2p}{p-1} + 1 \), \( a_1 = \nu a_0 + \frac{2n}{n-2} \). Then we get

\[ \left( \|D S\|_{n+1} \right) \left( \|D S\|_{n+1} \right) \left[ 1 + a_0 C(p, n) B (1 + \|S\|_{\infty}^2 \|D S\|_{\infty}^2) \right] \left( \|D S\|_{\frac{2n}{n+1}} \right) \left( \|D S\|_{n+1} \right), \]

Hence

\[ 1 = \lim_{n \to +\infty} \left( \|D S\|_{\frac{2n}{n+1}} \right) \left( \|D S\|_{n+1} \right) \leq \prod_{i=1}^{\infty} \left( 1 + C(p, n) a_i B (1 + \|S\|_{\frac{2}{\infty}}^2 \|D S\|_{\frac{2}{\infty}}^2) \right) \left( \|D S\|_{\frac{2n}{n+1}} \right) \left( \|D S\|_{n+1} \right), \]

The Hölder inequality \( \|D S\|_{a_0} \leq \|D S\|_{1}^{\frac{1}{2}} \|D S\|_{\frac{1}{2}} \), gives

\[ \|D S\|_{\frac{1}{2}} \leq \prod_{i=1}^{\infty} \left( 1 + C(p, n) a_i B (1 + \|S\|_{\frac{2}{\infty}}^2 \|D S\|_{\frac{2}{\infty}}^2) \right) \|D S\|_{\frac{1}{2}}, \]

If \( \|D S\|_{\infty} \geq \|S\|_{\infty} \), then inequality \( (\ast) \) gives

\[ \|D S\|_{\infty} \leq \prod_{i=1}^{\infty} \left( 1 + C(p, n) a_i B (1 + \|S\|_{\frac{2}{\infty}}^2 \|D S\|_{\frac{2}{\infty}}^2) \right) \|D S\|_{\frac{1}{2}}, \]

If \( \|D S\|_{\infty} \leq \|S\|_{\infty} \), then inequality \( (\ast) \) gives

\[ \|D S\|_{\frac{1}{2}} \leq \left( \frac{\|S\|_{\infty}}{\|D S\|_{\infty}} \right)^{\frac{n}{(n-2)p}} \prod_{i=1}^{\infty} \left( 1 + C(p, n) a_i B \right)^{\frac{n}{(n-2)p}}, \]

hence

\[ \|D S\|_{\infty} \leq C(p, n) \left( \lambda_1 + \|R\|_{2p} \right)^{\frac{n}{(n-2)p}} \left( \|S\|_{\frac{2}{\infty}}^2 \right)^{\frac{2p-n}{2p-n+2p}}. \]

At this stage note that, by Lemma 4.1 we have

\[ \|D S\|_{2}^2 = \|\Delta S_{\text{ph}}\|_{2} < \|S\|_{L^2} \leq \|\text{Ric}'\|_{2} < \|S\|_{L^2} \]

\[ \leq \|S\|_{\infty}^2 + \int_M \left( \frac{\text{Ric} - (n-1)}{\text{Vol } M} \right)^+ |S| \leq (\|S\|_{\infty}^2 + \|\text{Ric}'\|_{2}) |S|_{\infty}^2. \]

Since we have \( \frac{2p-n}{2p-n+2p} \leq 1 \), we get the result. \( \square \)
4.3. Critical points of the first eigenfunction. By Proposition 4.2 the section $S_f = \nabla f + fe$ of $E$ satisfies $\|D^ES_f\|_{\infty} \leq C(p,n,A)(|\lambda_1 - n| + \overline{p}_p)^{\frac{1}{\gamma}} \|S_f\|_{\infty}$. Since we can suppose the pinching on $|\lambda_1 - n|$ and $\overline{p}_p$ small enough to have $C(p,n,A)(|\lambda_1 - n| + \overline{p}_p)^{\frac{1}{\gamma}} \leq 1/4$, the previous inequality and Theorem 1.6 give

$$\inf |S_f| \geq \left[ 1 - C(p,n,A)(|\lambda_1 - n| + \overline{p}_p)^{\frac{1}{\gamma}} \right]\|S_f\|_{\infty} > C(p,n,A)(|\lambda_1 - n| + \overline{p}_p)^{\frac{1}{\gamma}} \|S_f\|_{\infty} \geq \|D^ES_f\|_{\infty}$$

We infer that if $x_0$ is a critical point of $f$ then by Lemma 4.1 we have

$$|Df_{x_0}(X,X) + f(x_0)| = |(D^ES_f, X)_E| \leq \|D^ES_f\|_{\infty} < |S_f(x_0)| = |f(x_0)|,$$

for any unit vector $X$ of $T_{x_0}M$. Hence we have $-|f(x_0)| - f(x_0) < Df_{x_0}(X,X) < |f(x_0)| - f(x_0)$ for any critical point $x_0$ of $f$. So the only critical points of $f$ are non degenerate global extrema, which implies that $M$ is homeomorphic to $S^n$ by the Reeb’s theorem.

References


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