Groups of smooth diffeomorphisms of Cantor sets embedded in a line

Dominique Malicet, Emmanuel Militon

25 février 2019

Abstract

Let K be a Cantor set embedded in the real line $\mathbb{R}$. Following Funar and Neretin, we define the diffeomorphism group of K as the group of homeomorphisms of K which locally look like a diffeomorphism between two intervals of $\mathbb{R}$. Higman-Thompson's groups V_n appear as subgroups of such groups. In this article, we prove some properties of this group. First, we study the Burnside problem in this group and we prove that any finitely generated subgroup consisting of finite order elements is finite. This property was already proved by Rover in the case of the groups V_n. We also prove that any finitely generated subgroup H without free subsemigroup on two generators is virtually abelian. The corresponding result for the groups V_n was unknown to our knowledge. As a consequence, those groups do not contain nilpotent groups which are not virtually abelian.

1 Introduction

We call *Cantor set* any compact totally disconnected set K such that any point of K is an accumulation point.

When we want to study the dynamics of the action of a group G on a closed surface, it is convenient to look at a minimal subset of its action. Recall that a G-invariant closed nonempty subset K of our surface is called a minimal subset for the action of G on our surface if every orbit of points in K is dense in K. This is equivalent to saying that K is minimal for the inclusion relation among G-invariant closed nonempty subsets. Zorn's lemma ensures that such sets always exist. A typical case which can occur is the case where this minimal subset turns out to be a Cantor set K.

In this article, we will restrict ourself to the case where our Cantor set is embedded in a line, *i.e.* embedded in a one-dimensional submanifold diffeomorphic to $\mathbb{R}$.

We will give two equivalent definitions of the group we are interested in. Let r be an integer greater than or equal to 1 or $+\infty$.

Definition 1.1. *Let K be a Cantor set contained in a line L which is C^r-embedded in a manifold M with $\dim(M) \geq 2$. We call group of C^r-diffeomorphisms of K the group of restrictions to K of C^r-diffeomorphisms f of M such that $f(K) = K$. We will denote this group by $\mathfrak{Diff}^r(K)$.*

Remarks:

1. The isomorphism class of this group is independent of the embedding of the line L in M and of the manifold M, as long as $\dim(M) \geq 2$. This is a consequence of the second definition below, which is independent of L and M, and of the equivalence between the two definitions. However, if we look at the same group in the case where M is a circle, we only obtain a strict subgroup of the latter group as elements of the group have to preserve a cyclic order.
2. If G is a group acting on a manifold M by C^r diffeomorphisms with such a Cantor set K as a minimal set, then there exists a nontrivial morphism $G \to \mathfrak{Diff}^r(K)$. If we understand well the group $\mathfrak{Diff}^r(K)$, we can obtain information on which group can act on M with such a minimal invariant set.

We now give a second definition of our group: the group $\mathfrak{Diff}^r(K)$ is the group of homeomorphisms of K which locally coincide with a C^r-diffeomorphism of an open interval of $\mathbb{R}$ (see precise definition below). We will prove the equivalence between the two definitions in Section 2 (see Proposition 2.1 for a precise statement).

Definition 1.2. *Let K be a Cantor set contained in $\mathbb{R}$. The group $\mathfrak{Diff}^r(K)$ is the group of homeomorphisms f of K such that, for any point x in K, there exists an open interval I of $\mathbb{R}$ and a C^r-diffeomorphism $\tilde{f} : I \to \tilde{f}(I)$ such that $\tilde{f}_{|I \cap K} = f_{|I \cap K}$.*

Remarks: we can adapt this second definition to other kinds of regularity. For instance, in this article, we will denote by $\mathfrak{Diff}^{1+Lip}(K)$ the group of homeomorphisms of K which locally coincide with a C^{1+Lip}-diffeomorphism between two intervals of $\mathbb{R}$, that is a C^1-diffeomorphism $\tilde{f}$ such that $\log(\tilde{f}')$ is Lipschitz continuous.

The generalizations of those two definitions to the case $r = 0$ are not equivalent, but we will use this second definition to define the group $\mathfrak{Diff}^0(K)$: it is the group of homeomorphisms of K which coincide locally with a homeomorphism between two intervals of $\mathbb{R}$.

In the article [3], Funar and Neretin have computed these groups in many cases and provided examples of Cantor sets for which these groups are trivial. They have in particular computed this group in the case where K is the standard ternary Cantor set, which we call K_2. Let us recall first the construction of K_2. Start with the segment $[0, 1]$. Cut this interval into three equal pieces $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$. Now, throw out the middle segment: we obtain a new compact set $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now remove the middle third of each of these intervals : we obtain the compact set $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Then repeat the procedure for each of the obtained intervals. We obtain a decreasing sequence of compact sets : the intersection of this sequence is the set K_2 (see Figure 1). More generally, if we remove $n - 1$ regularly spaced intervals at each step instead of one, we obtain a Cantor set which we denote by K_n.

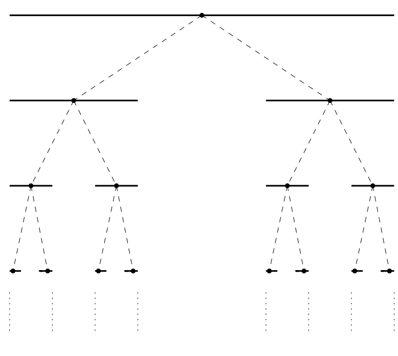

Figure 1 – The first steps of construction of K_2 and bijection of the intervals with the vertices of a binary tree

For convenience, we will call *interval of* K_2 the intersection of K_2 with one of the intervals appearing in this construction. The set of intervals of K_2 is a basis of the topology of K_2. This basis consists of clopen sets, *i.e.* sets which are closed and open. The set of intervals of K_2 are in bijective correspondence with the vertices of a binary tree (see Figure 1).

We now give a procedure which produces elements of $\mathfrak{Diff}^\infty(K_2)$ (see Figure 2 for an example of a diffeomorphism of K_2).

Step 1: Choose two finite partitions of K_2 by intervals of K_2 which have the same cardinality. We denote those partitions by $\{I_i,\ 1 \leq i \leq m\}$ and $\{J_j,\ 1 \leq j \leq m\}$.

Step 2: Choose a bijection between those two partitions. This enables us to construct an element f of the group $\mathfrak{Diff}^\infty(K_2)$ in the following way. If the interval I_i is sent to the interval J_j under the chosen bijection, the restriction of f to I_i is the unique orientation preserving affine map which sends the interval I_i onto the interval J_j. Such a map sends $I_i \cap K_2$ to $J_j \cap K_2$.

Step 3: Chose a subset A of the second partition $\{J_j,\ 1 \leq j \leq m\}$ and flip each of the intervals in A, *i.e.* compose the diffeomorphism obtained during Step 2 by the diffeomorphism whose restriction to J_j is the identity if $J_j \notin A$ and is the symmetry with respect to the center of J_j if $J_j \in A$.

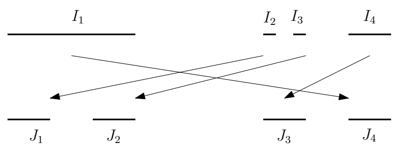

Figure 2 – An example of diffeomorphism of K_2

Of course, we can define a similar algorithm to construct elements of $\mathfrak{Diff}^r(K_n)$.

Theorem (Funar-Neretin). *For any $r \geq 1$, the group $\mathfrak{Diff}^r(K_n)$ is the group consisting of elements constructed following the above procedure.*

Strictly speaking, Funar and Neretin proved only the case where $n = 2$ but their proof also applies in the case of the Cantor sets K_n for $n > 2$. Notice that this group does not depend on the regularity $r \geq 1$: this seems to be a consequence of the "regular" shape of this Cantor set.

Inside $\mathfrak{Diff}^r(K_n)$, there is a natural subgroup : the subgroup consisting of elements which are constructed using only the first two steps of the above procedure. This subgroup is the well-known Higman-Thompson group V_n.

In this article, we prove some general results about the groups $\mathrm{diff}^r(K)$. These theorems tend to prove that those groups share common features with rank one simple Lie groups. In what follows, we fix a Cantor set K embedded in $\mathbb{R}$.

1.1 Burnside property

Definition 1.3. *A group G is* periodic *if any element of the group G has a finite order.*

In 1902, Burnside asked the question whether there existed finitely generated periodic groups which are infinite (see [2]). Nine years later, Schur managed to prove that any finitely generated periodic group which is a subgroup of $GL_n(\mathbb{C})$ has to be finite. Much later, in the 60's, Golod and Shafarevich proved in [4] that there existed infinite finitely generated periodic groups. Many more examples were constructed later.

Theorem 1.4. *Any periodic finitely generated subgroup of $\mathfrak{Diff}^{1+Lip}(K)$ is finite.*

We are not able to lower the regularity to C^1 in this theorem for the moment. However, observe that the same theorem is false in the case of the group of homeomorphisms of a Cantor set as any finitely generated group is a subgroup of this group. To see this, observe that any infinite countable group G acts continuously and faithfully on $\{0,1\}^G$, which is a Cantor set for the product topology and recall that any two Cantor sets are homeomorphic.

We prove Theorem 1.4 in Section 3 of this article.

As a consequence, as the Higman-Thompson groups V_n are subgroups of groups of C^∞-diffeomorphisms of Cantor sets, we have a new proof of the following theorem by Rover.

Theorem (Rover)**.** *Let $n \geq 2$. Any finitely generated periodic subgroup of V_n is finite.*

The proof of Theorem 1.4 relies on an adaptation of standard 1-dimensional tools from dynamical systems, namely Sacksteder's theorem and the Thurston stability theorem.

1.2 Subgroups without free subsemigroups on two generators

In this section, we will look at finitely generated subgroups of the group $\mathfrak{Diff}^{1+Lip}(K)$ without free subsemigroups on two generators. This category of groups contain all the finitely generated groups whose growth is subexponential.

A special class of examples of such groups is given by nilpotent finitely generated subgroups, which we define below.

Fix a group G. If H and H' are subgroups of G, we define $[H, H']$ as the subgroup of G generated by elements of the form $[h, h'] = hh'h^{-1}h'^{-1}$, where $h \in H$ and $h' \in H'$.

We define a sequence $(G_n)_{n \geq 1}$ of subgroups of G by the following relations:

$$G_1 = G$$
$$\forall n \geq 1, G_{n+1} = [G_n, G].$$

A group G is said to be *nilpotent* if there exists $n \geq 1$ such that $G_n = \{1\}$. If the group G is nilpotent and nontrivial, its *order* is the smallest integer n such that G_n is nontrivial and G_{n+1} is trivial. A typical example of finitely generated nilpotent group is the Heisenberg group $\mathcal{H}$ with integer coefficient. This group $\mathcal{H}$ is the group of upper triangular 3×3 matrices with integral coefficients and 1's on the diagonal.

The following theorem states that subgroups of $\mathfrak{Diff}^{1+Lip}(K)$ without free subsemigroups on two generators are close to being abelian.

Theorem 1.5. *Let G be a finitely generated subgroup of $\mathfrak{Diff}^{1+Lip}(K)$ without free subsemigroups on two generators. Then the group G is virtually abelian.*

Recall that, by definition, a group is virtually abelian if it contains a finite-index subgroup which is abelian. We cannot hope for a better conclusion in Theorem 1.5. Indeed, take $K = K_2$ the standard ternary Cantor set. Let S be the group of diffeomorphisms of K_2 which permute the intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, $[\frac{8}{9}, 1]$ using for each of those intervals the unique orientation-preserving affine map which sends this interval to the other one. The group S is isomorphic to the finite group $\mathcal{S}_4$, the symmetric group on 4 elements. Take also an infinite order element f of $\mathfrak{Diff}^{\infty}(K_2)$ which is supported in $[0, \frac{1}{9}] \cap K_2$, meaning that it pointwise fixes the points outside $[0, \frac{1}{9}] \cap K_2$. Then the subgroup of $\mathfrak{Diff}^{\infty}(K_2)$ generated by f and S is virtually abelian : it contains the group $\mathbb{Z}^4$ as a finite index subgroup. But it is not abelian. Notice that, with this kind of construction, we can obtain any virtually abelian group as a subgroup of a group of diffeomorphisms of a Cantor set.

Observe that finitely generated groups with a subexponential growth do not contain any free subsemigroup on two generators. Hence subgroups of $\mathfrak{Diff}^{1+Lip}(K)$ with nonexponential growth are virtually abelian.

Notice that Theorem 1.5 implies Theorem 1.4 as periodic groups do not contain any free subsemigroup on two generators and finitely generated abelian periodic groups are known to be finite. However, we use Theorem 1.4 (and even a stronger version of it which is Proposition 3.10) to prove Theorem 1.5. That is why we will first prove Theorem 1.4 in this article.

If we only assumed that our group did not contain any free subgroups, we could not prove that our group is virtually abelian. Indeed, the Thompson group F is contained in V_2 and is hence a subgroup of the group of diffeomorphisms of the standard ternary Cantor set. But this group is finitely generated, does not contain any free subgroup on two generators and is not Abelian. For more information about the group F and references for proofs of those results, see Section 1.5 of [8]. The best we can hope for finitely generated subgroups without a free subgroup on two generators is that they have a finite orbit.

We prove Theorem 1.5 in Section 4 of this article.

Observe that the corresponding statement for Higman-Thompson's groups V_n was unknown, as far as we know. We state it as a corollary.

Corollary 1.6. *Let $n \geq 2$. Any subgroup of the group V_n without free subsemigroup on two generators is virtually abelian.*

As the derived subgroup of a finitely generated nilpotent group is finitely generated and has no subsemigroups on two generators, we obtain the following statement.

Corollary 1.7. *Any finitely generated nilpotent subgroup of $\mathfrak{Diff}^{1+Lip}(K)$ is virtually abelian.*

In particular, there is no Heisenberg group with integer coefficients as a subgroup of $\mathfrak{Diff}^{r}(K)$ for $r \geq 2$.

Higman-Thompson's groups V_n were already known to satisfy this corollary by a result by Bleak, Bowman, Gordon Lynch, Graham, Hughes, Matucci and Sapir [1] about distorted cyclic subgroups.

As a consequence of this corollary, we obtain the following statement, which is related to the Zimmer conjecture.

Theorem 1.8. *Let $r \geq 2$ and Γ be a finite index subgroup of $SL_n(\mathbb{Z})$ (or any almost simple* *group which contains a nonabelian infinite nilpotent group). Any morphism* $\Gamma \to \mathfrak{Diff}^r(K)$ *has a finite image.*

Navas proved in [9] that subgroups of C^{1+Lip}-diffeomorphisms of the half-line without free subsemigroups on two generators are abelian. To prove Theorem 1.5, we try to adapt his techniques. However, this adaptation is not easy as groups of diffeomorphisms of the half-line preserve an order on the half-line whereas our groups do not a priori preserve any order on our Cantor set. Moreover, Navas is able to lower the regularity to C^{1+bv} whereas we have to stick to the C^{1+Lip} regularity.

Acknowledgement: The second author wants to thank Isabelle Liousse for a conversation which was the origin of this article.

2 Equivalence between the two definitions

Let $r \geq 1$ be an integer or $r = \infty$. In this section, we prove the equivalence between Definitions 1.1 and 1.2 of $\mathfrak{Diff}^r(K)$.

Let M be a differential manifold with $\dim(M) \geq 2$. Let L be a real line which is C^r-embedded in M. We identify the line L with the real line $\mathbb{R}$. Denote by $\mathfrak{Diff}^r(K)_1$ the group of C^r-diffeomorphisms of K according to Definition 1.1, that is the group of restrictions to K of C^r-diffeomorphisms of M which preserve K. Denote by $\mathfrak{Diff}^r(K)_2$ the group of C^r-diffeomorphisms of K according to Definition 1.2, that is the group of homeomorphism of $K \subset L = \mathbb{R}$ which locally coincide with C^r-diffeomorphisms between two intervals of $\mathbb{R}$.

We prove the following statement.

Proposition 2.1. *The map*

$$\mathfrak{Diff}^r(K)_1 \quad \mapsto \quad \mathfrak{Diff}^r(K)_2$$

which to a homeomorphism f *of* K *in* $\mathfrak{Diff}^r(K)_1$*, associates* f*, is well-defined, as it takes value in* $\mathfrak{Diff}^r(K)_2$*, and is onto.*

This amounts to showing that, if g is a C^r-diffeomorphism of M which preserves K, then the restriction $g_{|K}$ belongs to $\mathfrak{Diff}^r(K)_2$ and that, if we denote by f a homeomorphism of K in $\mathfrak{Diff}^r(K)_2$, then there exists a C^r-diffeomorphism g of M such that $g_{|K} = f$.

Proof. Let g be a C^r diffeomorphism of M which preserves K. Take a chart φ defined on an open subset U of M onto $\mathbb{R}^{\dim(M)}$ such that $K \subset U$ and $\varphi(L \cap U) = \mathbb{R} \times \{0\}^{\dim(M)-1}$. For instance, you can take as open set U a tubular neighbourhood of a segment of L which contains K. Finally, denote by Π the "projection on L" $\varphi^{-1} \circ p_1 \circ \varphi$, where $p_1 : \mathbb{R}^{\dim(M)} \to \mathbb{R}$ is the projection on the first coordinate.

Let $x_0 \in K$. We will prove that the differential of $\Pi \circ g_{|L}$ at the point x_0 does not vanish. Hence, by the inverse function theorem, the map $\Pi \circ g_{|L}$ is a C^rdiffeomorphism of an open neighbourhood I of the point x_0 in L onto an open neighbourhood J of the point $g(x_0)$ in L. Moreover, $g(I \cap K) \subset K \subset U \cap L$ so that $\Pi \circ g_{|I \cap K} = g_{|I \cap K}$ and the map $g_{|K}$ satisfies Definition 1.2.

It remains to show that $d(\Pi \circ g_{|L})(x_0) \neq 0$. As the compact set K is a Cantor set, there exists a sequence $(y_n)_{n \in \mathbb{N}}$ of elements of $K \setminus \{x_0\}$ which converges to the point x_0. Observe that, for any n, the point $g(y_n)$ belongs to $K \subset L \cap U$. Hence the partial derivative

$$\frac{\partial}{\partial x_1}(\varphi \circ g \circ \varphi^{-1})(\varphi(x_0))$$

belongs to $\mathbb{R} \times \{0\}$ and does not vanish as g is a diffeomorphism. Hence

$$\frac{\partial}{\partial x_1}(p_1 \circ \varphi \circ g \circ \varphi^{-1})(\varphi(x_0)) \neq 0$$

and $d(\Pi \circ g_{|L})(x_0) \neq 0$.

Now, let f be a homeomorphism of K which satisfies Definition 1.2. By compactness of K, there exists a partition $(K_i)_{1 \leq i \leq l}$ of K such that

1. Each subset K_i is a clopen subset of K.
2. For any index $1 \leq i \leq l$, there exists an open interval I_i of $\mathbb{R}$ such that $K_i = I_i \cap K$.
3. The intervals I_i are pairwise disjoint.
4. For any $1 \leq i \leq l$, there exists a diffeomorphism $\tilde{f}_i$ defined on I_i onto an open interval J_i such that $\tilde{f}_{i|K_i} = f_{|K_i}$.
5. The intervals J_i are pairwise disjoint.

To obtain such a partition, take first any cover $(K'_i)_{1 \leq i \leq l'}$ by clopen subsets such that conditions 1., 2. and 4. are satisfied. Takes finite partitions of the clopen sets $K'_i \setminus \cup_{j<i} K'_j$ and throw away empty sets to obtain a partition such that conditions 1., 2. and 4. are satisfied. We cannot take the sets $K'_i \setminus \cup_{j<i} K'_j$ directly as those sets might not satisfy the second condition. Finally, shrink the obtained intervals I_i in such a way that the two remaining conditions hold.

Take closed intervals $I'_i \subset I_i$ in such a way that the compact set K_i is contained in the interior of the interval I'_i and let $J'_i = \tilde{f}_i(I'_i)$. Finally, use the isotopy extension property (Theorem 3.1 p. 185 in [6]) to extend the map

$$\begin{array}{rccc} \tilde{f}: & \bigcup_{i=1}^{l} I'_i & \to & \bigcup_{i=1}^{l} J'_i \\ & x \in I_i & \mapsto & \tilde{f}_i(x) \end{array}$$

to a diffeomorphism g in $\mathrm{Diff}^r(M)$. In [6], the isotopy extension property is stated only for one disk but, with an induction, it is not difficult to prove this property for a union of disjoint closed disks (here closed intervals, which are one-dimensional disks). □

3 Burnside property

In this section, we prove Theorem 1.4. To prove it, we will use Definition 1.2 of the group $\mathfrak{Diff}^{1+Lip}(K)$. Hence we see our Cantor set K as a subset of the real line $L = \mathbb{R}$.

Before starting the actual proof of Theorem 1.4, we need some definitions.

First, elements of $\mathfrak{Diff}^1(K)$ have a well-defined derivative at each point of K. Indeed, fix a diffeomorphism f in $\mathfrak{Diff}^1(K)$ and a point x_0 of K. Then there exists an open interval I of $\mathbb{R}$ which contains the point x_0 and a C^1-diffeomorphism $\tilde{f}: I \to \tilde{f}(I)$ such that $f_{I \cap K} = \tilde{f}_{I \cap K}$. Then the derivative $\tilde{f}'(x_0)$ does not depend on the chosen extension $\tilde{f}$. Indeed,

$$\tilde{f}'(x_0) = \lim_{\substack{x \to x_0 \\ x \in K}} \frac{f(x) - f(x_0)}{x - x_0}$$

and the right-hand side of this equlity depends only on f. We call this number the derivative of f at x_0 and we denote it by $f'(x_0)$.

As we can define the notion of derivative for the elements of our group, we also have a notion of hyperbolic fixed point which is recalled in the following definition.

Definition 3.1. *Let f be a diffeomorphism in $\mathfrak{Diff}^1(K)$. Let x_0 be a point of K. We say that the point x_0 is a hyperbolic fixed point for f if*

1. $f(x_0) = x_0$.
2. $|f'(x_0)| \neq 1$.

The proof of Theorem 1.4 relies on the following propositions. This first proposition is a consequence of a theorem by Sacksteder.

Proposition 3.2. *Let G be a finitely generated subgroup of $\mathfrak{Diff}^{1+Lip}(K)$. Then one of the following properties holds.*

1. *The group G contains an element with a hyperbolic fixed point.*
2. *Any invariant minimal subset of the action of G on K is a finite subset of K.*

The second step is an adaptation of a theorem by Thurston which is called the Thurston stability theorem. The above proposition allows us to prove this second proposition.

Proposition 3.3. *Let G be a finitely generated subgroup of $\mathfrak{Diff}^{1+Lip}(K)$. Suppose that G contains no element with a hyperbolic fixed point. Then one of the following properties holds.*

1. *There exists a finite index subgroup G_1 of G and a nontrivial morphism $G' \to \mathbb{R}$.*
2. *The group G is finite.*

End of the proof of Theorem 1.4. Let G be a finitely generated periodic subgroup of $\mathfrak{Diff}^{1+Lip}(K)$. Observe that any element f of $\mathfrak{Diff}^{1+Lip}(K)$ with a hyperbolic fixed point x_0 has to be an infinite order element: the sequence $((f^n)'(x_0))_n = ((f'(x_0))^n)_n$ is infinite. Hence, the group G cannot contain any element with a hyperbolic fixed point and, by Proposition 3.3, the group G has to be finite. □

3.1 Proof of Proposition 3.2

Before proving the proposition, we have to recall the definition of a pseudogroup of diffeomorphisms of the real line $\mathbb{R}$.

Definition 3.4. *A set Γ of C^r-diffeomorphisms $g : \mathrm{dom}(g) \to \mathrm{ran}(g)$ between two open subsets $\mathrm{dom}(g)$ and $\mathrm{ran}(g)$ of $\mathbb{R}$ is called a pseudogroup of C^r-diffeomorphisms of $\mathbb{R}$ if*

1. *The set Γ is stable under composition, that is, if two elements g and h belong to Γ with $\mathrm{ran}(h) \subset \mathrm{dom}(g)$, then the composition gh of those elements belongs to Γ.*
2. *The set Γ is stable under inverses, that is, for any element g in Γ, its inverse g^{-1} belongs to Γ.*
3. *The identity of $\mathbb{R}$ belongs to Γ.*
4. *The set Γ is stable under restrictions, that is, if g is an element of Γ and A is an open subset of $\mathrm{dom}(g)$ then the diffeomorphism $g_{|A} : A \to g(A)$ belongs to Γ.*

Let Γ be a pseudogroup of diffeomorphisms of $\mathbb{R}$. A subset A of $\mathbb{R}$ is *invariant* under Γ if, for any point x of A and any element g of Γ, we have

$$x \in \mathrm{ran}(g) \Rightarrow g(x) \in A.$$

A compact subset A of $\mathbb{R}$ is a *minimal invariant set* for the action of Γ if the set A is nonempty and invariant under Γ and minimal for the inclusion relation among invariant nonempty compact subsets of $\mathbb{R}$. A subset $\mathcal{S}$ of Γ is called a *generating set* of Γ if any element g of Γ is the restriction of a product of elements of $\mathcal{S}$. We say that a pseudo-group Γ is *finitely-generated* if it admits a finite generating set.

Proposition 3.2 is a consequence of the following theorem by Sacksteder (see [8][Theorem 3.2.2 p.91] for a proof).

Theorem 3.5 (Sacksteder). *Let $\mathcal{G}$ be a pseudogroup of C^{1+Lip}-diffeomorphisms of $\mathbb{R}$. Suppose that*

1. *The pseudogroup $\mathcal{G}$ has a Cantor set K' as a minimal set of its action on the line.*
2. *There exists a finite set $\mathcal{S}$ of elements of $\mathcal{G}$ which generates $\mathcal{G}$ whose domains have a compact intersection with K'.*

Then the pseudo-group $\mathcal{G}$ contains an element with a hyperbolic fixed point.

To apply this theorem, we need to make a connection between our group G and a pseudo-group. This is the intention of the following proposition, which is roughly a consequence of the equivalence between the two definitions of the group $\mathfrak{Diff}^r(K)$.

Proposition 3.6. *Let G be a finitely generated subgroup of $\mathfrak{Diff}^{1+Lip}(K)$. Then there exists a pseudogroup $\mathcal{G}$ of C^{1+Lip}-diffemorphisms of the line L such that*

1. *For any diffeomorphism h of the pseudo-group $\mathcal{G}$ and for any point x of $\mathrm{dom}(h) \cap K$, there exists an element g of the group G such that $h_{|K} = g$ on a neighbourhood of x in K.*
2. *For any element g in G and any point x in K, there exists an element h of $\mathcal{G}$ such that x belongs to $\mathrm{dom}(h)$ and $g = h_{|K}$ on a neighbourhood of x in K.*
3. *There exists a finite generating set $\mathcal{S}$ of $\mathcal{G}$ such that, for any element h in $\mathcal{S}$, the set $\mathrm{dom}(h) \cap K$ is compact.*

In particular, the pseudo-group $\mathcal{G}$ preserves K and, for any point x of K, the orbit of x under the action G is also the orbit of x under the action of $\mathcal{G}$. The last property will enable us to apply Sacksteder's theorem. Before proving the above proposition, we use it to prove Proposition 3.2.

Proof of Proposition 3.2. Denote by $\mathcal{G}$ the pseudo-group associated to G as in Proposition 3.6. Let $K_m \subset K$ be a minimal invariant set for the action of G on the Cantor set K. Then the set K_m is a minimal invariant set for the action of the pseudo-group $\mathcal{G}$. Suppose that the set K_m is not finite. Then the set K'_m of accumulation points of K_m is closed and nonempty. As the set K'_m is also G-invariant, by minimality of K_m, we have $K'_m = K_m$ and the set K_m is a Cantor set. By Theorem 3.5, there exist $x \in K_m \subset K$ and an element $h \in \mathcal{G}$ such that $x \in \mathrm{dom}(h)$ and x is a hyperbolic fixed point for h. Hence, by Proposition 3.6, some element of G has a hyperbolic fixed point. □

Proof of Proposition 3.6. Let g be a diffeomorphism in $\mathfrak{Diff}^{1+Lip}(K)$. By Definition 1.2, there exists a diffeomorphism $\hat{g}$ from an open neighbourhood O_g of K in L to another neighbourhood of K in L. For each connected component $O_{i,g}$ of O_g (which is an open interval with compact intersection with K), let $\gamma_{i,g} = \hat{g}_{|O_{i,g}}$. If S is a symmetric finite generating set of G, let $\mathcal{S}$ be the finite set consisting of diffeomorphisms of the form $\gamma_{i,s}$, where $s \in S$.

We claim that, if $\mathcal{G}$ denotes the pseudo-group generated by $\mathcal{S}$, then $\mathcal{G}$ satisfies the wanted properties.

Let us prove it by induction on word length. Fix $x \in K$. For any element g in the group G, let us denote by $l_S(g)$ the minimal number of factors required to write g as a product of elements of S. Let us prove the following statement by induction on n: for any element $g \in G$ with $l_S(g) = n$, there exists an element h of $\mathcal{G}$ such that $g = h_{|K}$ on a neighbourhood of x in K.

For $n = 1$, this property holds by construction of $\mathcal{S}$. Suppose this property is true for some n and let us prove it for $n+1$. Let $g \in G$ and suppose $l_S(g) = n+1$. Write $g = sg_1$, where $l_S(g_1) = n$ and s belongs to S. By induction hypothesis, there exists $h \in \mathcal{G}$ such that $g_1 = h$ on a neighbourhood of x. Let $O_{i,s}$ be the connected component of O_s which contains $g_1(x)$. Then $g = \gamma_{i,s} h_{|K}$ on a neighbourhood of x in K. This completes the induction.

For any element h in $\mathcal{G}$ and any point x in $\mathrm{dom}(h)$, we denote by $l_{\mathcal{S},x}(h)$ the minimal number of factors required to write h as a product of elements of $\mathcal{S}$ on a neighbourhood of x. To finish the proof of Proposition 3.6, it suffices to prove the following statement by induction on n. For any element h in $\mathcal{G}$ and for any point x in $\mathrm{dom}(h) \cap K$, if $l_{\mathcal{S},x}(g) \leq n$, then there exists an element g of G such that $g = h_{|K}$ on a neighbourhood of x in K. This induction is straightforward to carry out and is left to the reader. □

3.2 Proof of Proposition 3.3

Let G be a finitely generated subgroup of $\mathfrak{Diff}^{1+Lip}(K)$ which does not contain any element with a hyperbolic fixed point. Suppose further that there is no nontrivial morphism from a finite index normal subgroup G_1 of G to $\mathbb{R}$. Let us prove that G is a finite group. We will use the following lemma.

Lemma 3.7. *For any point x in K, there exists a G-invariant clopen neighbourhood U of x such that the action of G on U factors to a finite group action.*

Before proving this lemma, we use it to prove Proposition 3.3.

Proof of Proposition 3.3. By Lemma 3.7, there exists a cover $(U_1, U_2, \ldots, U_r)$ of K by G-invariant clopen sets on which the action of G factors to a finite group action. Changing U_i to

$$U_i - \bigcup_{1 \leq j \leq i-1} U_j$$

if necessary, we can suppose that the sets U_j are pairwise disjoint. For any i, we denote by G_i the group of restrictions to U_i of elements of G. The groups G_i are finite and the restriction maps define a morphism

$$G \to \prod_{i=1}^{r} G_i.$$

This morphism is one-to-one as the open sets U_i cover K. Hence G is finite. □

Now, we prove Lemma 3.7. This lemma will be a consequence of the following lemma which will be proved afterwards.

Lemma 3.8. *For any minimal set $K_{min} \subset K$ for the action of G on K, the set K_{min} is finite and there exists a G-invariant clopen neighbourhood U of K_{min} on which the action factors to a finite group action.*

Proof of Lemma 3.7. Take the closure $\overline{G.x}$ of the orbit of x under the group G. Take a minimal set K_{min} of the action of G on the compact set $\overline{G.x}$. Apply Lemma 3.8 to find a G-invariant clopen neighbourhood U of K_{min} on which the action factors to a finite group action. As $K_{min} \subset \overline{G.x}$, there exists g in G such that $g(x) \in U$. As U is G-invariant, the point x belongs to U and the lemma is proved. □

We now prove Lemma 3.8.

Proof of Lemma 3.8. By Proposition 3.2, the compact set K_{min} has to be a finite set (recall we supposed that no element of G had a hyperbolic fixed point). Let G_2 be the subgroup of G consisting of diffeomorphisms which pointwise fix K_{min}. It is a finite index subgroup of G and it is finitely generated as a finite index subgroup of a finitely generated group. Fix $x \in K_{min}$. Notice that the derivative of any diffeomorphism in G_2 at x is either 1 or -1 as we supposed that the group G contained no elements with a hyperbolic fixed point. Take the subgroup G_1 of G_2 consisting of elements whose derivative at x is 1: it is still a finite index (finitely generated) subgroup of G.

Claim : There exists a G_1-invariant clopen neighbourhood of x on which G_1 acts trivially.

Suppose this claim holds and let us see how to finish the proof of Lemma 3.8.

By the claim, there exists a G_1-invariant clopen neighbourhood U' of x on which G_1 acts trivially. Observe that the open set

$$U = \bigcup_{g \in G} g(U')$$

is a G-invariant clopen neighbourhood of K_{min}. As G_1 is a normal subgroup of G, it acts trivially on U. As G_1 is a finite index subgroup of G, the action of G on U factors to a finite group action.

It remains to prove the claim. Suppose there exists no neighbourhood of x which is pointwise fixed by all the elements of G_1. Then there exists a sequence (y_n) of elements of K which converges to x such that, for any n, there exists an element g_n in G_1 such that $g_n(y_n) \neq y_n$. We then use Lemma 3.9 below which a slight modification of the Thurston stability theorem. It implies that there exists a nontrivial morphism $G_1 \to \mathbb{R}$, which is impossible by hypothesis. □

In the above proof, we used the following lemma which is a variant of the Thurston stability theorem.

Lemma 3.9. *Let G be a finitely generated subgroup of $\mathfrak{Diff}^1(K)$. Suppose there exists a point x_0 of K which is fixed under all the elements of G such that, for any diffeomorphism g in G, $g'(x_0) = 1$. Suppose there exists a sequence $(y_n)_n$ of points of K converging to x_0 such that, for any n, we can find a diffeomorphism g_n in G such that $g_n(y_n) \neq y_n$. Then there exists a non-trivial morphism $G \to \mathbb{R}$.*

As this lemma is slightly different than the standard result (see [13]), we prove it. We adapt the proof from Schachermayer (see [12]).

Proof. Take a symmetric finite generating set $\{s_1, s_2, \ldots, s_r\}$ of G. For any i, let $\tilde{s}_i$ be a diffeomorphism of a neighbourhood U_i of x_0 in the line L (onto its image) such that

$$\tilde{s}_{i|K \cap U_i} = s_{i|K \cap U_i}.$$

Let U be a neighbourhood of x_0 in L which is contained in all the open sets U_i and which is diffeomorphic to $\mathbb{R}$. In what follows, we identify the set U with the real line $\mathbb{R}$ Take n sufficiently large so that $y_n \in U$ and, for any i, $s_i(y_n) \in U$ and let

$$M_n = \max_{1 \leq i \leq r} |s_i(y_n) - y_n|,$$

where $|.|$ is the absolute value on $\mathbb{R}$. By hypothesis on the sequence $(y_n)_n$, $M_n \neq 0$ for any n. Replacing the sequence (y_n) by one of its subsequence if necessary, we can suppose that, for any i, the sequence $\left(\frac{s_i(y_n)-y_n}{M_n}\right)_n$ converges to some real number L_i.

<u>Claim</u>: For any $g = s_{i_1} \dots s_{i_l} \in G$, with $1 \leq i_k \leq r$ for any k, the sequence $\left(\frac{g(y_n)-y_n}{M_n}\right)_n$ converges to

$$L(g) = \sum_{k=1}^{l} L_{i_k}.$$

This claim provides a morphism $L : G \to \mathbb{R}$. This morphism is non-trivial as, by definition of M_n,

$$\max_{1 \leq i \leq r} (|L(s_i)|) = 1.$$

Hence it suffices to prove the claim to complete the proof of the lemma.

We prove the claim by induction on l. The claim is true for $l = 1$ by definition of the L_i's.

Suppose the claim holds for some integer $l \geq 1$. Let $g = s_{i_1} \dots s_{i_{l+1}}$ with, for any k, $1 \leq i_k \leq r$. We denote $h = s_{i_2} \dots s_{i_{l+1}}$ so that $g = s_{i_1} h$. Then

$$\frac{g(y_n) - y_n}{M_n} = \frac{s_i(h(y_n)) - h(y_n)}{M_n} + \frac{h(y_n) - y_n}{M_n}.$$

The quantity $\frac{h(y_n)-y_n}{M_n}$ converges to

$$L(h) = \sum_{k=2}^{l+1} L_{i_k}$$

by induction hypothesis. Moreover,

$$\frac{|(s_{i_1}(h(y_n)) - h(y_n)) - (s_{i_1}(y_n) - y_n)|}{M_n} \leq \max_{t \in [0,1]} \left|\tilde{s}'_{i_1}(y_n + t(h(y_n) - y_n)) - 1\right| \frac{|h(y_n) - y_n|}{M_n}.$$

As the sequence $(\frac{|h(y_n)-y_n|}{M_n})_n$ converges and as

$$\lim_{n \to +\infty} \left|\tilde{s}'_{i_1}(y_n + t(h(y_n) - y_n)) - 1\right| = 0$$

uniformly in $t \in [0,1]$, this quantity converges to 0 and

$$\lim_{n \to +\infty} \frac{s_{i_1}(h(y_n)) - h(y_n)}{M_n} = L_{i_1}.$$

□

3.3 A generalization of Theorem 1.4

In the rest of the article, we will need the following generalization of Theorem 1.4.

Proposition 3.10. *Let F be a closed subset of K and G be a subgroup of $\mathfrak{Diff}^{1+Lip}(K)$ which consists of elements which preserve F. Denote by $G(F)$ the group of restrictions to F of elements of G. Suppose the group $G(F)$ is periodic. Then*

1. *The group $G(F)$ is finite.*
2. *Let G_1 be the subgroup of G consisting of elements which pointwise fix F and have a positive derivative at each point of F. Then G_1 is a finite index subgroup of G.*

Observe that the second conclusion of this proposition implies the first one. As the proof of this proposition is sometimes really similar to the proof of Theorem 1.4, we will skip some details.

Proof. First, let us prove by contradiction that any minimal invariant set for the action of $G(F)$ on F is finite. Suppose the action of the group $G(F)$ on F has an infinite minimal invariant subset $K_1 \subset F$. Then the set K_1 has to be a Cantor set. Hence, by Proposition 3.2, the action of $G(F)$ on K_1 has a hyperbolic fixed point. It is impossible as the group $G(F)$ consists of finite order elements by hypothesis.

We then need the following claim.

Claim 3.11. *The action of the group $G(F)$ on F has only finite orbits.*

Proof. Suppose there exists a point p of F such that the orbit $G.p$ is infinite. Take a minimal invariant set $K_2 \subset \overline{G.p}$ for the action of $G(F)$ on F. We just saw that the set K_2 has to be finite. Take the finite index subgroup G_2 of $G(F)$ consisting of elements which pointwise fix the finite set K_2 and have a positive derivative at each point of K_2. As the set K_2 is accumulated by an infinite orbit under the action of G_2, by Lemma 3.9, either there exists an element of G_2 with a hyperbolic fixed point or there exists a non-trivial morphism $G_2 \to \mathbb{R}$. None of those possibilities can occur as the group G_2 consists of finite order elements. □

Hence any orbit of the action of $G(F)$ on F is finite. As in the proof of Proposition 3.3 and as the group $G(F)$ is finitely generated, the stabilizer of any point of F is locally constant. More precisely, if a finitely generated subgroup of $G(F)$ pointwise fixes a point of F, then it pointwise fixes a neighbourhood of this point: otherwise, the group $G(F)$ would have an infinite orbit. Hence, using the compactness of F, we deduce that the group $G(F)$ is finite.

Let us prove the second point of the proposition. As the group $G(F)$ is finite, the subgroup G_3 of G consisting of elements which pointwise fix F is a finite index subgroup of G : it is the kernel of the restriction morphism $G \to G(F)$. Hence it suffices to prove that the group G_1 is a finite index subgroup of G_3.

Observe that, as elements of G_3 pointwise fixes F, the derivative of any element of G_3 at each accumulation point of F is equal to one. Let us denote by F' the set of accumulation points of F. As the group G_3 is finitely generated, there exists a neighbourhood U of F' such that the derivative of any element of G_3 is positive on U. Observe that the set $F \setminus U$ is compact and consists of isolated points : this set is finite. Moreover, the group G_1 is the kernel of the morphism

$$\begin{array}{ccc} G_3 & \to & \{-1,1\}^{F \setminus U} \\ g & \mapsto & (\mathrm{sgn}(g'(x)))_{x \in F \setminus U} \end{array},$$

where, for any real number $\lambda \neq 0$, $\mathrm{sgn}(\lambda)=1$ if $\lambda>0$ and $\mathrm{sgn}(\lambda)=-1$ if $\lambda<0$. As the group $\{-1,1\}^{F\setminus U}$ is finite, the group G_1 is a finite index subgroup of G_3. $\square$

4 Groups without free subsemigroups on two generators

In this section, we prove Theorem 1.5. As in the proof of Theorem 1.4, we will use Definition 1.2 of the group $\mathfrak{Diff}^{1+Lip}(K)$. Hence we see our Cantor set K as a subset of $\mathbb{R}$.

We fix a group G satisfying the hypothesis of Theorem 1.5: the subgroup G of $\mathfrak{Diff}^{1+Lip}(K)$ does not contain any free subsemigroup on two generators.

The proof is divided in three steps which correspond to subsections 4.2, 4.3 and 4.4.

1. First, we find a finite index subgroup G_1 of G such that any minimal invariant set for the action of G_1 on K is a fixed point.
2. Then we prove that any element of the derived subgroup G_1' of G_1 pointwise fixes a neighbourhood of $\mathrm{Fix}(G_1)$. This is the main step of the proof which heavily relies on distortion estimates.
3. We deduce the theorem from the two above steps.

The following subsection is devoted to a useful preliminary result.

4.1 A preliminary result

We will often need the following result. For any element g in $\mathfrak{Diff}^1(K)$, we denote by $\mathrm{Per}(g)$ the set of periodic points of g, *i.e.* the set of points p of K such that there exists an integer $n \geq 1$ such that $g^n(p)=p$.

Lemma 4.1. *For any element g in $\mathfrak{Diff}^1(K)$, there exists $N \geq 1$ such that*

$$\mathrm{Per}(g)=\left\{p \in K,\ g^N(p)=p\right\}.$$

Proof. Fix an element g in $\mathfrak{Diff}^1(K)$ and define

$$\begin{array}{rccl} T: & \mathrm{Per}(g) & \to & \mathbb{R}_+ \\ & p & \mapsto & T(p)=\min\left\{T \geq 1,\ g^T(p)=p\right\} \end{array}.$$

This lemma is a consequence of the two following claims.

Claim 4.2. *For any point p in $\mathrm{Per}(g)$, there exists an open neighbourhood U of the point p such that $U \cap \mathrm{Per}(g)=U \cap \overline{\mathrm{Per}(g)}$ and $T_{|U\cap \mathrm{Per}(g)}$ is bounded.*

Claim 4.3. $\overline{\mathrm{Per}(g)}=\mathrm{Per}(g)$.

By Claim 4.2 and Claim 4.3, the set $\mathrm{Per}(g)$ is compact and the function T is locally bounded on $\mathrm{Per}(g)$. Hence the function T is bounded by an integer M. It suffices to take $N=M!$ to prove the lemma. $\square$

Proof of Claim 4.2. By definition of T, for any point p of $\mathrm{Per}(g)$, $g^{T(p)}(p)=p$. Recall that the set K is contained in $\mathbb{R}$. Fix a point p in $\mathrm{Per}(g)$. By Definition 1.2, there exists an open interval I' of $\mathbb{R}$ which contains the point p and a homeomorphism $\tilde{h}_1 : I' \to \tilde{h}_1(I)$ such that $\tilde{h}_{1|I' \cap K} = g^{T(p)}_{|I \cap K}$. The homeomorphism $\tilde{h}_1$ is not necessarily orientation-preserving but there exists an open interval I of $\mathbb{R}$ which contains the point p and an orientation-preserving homeomorphism $\tilde{h} : I \to \tilde{h}(I)$ such that $g^{2T(p)}_{|I \cap K} = \tilde{h}_{|I \cap K}$.

Let $U = I \cap K$. We will prove that, for any point x in U, either $g^{2T(p)}(x) = x$ or $x \notin \mathrm{Per}(g)$. This proves the claim as

$$\mathrm{Per}(g) \cap U = \left\{ x \in U,\ g^{2T(p)}(x) = x \right\}$$

is closed in U and T is bounded by $2T(p)$ on U.

Take a point x in U and suppose that $g^{2T(p)}(x) \neq x$. Then, as $\tilde{h}$ is an increasing map which fixes the point p, either

$$\left\{ g^{2nT(p)}(x), n > 0 \right\} = \left\{ \tilde{h}^n(x),\ n > 0 \right\}$$

is infinite and contained in U or

$$\left\{ g^{2nT(p)}(x), n < 0 \right\} = \left\{ \tilde{h}^n(x),\ n < 0 \right\}$$

is infinite and contained in U. In either cases $x \notin \mathrm{Per}(g)$. □

Proof of Claim 4.3. Let p be a point of $\overline{\mathrm{Per}(g)}$. Then the closure of the orbit of p under the action of the diffeomorphism g contains a a minimal set F. By Proposition 3.2, this set F has to be a periodic orbit of g and $F \subset \mathrm{Per}(g)$: there is no periodic point of g in a neighbourhood of a hyperbolic fixed point of g. Moreover, as, by Claim 4.2, the set $\mathrm{Per}(g)$ is open in $\overline{\mathrm{Per}(g)}$, there exists $n > 0$ such that the point $g^n(p)$ belongs to $\mathrm{Per}(g)$. Hence the point p belongs to $\mathrm{Per}(g)$. □

4.2 Definition of the finite index subgroup G_1

This section is devoted to the proof of the following proposition. For any subgroup H of $\mathfrak{Diff}^{1+Lip}(K)$, we denote by $\mathrm{Fix}(H)$ the subset of K consisting of points which are fixed under all the elements of the group H.

Proposition 4.4. *There exists a finite index subgroup G_1 of G such that the two following properties hold.*

1. *Any minimal invariant subset for the action of G_1 on K is a point of* $\mathrm{Fix}(G_1)$.
2. *For any diffeomorphism g in the group G_1 and any point p of* $\mathrm{Fix}(G_1)$*, we have*

$$g'(p) > 0.$$

We start the proof of this proposition by the following lemma, which is more or less a consequence of Sacksteder's Theorem.

Lemma 4.5. *Any minimal invariant subset for the action of the group G on K is finite.*

Proof. If one looks closely at the proof of Sacksteder's theorem, one can see that it implies directly that, if the action of the group G on the Cantor set K has an infinite minimal invariant set, then the group G contains a free subsemigroup on two generators. We provide here a proof which uses Sacksteder's theorem.

Suppose that the action of the group G on K has an infinite minimal subset K_{min}. We want to prove that the group G contains a free subsemigroup on two generators. This will give a contradiction and will complete the proof of the lemma. To do this, we use the following classical lemma (see [5][Proposition 2 p.188] for a proof).

Lemma 4.6 (Positive ping-pong lemma). *Let H be a group acting on a set E. Assume there exist elements h_1 and h_2 of H as well as disjoint nonempty subsets A and B of E such that*

$$\begin{cases} h_1(A \cup B) \subset A \\ h_2(A \cup B) \subset B \end{cases}.$$

Then the subsemigroup of G generated by h_1 and h_2 is free.

First, as the set K_{min} is infinite, it contains accumulation points. As the set of accumulation points of K_{min} is closed and invariant under the action of the group G and as K_{min} is a minimal invariant set, we deduce that the set K_{min} is a Cantor set. Then, by Proposition 3.2, there exists an element h of G with a hyperbolic fixed point $p \in K_{min}$. Taking h^{-1} instead of h if necessary, we can suppose that $|h'(p)| < 1$. Taking the definition of a diffeomorphism of K, we know that there exists a C^{1+Lip} diffeomorphism $\tilde{h}$ from an open interval I of $\mathbb{R}$ which contains the point p to an open interval $\tilde{h}(I)$ such that $\tilde{h}_{|I\cap K} = h_{|I\cap K}$. Moreover, choose the interval I sufficiently small so that $\sup_{x\in I} |\tilde{h}'(x)| < 1$. Hence the sequence of sets $(\tilde{h}^n(I))_{n\geq 0}$ is decreasing and

$$\cap_{n\in\mathbb{N}} \tilde{h}^n(I) = \{p\}.$$

As K_{min} is a minimal Cantor set, there exists a point p' in $G.p \cap (I \setminus \{p\})$. Fix an element g of the group G such that $g(p) = p'$. Let $\tilde{g}$ be a diffeomorphism between two intervals of $\mathbb{R}$ which coincides with g on a neighbourhood of p. Take an integer N_1 sufficiently large so that $\tilde{g} \circ \tilde{h}^{N_1}$ is defined on I, $\sup_{x\in I} |(\tilde{g} \circ \tilde{h}^{N_1})'(x)| < 1$ and $g \circ h^{N_1}(I \cap K) \subset I \cap K$. The map $h_1 = g \circ h^{N_1}_{|I\cap K}$ has a unique fixed point p': indeed, the map $\tilde{g} \circ \tilde{h}^{N_1}$ has a fixed point on I which is the limit of any positive orbit under the action of $\tilde{g} \circ \tilde{h}^{N_1}$. Hence this fixed point belongs to K. Observe that $h_1(p) = g(p) \neq p$. Hence there exists an open interval J_1 which contains the point p such that $h_1(J_1)$ is disjoint from J_1. Finally take an open interval $J_2 \subset I$ which contains the point p' and is disjoint from the interval J_1. Then there exist integers $N_2 > 0$ and $N_3 > 0$ such that

$$h^{N_2}(J_1 \cup J_2) \subset J_1$$

and

$$h_1^{N_3}(J_1 \cup J_2) \subset J_2.$$

By the positive ping-pong lemma, the group G contains a free semigroup on two generators. □

Now, we can finish the proof of Proposition 4.4

End of the proof of Proposition 4.4. By Lemma 4.5, any minimal subset for the action of the group G on the Cantor set K is contained in the set

$$F = \bigcap_{g\in G} \mathrm{Per}(g).$$

By Lemma 4.1, the set F is a closed subset of K. Moreover, it is invariant under the action of G. Let us denote by $G(F)$ the group of restrictions to F of elements of G. By definition of F and by Lemma 4.1, the group $G(F)$ consists of finite order elements. By Proposition 3.10, the group $G(F)$ is finite. Moreover, let G_1 be the subgroup of G consisting of elements which pointwise fix F and whose derivative at each point of F is positive. Then the group G_1 is a finite index subgroup of G by the same proposition.

Let us check that this group G_1 satisfies the wanted property. Let K_{min} be a minimal invariant subset of the action of the group G_1 on the Cantor set K. Then the set

$$M = \bigcup_{g \in G} g(K_{min})$$

is a closed G-invariant subset of K which consists of a finite number of copies of K_{min}. Any G-orbit of a point in this set M is dense in M: it is a minimal subset for the action of G on K. Hence, by Lemma 4.5,

$$K_{min} \subset M \subset F \subset \mathrm{Fix}(G_1).$$

□

4.3 Behaviour of individual elements of G_1'

Let us fix a subgroup G_1 of G which satisfies Proposition 4.4 for the rest of this section. We prove the following result. Recall that the derived subgroup H' of a group H is the subgroup of H generated by the commutators of elements of H, *i.e.* elements of the form $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ with $h_1,\ h_2 \in H$.

Proposition 4.7. *Any element of the derived subgroup G_1' of the group G_1 pointwise fixes a neighbourhood of* $\mathrm{Fix}(G_1)$.

In this proposition, the neighbourhood can a priori depend on the chosen element of the group G_1'.

We split the proof of Proposition 4.7 into two steps.

1. We first prove that any point of $\mathrm{Fix}(G_1)$ is accumulated by points of $\mathrm{Fix}(G_1')$.
2. Then we use this first step to prove Proposition 4.7.

Throughout the proof of the proposition, we will use the two following definitions.

Definition 4.8. *Let I be an interval of $\mathbb{R} \supset K$ and f be an element of $\mathfrak{Diff}^{1+Lip}(K)$. We say that*

- *f is* monotonous *on I if there exists a C^{1+Lip}-diffeomorphism $\tilde{f} : I \to \tilde{f}(I)$ such that $f_{|I\cap K} = \tilde{f}_{I\cap K}$.*
- *f is* increasing *on I if there exists an orientation-preserving (*i.e. *increasing) C^{1+Lip}-diffeomorphism $\tilde{f} : I \to \tilde{f}(I)$ such that $f_{|I\cap K} = \tilde{f}_{I\cap K}$.*

Let A be a subset of $\mathbb{R}$ and p be a point of A. We call *left-neighbourhood* (respectively *right-neighbourhood*) of the point p in A any subset of $\mathbb{R}$ which contains a set of the form $[p-\alpha, p) \cap A$ (resp. $(p, p+\alpha] \cap A$), for some $\alpha > 0$.

We say that the point p is *accumulated on the left* (respectively *accumulated on the right*) by the set A if, for any $\alpha > 0$, $[p-\alpha, p) \cap A \neq \emptyset$ (resp. $(p, p+\alpha] \cap A \neq \emptyset$. Equivalently, a point p is accumulated on the left (resp. right) by the set A if any left-neighbourhood (resp. right-neighbourhood) of the point p is nonempty.

We say that the point p is isolated on the left (resp. on the right) in A if it is not accumulated on the left (resp. on the right) by the set A or, equivalently, if it has a left-neighbourhood (resp. right-neighbourhood) in A which is empty.

a. First step

We formulate this first step as a proposition.

Proposition 4.9. *Let p be a point of* $\mathrm{Fix}(G_1)$.

If the point p is accumulated on the left by points of K, then the point p is accumulated on the left by points of $\mathrm{Fix}(G'_1)$.

If the point p is accumulated on the right by points of K, then the point p is accumulated on the right by points of $\mathrm{Fix}(G'_1)$.

Proof of Proposition 4.9. Observe that it suffices to prove the following two properties

1. Any point of $\mathrm{Fix}(G_1)$ which is isolated on the left in $\mathrm{Fix}(G_1)$ but not in K is accumulated on the left by points of $\mathrm{Fix}(G'_1)$.
2. Any point of $\mathrm{Fix}(G_1)$ which is isolated on the right in $\mathrm{Fix}(G_1)$ but not in K is accumulated on the right by points of $\mathrm{Fix}(G'_1)$.

As the proofs of those two properties are similar, we will only prove the first one. Hence let p be a point of $\mathrm{Fix}(G_1)$ which is isolated on the left in $\mathrm{Fix}(G_1)$ but not in K.

To start the proof, we will define a "local minimal invariant set" K_1 in a left-neighbourhood of p. The set K_1 accumulates to p and we will see later that any element of G'_1 fixes the points of K_1 in a left-neighbourhood of p.

As the group G_1 is a finite index subgroup of G and the group G is finitely generated, then the group G_1 is finitely generated. Fix a finite symmetric generating set S of G_1 and $\alpha_0 > 0$ small enough so that $\mathrm{Fix}(G_1) \cap [p - \alpha_0, p) = \emptyset$ and, for any element s of S, the diffeomorphism s is monotonous on $[p - \alpha_0, p]$. Then any element of S is increasing on $[p - \alpha_0, p]$ as, by Proposition 4.4, any element of G_1 has a positive derivative at the point p.

Consider the set $\mathcal{M}$ of closed nonempty subsets A of $(p - \alpha_0, p) \cap K$ such that, for any element s of the generating set S

$$s(A) \cap (p - \alpha_0, p) \subset A.$$

This last property is an analogue of a "local invariance" property. Of course, the set $\mathcal{M}$ is nonempty as the set $(p - \alpha_0, p) \cap K$ belong to $\mathcal{M}$.

Take a point p_1 in $(p - \alpha_0, p) \cap K$. Let

$$p_2 = \max\{s(p_1),\ s \in S\}.$$

Observe that, necessarily, $p_2 > p_1$: otherwise, the point p_2 would be fixed under any element of S hence any element of G_1, in contradiction with the definition of α_0. To define the set K_1, we need the following lemma.

Lemma 4.10. *For any closed set A in $\mathcal{M}$,*

$$A \cap [p_1, p_2] \neq \emptyset$$

and the point p is accumulated by points of A.

Before proving Lemma 4.10, let us see how to construct the set K_1.

The set $\mathcal{M}$ is partially ordered by the set inclusion relation. Moreover, by compactness and Lemma 4.10, for any totally ordered family $(A_i)_{i\in I}$ of elements of $\mathcal{M}$, the set

$$\bigcap_{i\in I} A_i$$

is nonempty and belongs to $\mathcal{M}$: this set is a lower bound for this totally ordered family. By Zorn's lemma, the set $\mathcal{M}$ contains a minimal element for the inclusion relation. We denote by K_1 this element of $\mathcal{M}$. We can see it as a minimal invariant set for the left-germ of G_1 at p.

Now, let us prove Lemma 4.10.

Proof of Lemma 4.10. Denote by p' a point in $A \cap (p - \alpha_0, p)$.

If the point p' belongs to $[p_1, p_2]$, there is nothing to prove.

Suppose now that the point p' belongs to the interval (p_2, p). Denote by $\mathcal{B}$ the set of elements g of G_1 with the following property. There exists a family $(s_i)_{1\leq i\leq p}$ of elements of S such that $g = s_1 s_2 \dots s_p$ and

$$\forall 1 \leq i \leq p,\ s_i s_{i+1} \dots s_p([p', p] \cap K) \subset (p - \alpha_0, p].$$

Now let B be the subset of $K \cap (p - \alpha_0, p)$ defined by

$$B = \{b(p'), b \in \mathcal{B}\}.$$

Observe that, as the subset A belongs to $\mathcal{M}$, the subset B is contained in A.

Let $m = \inf(B \cap [p_1, p])$. It suffices to prove that the element m belongs to $[p_1, p_2)$. We will do it by contradiction. Suppose that $p_2 \leq m$. Then, as the point m does not belong to $\mathrm{Fix}(G_1)$, there exists an element s of S such that $s(m) < m$ (recall that the set S is symmetric). As $p_2 \leq m$ and by construction of the point p_2, we have $p_1 \leq s(m)$ and $p - \alpha_0 < s(m)$. Hence, if x is a point of B close to m, then $s(x)$ is also a point of B which belongs to $[s(m), m)$, in contradiction with the definition of m.

If $p - \alpha_0 < p' < p_1$, the proof is analogous. Namely look at the same set B but look at the supremum of $B \cap [p_1, p_2]$ instead of the infimum and prove that it belongs to $(p_1, p_2]$.

Observe that we can take the point p_1 as close as we want to the point p. Hence the point p is accumulated by points of A. □

Let H be any subgroup of $\mathfrak{Diff}^1(K)$, F be a closed subset of K and q be a point of $\mathrm{Fix}(H)$ which is accumulated on the left by F. On the group H, we define the following equivalence relation $\equiv_{F,q,-}$. For any elements g_1 and g_2 of the group H,

$$g_1 \equiv_{F,q,-} g_2 \quad \Leftrightarrow \quad \exists \alpha > 0,\ g_{1|F\cap[q-\alpha,q]} = g_{2|F\cap[q-\alpha,q]}.$$

We denote by $\mathcal{H}(F)$ the group $H/\equiv_{F,q,-}$. This is the "group of left-germs at q of elements of H restricted to F". In particular we set $\mathcal{G}_1(K_1) = G_1/\equiv_{K_1,p,-}$. This is the "group of left-germs at p of elements of G_1 restricted to K_1".

The following proposition completes the proof of the first step. □

Proposition 4.11. *1. The group $\mathcal{G}_1(K_1)$ is abelian.*

2. Any element of G_1 either fixes all the points of K_1 or has no fixed point on a left-neighbourhood of p (which depends on the element of G_1).

We obtain immediately the following corollary which will be useful later.

Corollary 4.12. *Let p be a point of $\mathrm{Fix}(G_1)$ which is isolated on the left in $\mathrm{Fix}(G_1)$ but accumulated on the left by K. Then there exists an element h of G_1 and $\alpha > 0$ such that*

1. *the diffeomorphism h is increasing on $[p-\alpha, p]$.*
2. *$\forall x \in [p-\alpha, p] \cap K, h(x) > x$.*

Of course, we have analogous statements for points which are isolated on the right in $\mathrm{Fix}(G_1)$.

To prove Proposition 4.11, we need the following lemma.

Lemma 4.13. *Let H be a subgroup of $\mathfrak{Diff}^1(K)$, F be a closed subset of K and q be a point of $\mathrm{Fix}(H)$ which is accumulated on the left by F. Suppose that there exists $\alpha_0 > 0$ such that, for any diffeomorphism h in H, the following property holds.*

For any element h of the group H and any $\alpha \in (0, \alpha_0)$, if the diffeomorphism h is increasing on $[q-\alpha, q]$ and if the diffeomorphism h has a fixed point x in $[q-\alpha, q] \cap F$ then the diffeomorphism h pointwise fixes $[x, q] \cap F$.

Then the group $\mathcal{H}(F) = H/\equiv_{F,q,-}$ is abelian.

To prove this lemma, we use the same techniques as in the proof of a famous theorem by Hölder, which states that any group of fixed-point-free homeomorphisms of the real line is abelian. We need the following definition for this proof.

Observe that we could have thought of a weaker and more natural hypothesis for Lemma 4.13: any element of $\mathcal{H}(F)$ has a representative with no fixed point in $V \setminus \{q\}$, where V is a left-neighbourhood of q in F. However, our proof does not work with this weaker hypothesis. This is due to the various speeds of convergence of the orbits to the point q that can exist. The hypothesis of Lemma 4.13 that we took avoids this problem.

Definition 4.14. *Let H be a group and $\preceq$ be an order on H. The order $\preceq$ is called*

1. total *if, for any elements h_1 and h_2 of H, either $h_1 \preceq h_2$ or $h_2 \preceq h_1$.*
2. biinvariant *if, for any elements h_1, h_2 and h_3 of H,*
$$h_1 \preceq h_2 \Rightarrow h_3 h_1 \preceq h_3 h_2$$
and
$$h_1 \preceq h_2 \Rightarrow h_1 h_3 \preceq h_2 h_3.$$
3. Archimedean *if, for any element h_1 and h_2 of H with $1 \prec h_2$ (*i.e. *$1 \preceq h_2$ and $h_2 \neq 1$), there exists an integer $n \geq 0$ such that $h_1 \preceq h_2^n$.*

The main idea of the proof of Lemma 4.13 is to apply the following lemma (see [8][Proposition 2.2.29 p.40] for a proof of this lemma).

Lemma 4.15. *Any group which admits a bi-invariant total Archimedean order is a subgroup of $(\mathbb{R}, +)$.*

Proof of Lemma 4.13. We define an order $\preceq$ on the group $\mathcal{H}(F)$ in the following way. For any elements ξ and η which are respectively represented by elements g_ξ and g_η of H, we have $\xi \preceq \eta$ if and only if there exists $\alpha > 0$ such that, for any point x of $F \cap (q-\alpha, q)$, we have $g_\xi(x) \leq g_\eta(x)$. We will prove that this defines a biinvariant total Archimedean order on $\mathcal{H}(F)$. By Lemma 4.15, this implies that the group $\mathcal{H}(F)$ is abelian and proves Lemma 4.13.

Let ξ_1, ξ_2 and ξ_3 be elements of $\mathcal{H}(F)$ with $\xi_1 \preceq \xi_2$. Let us prove that $\xi_1\xi_3 \preceq \xi_2\xi_3$ and that $\xi_3\xi_1 \preceq \xi_3\xi_1$.

Take elements g_1, g_2 and g_3 of the group H which respectively represent the elements ξ_1, ξ_2 and ξ_3. Take $\alpha > 0$ small enough such that

1. The diffeomorphisms g_1, g_2 and g_3 are increasing on $[q-\alpha, q]$.
2. For any point x in $[q-\alpha, q) \cap F$, $g_1(x) \leq g_2(x)$.

Take $\alpha' > 0$ small enough so that $g_3([q-\alpha', q] \cap F) \subset [q-\alpha, q] \cap F$. Then, for any point x in $[q-\alpha', q] \cap F$,

$$g_1(g_3(x)) \leq g_2(g_3(x))$$

and $\xi_1\xi_3 \preceq \xi_2\xi_3$. Now, take $\alpha'' > 0$ small enough such that $g_1([q-\alpha'', q] \cap F) \subset [q-\alpha, q] \cap F$ and $g_2([q-\alpha'', q] \cap F) \subset [q-\alpha, q] \cap F$. Then, as the diffeomorphism g_3 is increasing on $[q-\alpha, q]$, for any point x of $[q-\alpha'', q] \cap F$,

$$g_3(g_1(x)) \leq g_3(g_2(x))$$

and $\xi_3\xi_1 \preceq \xi_3\xi_2$.

We have just proved that the order $\preceq$ is biinvariant. Let us explain why this order is total. Let ξ and η be elements of $\mathcal{H}(F)$ which are respectively represented by elements g_ξ and g_η of H. By hypothesis of Lemma 4.13, either the diffeomorphism $g_\xi \circ g_\eta^{-1}$ is equal to the identity on $F \cap (q-\alpha, q)$ for $\alpha > 0$ small enough or this diffeomorphism displaces all the points of F in a left-neighbourhood of q. Hence either $\xi\eta^{-1} \succeq 1$ or $\xi\eta^{-1} \preceq 1$. By invariance of the order $\preceq$ under right-multiplication, we deduce that either $\xi \succeq \eta$ or $\xi \preceq \eta$. The order $\preceq$ is total.

Now, let us prove that it is Archimedean. Let ξ_1 and ξ_2 be elements of $\mathcal{H}(F)$ such that $\xi_1 \succ 1$. If $\xi_2 \preceq 1$, then $\xi_2 \preceq 1 \prec \xi_1$ so we suppose that $\xi_2 \succ 1$ in what follows.

Take respective representatives g_1 and g_2 of ξ_1 and ξ_2 in H. Take $\alpha \in (0, \alpha_0)$ small enough such that

1. The diffeomorphisms g_1 and g_2 are increasing on $[q-\alpha, q]$.
2. For any point x in $[q-\alpha, q) \cap F$, $g_1(x) > x$ and $g_2(x) > x$.

Fix a point x_0 in $F \cap [q-\alpha, q)$. Observe that any positive orbit under g_1 of points of $F \cap [q-\alpha, q)$ converges to the point q. Hence there exists $k > 0$ such that $g_2(x_0) < g_1^k(x_0)$, which can be rewritten

$$g_2(x_0) < g_1^k g_2^{-1}(g_2(x_0)).$$

As the diffeomorphism $g_1^k g_2^{-1}$ is increasing on $[g_2(x_0), q]$ then, by hypothesis of Lemma 4.13, one of the following occurs.

1. Either the diffeomorphism $g_1^k g_2^{-1}$ pointwise fixes a left-neighbourhood of the point q in F. In this case $\xi_1^k \xi_2^{-1} = 1$.
2. Or it has no fixed point in $[g_2(x_0), q] \cap F$ in which case

$$\forall x \in F \cap [g_2(x_0), p], \; g_1^k \circ g_2^{-1}(x) > x.$$

In either case, $\xi_1^k \xi_2^{-1} \succeq 1$ and, by invariance of the relation $\preceq$ under right-multiplication, $\xi_1^k \succeq \xi_2$. The order $\preceq$ is Archimedean. □

Proof of Proposition 4.11. We denote by K_1' the set of accumulation points of K_1. By minimality of K_1, observe that either $K_1' \cap (p-\alpha_0, p) = \emptyset$ or $K_1' \cap (p-\alpha_0, p) = K_1$. Indeed, if the set $K_1' \cap (p-\alpha_0, p)$ is nonempty, then $K_1' \cap (p-\alpha_0, p)$ is an element of $\mathcal{M}$ which is contained in K_1, hence which is equal to K_1 by minimality of K_1. This remark splits the proof into two cases.

First case: $K_1' \cap (p-\alpha_0, p) = \emptyset$. As any point of the set K_1 is isolated, any element of G_1 either pointwise fixes all the points of K_1 or displaces all the points of K_1 on a left neighbourhood of the point p (neighbourhood which a priori depends on the element of G_1). Lemma 4.13 implies that the group $\mathcal{G}_1(K_1)$ is abelian.

Second case: $K_1' \cap (p-\alpha_0, p) = K_1$. In this second case, it also suffices to prove that any element of G_1 satisfies the hypothesis of Lemma 4.13. However, proving this fact is harder than in the first case.

Suppose for a contradiction that there exists $\alpha > 0$ and a diffeomorphism g_1 in G_1 such that

1. The diffeomorphism g_1 is increasing on $[p-\alpha, p]$.
2. There exists a point x in $K_1 \cap [p-\alpha, p)$ such that $g_1(x) = x$.
3. The diffeomorphism g_1 does not fix all the points of $[x, p] \cap K_1$.

Let $p_1 < p_2$ be two points of $[x,p] \cap \mathrm{Fix}(g_1)$ such that $K_1 \cap (p_1, p_2) \neq \emptyset$ and, for any point x in $K_1 \cap (p_1, p_2)$, $g_1(x) \neq x$. In this second case, Proposition 4.11 is a consequence of the two following lemmas.

Lemma 4.16. *There exists an element h of G_1 such that $p_1 < h(p_1) < p_2$ and the diffeomorphism h is increasing on $[p_1, p]$.*

Before stating the second lemma, we need a definition which is a generalization of a standard definition for pseudogroups or groups of homeomorphisms of the real line (see [8][Definition 2.2.43 p.52]).

Definition 4.17. *Two elements g and h of $\mathfrak{Diff}^{1+Lip}(K)$ are* crossed *if there exists points $p_1 < p_2$ of the Cantor set K such that*

1. *The diffeomorphisms g and h are increasing on $[p_1, p_2]$.*
2. *The points p_1 and p_2 are fixed under g but the diffeomorphism g has no fixed point in (p_1, p_2).*
3. *Either we have $p_1 < h(p_1) < p_2$ or $p_1 < h(p_2) < p_2$.*

Observe that the element g_1 which is defined above and the element h which is given by Lemma 4.16 are crossed. Now, it suffices to use Lemma 4.18 below to obtain the wanted contradiction.

Lemma 4.18. *If two elements g and h of $\mathfrak{Diff}^{1+Lip}(K)$ are crossed, then the group generated by g and h contains a free semi-group on two generators.*

□

Now, let us prove Lemmas 4.16 and 4.18.

Proof of Lemma 4.16. Denote by $\mathcal{B}$ the set of elements g of G_1 with the following property. There exists a family $(s_i)_{1 \leq i \leq p}$ of elements of S such that $g = s_1 s_2 \dots s_p$ and

$$\forall 1 \leq i \leq p,\ s_i s_{i+1} \dots s_p([p_1, p] \cap K) \subset (p - \alpha_0, p].$$

Now let A be the closure in K_1 of the subset of $K \cap (p-\alpha_0, p)$ defined by

$$B = \{b(p_1), b \in \mathcal{B}\}.$$

Observe that, as the subset K_1 belongs to the collection $\mathcal{M}$, we have $A \subset K_1$. Moreover, the set A belongs to the collection $\mathcal{M}$ so that $A = K_1$, by minimality of K_1. Hence the set B is dense in K_1 and $B \cap (p_1, p_2) \neq \emptyset$ as $K_1 \cap (p_1, p_2) \neq \emptyset$. This proves Lemma 4.16. □

The proof of Lemma 4.18 is similar to [8][Lemma 2.2.44].

Proof of Lemma 4.18. Suppose for instance that $p_1 < h(p_1) < p_2$. Moreover, as the diffeomorphism g is increasing on $[p_1, p_2]$, taking g^{-1} instead of g if necessary, we can suppose that

$$\forall x \in (p_1, p_2) \cap K, \ g(x) < x.$$

Observe that, for any point x in $(p_1, p_2) \cap K$, the sequence $g^n(x)$ converges to the point p_1. In particular, the point p_1 is accumulated on the right by points of K. Take $\alpha > 0$ small enough so that

1. $p_1 + \alpha \in K$ and the diffeomorphism h is monotonous on $[p_1, p_1 + \alpha]$.
2. $h(p_1 + \alpha) < p_2$.
3. The sets $[p_1, p_1 + \alpha]$ and $h([p_1, p_1 + \alpha] \cap K)$ are disjoint.

Take a sufficiently large integer n such that $g^n(h(p_1 + \alpha)) < p_1 + \alpha$. Let $f_1 = g^n$ and $f_2 = h \circ g^n$. Observe that

$$f_1([p_1, p_1 + \alpha] \cap K \cup h([p_1, p_1 + \alpha] \cap K)) \subset [p_1, p_1 + \alpha] \cap K$$

and that

$$f_2([p_1, p_1 + \alpha] \cap K \cup h([p_1, p_1 + \alpha] \cap K)) \subset h([p_1, p_1 + \alpha] \cap K).$$

Now, by the positive ping-pong lemma (Lemma 4.6), the semigroup generated by f_1 and f_2 is free. □

b. Second step

Now, we are ready for the second step of the proof of Proposition 4.7. We will reformulate this second step as two propositions. Fix a point p in $\mathrm{Fix}(G_1)$ which is accumulated on the left by the set K. Denote by $\mathcal{G}_1$ the group of left-germs at p of elements of G_1: this the quotient of the group G_1 by the equivalence relation $\equiv_{p,-}$ defined for any elements g_1 and g_2 of G_1 by

$$g_1 \equiv_{p,-} g_2 \Longleftrightarrow \exists \alpha > 0, \ g_{1|[p-\alpha,p]\cap K} = g_{2|[p-\alpha,p]\cap K}.$$

Proposition 4.19. *Suppose that the point p is isolated on the left in* $\mathrm{Fix}(G_1)$. *Then the group $\mathcal{G}_1$ is abelian.*

Of course, we have an analogous statement in the case where the point p is isolated on the right in $\mathrm{Fix}(G_1)$.

In the case where the point p is accumulated on the left by points of $\mathrm{Fix}(G_1)$, we will prove the following stronger proposition.

Proposition 4.20. *Suppose the point p is accumulated on the left by points of* $\mathrm{Fix}(G_1)$. *Then there exists a left-neighbourhood L_p of the point p with the following properties.*

1. *$L_p \cap K$ is invariant under the action of G_1.*
2. *The action of G_1' on $L_p \cap K$ is trivial.*

Once again, there is an analogous statement for points which are accumulated on the right by points of $\mathrm{Fix}(G_1)$.

The two above propositions and there variants for right-neighbourhoods imply Proposition 4.7.

We will start by proving Proposition 4.19. Then we will prove Proposition 4.20.

To prove Proposition 4.19, we need the following lemma, which is a variant of Kopell lemma for diffeomorphisms of the half-line (see [7] to see this lemma and its proof).

Lemma 4.21. *Let g_1 and g_2 be elements of $\mathfrak{Diff}^{1+Lip}(K)$. Let $p \in \mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$ be a point accumulated on the left by points of K such that the elements g_1 and g_2 have a positive derivative at p. Suppose that*

1. *There exists $\alpha > 0$ such that $(p-\alpha, p) \cap \mathrm{Fix}(g_1) = \emptyset$ and such that g_2 is increasing on $(p-\alpha, p)$.*
2. *For any point x in $(p-\alpha, p) \cap K$, $g_1 g_2(x) = g_2 g_1(x)$.*

Then $(p-\alpha, p) \cap \mathrm{Fix}(g_2) = \emptyset$ or the diffeomorphism g_2 fixes all the points of K in a left-neighbourhood of the point p.

Of course, we have an analogous lemma for fixed points of g_1 which are isolated on the right. In the classical Kopell lemma, we only need that g_1 is C^{1+bv} and g_2 is C^1. In contrast, in our case, we can lower the regularity to C^{1+bv} but we still need both elements g_1 and g_2 to have a C^{1+bv} regularity. The proof of this lemma is closely related to the proof in the standard case.

Proof of Lemma 4.21. Take $\alpha > 0$ such that g_1 has no fixed point on $[p-\alpha, p)$ and such that there exist C^{1+Lip}-diffeomorphisms $\tilde{g}_1$ and $\tilde{g}_2$ defined on $[p-\alpha, p]$ such that $g_1{}_{|[p-\alpha,p]\cap K} = \tilde{g}_1{}_{|[p-\alpha,p]\cap K}$ and $g_2{}_{|[p-\alpha,p]\cap K} = \tilde{g}_2{}_{|[p-\alpha,p]\cap K}$. For $i = 1, 2$, denote by k_i the Lipschitz constant of $\log(|\tilde{g}_i'|)_{|[p-\alpha,p]}$ and by D the diameter of the compact subset K of $\mathbb{R}$.

Suppose that g_2 has a fixed point p' in $[p-\alpha, p)$. We will prove that there exists $M > 0$ such that, for any $k > 0$, $\sup_{x \in [p',p]\cap K} |(g_2^k)'(x)| \leq M$. From this, we will deduce that g_2 fixes the points of $[p', p] \cap K$.

Taking g_1^{-1} instead of g_1 if necessary, we can suppose that, for any point x in $[p-\alpha, p) \cap K$, $\tilde{g}_1(x) > x$. As the diffeomorphisms g_1 and g_2 commute the points $g_1^n(p')$, for $n \geq 0$ are fixed points of g_2 and form a sequence which converges to the point p. Hence $g_2'(p) = 1$.

For any $k \geq 0$ and any $n \geq 0$, $g_2^k = g_1^{-n} g_2^k g_1^n$. Hence, for any point x of $K \cap [p', p]$,

$$(*)\quad (g_2^k)'(x) = \frac{(g_1^n)'(x)}{(g_1^n)'(g_2^k(x))} . (g_2^k)'(g_1^n(x)).$$

Observe that the sequence $((g_2^k)'(g_1^n(x)))_n$ converges to $1 = (g_2^k)'(p)$ as $n \to +\infty$. Let us prove that

$$\frac{(g_1^n)'(x)}{(g_1^n)'(g_2^k(x))} \leq e^{k_1 D}.$$

Indeed,

$$\log(|\tfrac{(g_1^n)'(x)}{(g_1^n)'(g_2^k(x))}|) \quad = \quad \sum_{i=0}^{n-1} \log(|g_1'(g_1^i(x))|) - \log(|g_1'(g_1^i(g_2^k(x)))|).$$

For any index $i \geq 0$, denote by I_i the closed interval of $\mathbb{R}$ whose ends are $g_1^i(x)$ and $g_1^i(g_2^k(x))$ and denote by $n_0 \geq 0$ the integer such that the point x belongs to the interval $[g_1^{n_0}(p'), g_1^{n_0+1}(p'))$. Observe that, for any $i \geq 0$, $I_i \subset [g_1^{n_0+i}(p'), g_1^{n_0+i+1}(p'))$. Hence the intervals I_i, for $i \geq 0$ are pairwise disjoint and

$$\log(|\tfrac{(g_1^n)'(x)}{(g_1^n)'(g_2^k(x))}|) \quad \leq \quad k_1 \sum_{i=0}^{n-1} |I_i| \quad \leq \quad k_1 D.$$

Then, by $(*)$, for any $k > 0$, $\sup_{x \in [p',p] \cap K} |(g_2^k)'(x)| \leq M$, where $M = e^{k_1 D}$.

Now let us prove that the diffeomorphism g_2 fixes the points in $[p',p] \cap K$. Suppose for a contradiction that there exists a point x_0 in $[p',p] \cap K$ which is not fixed under g_2. Let I be the connected component of $[p',p] - \mathrm{Fix}(g_2)$ which contains x_0. Take a point y in $I - K$ and let (y_-, y_+) be the connected component of $I - K$ which contains y. Then, for any $k > 0$,

$$\frac{(\tilde{g}_2^k)'(y)}{(\tilde{g}_2^k)'(y_-)} \leq e^{k_2 D}.$$

Hence $(\tilde{g}_2^k)'(y) \leq M e^{k_2 D}$. This implies that, for any $k \geq 0$, $\sup_{x \in I} |(\tilde{g}_2^k)'(x)| \leq M e^{k_2 D}$. Hence, using the mean value theorem, we see that the diffeomorphism $\tilde{g}_2$ has to fix the points of I, a contradiction. □

Propostion 4.19 is a consequence of the following weaker Proposition.

Proposition 4.22. *Suppose that the point p is isolated on the left in* $\mathrm{Fix}(G_1)$*. Then the group $\mathcal{G}_1$ is metabelian,* i.e. *the group $\mathcal{G}_1'$ is abelian.*

Proof of Proposition 4.22. By Corollary 4.12, there exists an element f of G_1 and $\alpha_0' > 0$ such that the diffeomorphism f has no fixed point on $(p - \alpha_0', p)$ and is increasing on the interval $[p - \alpha_0', p]$. Taking f^{-1} instead of f if necessary, we can suppose that, for any point x in $(p - \alpha_0', p) \cap K$, $f(x) > x$.

Fix two nontrivial elements ξ_1 and ξ_2 of the group $\mathcal{G}_1'$. We want to prove that $\xi_1 \xi_2 = \xi_2 \xi_1$. Denote by $\mathcal{G}_2$ the subgroup of $\mathcal{G}_1$ generated by the elements ξ_1 and ξ_2. The advantage of considering this subgroup instead of $\mathcal{G}_1'$ is that the group $\mathcal{G}_2$ is finitely generated whereas the group $\mathcal{G}_1'$ might not be finitely generated.

Fix respective representative g_1 and g_2 of ξ_1 and ξ_2 in the group G_1' and let $K_2 \subset K$ the set of fixed points of the group G_2 generated by g_1 and g_2. Observe that, by Proposition 4.9, the set $\mathrm{Fix}(G_1') \subset K_2$ accumulates on the left of the point p. Take a point $p - \alpha_0' < p' < p$ such that

1. The diffeomorphisms g_1 and g_2 are increasing on $[p', p]$.
2. The point p' belongs to the set $\mathrm{Fix}(G_1')$.

An easy induction on wordlength proves that any element of G_2 is increasing on $[p', p]$.

We will distinguish two cases depending on the existence of a fixed point which is outside K_2 for an element of G_2.

First case: As the elements ξ_1 and ξ_2 are supposed to be nontrivial, the set $(K \setminus K_2) \cap [p', p]$ is nonempty. Suppose that, for any element g in the group G_2 and any connected component (p_1, p_2) of $[p', p] \setminus K_2$ which meets K, either

$$\mathrm{Fix}(g) \cap (p_1, p_2) = \emptyset$$

or g is equal to the identity on $(p_1, p_2) \cap K$. Take a connected component (p_1, p_2) of $[p', p] - K_2$ which meets K. Denote by $G_{2|(p_1, p_2)}$ be the group of restrictions to $(p_1, p_2) \cap K$ of elements of the group G_2. We now use the following lemma which is once again a straightforward consequence of Lemma 4.15.

Lemma 4.23. *Let I be an open interval of $\mathbb{R}$ which meets K and whose endpoints belong to the Cantor set K. Let G be a subgroup of $\mathfrak{Diff}^1(K)$. Suppose that*

1. *Any diffeomorphism in the group G preserves $I \cap K$ and is increasing on the interval I.*

2. Any element of G which has a fixed point in $I \cap K$ is equal to the identity on $I \cap K$. Then the group $G_{|I \cap K}$ of restrictions to $I \cap K$ of elements of G is abelian.

Proof. As any nontrivial element of the group G is increasing on I and has no fixed point on $I \cap K$, then, for any nontrivial element g of the group $G_{|I \cap K}$, either

$$\forall x \in I \cap K, \; g(x) > x$$

or

$$\forall x \in I \cap K, \; g(x) < x.$$

Let g and h be two elements of the group G. Hence, if there exists a point x_0 in $I \cap K$ such that $g(x_0) < h(x_0)$ then, for any point x of $I \cap K$, $g(x) < h(x)$: otherwise the diffeomorphism $g^{-1}h$ would be nontrivial and would have a fixed point in $I \cap K$, which is not possible. We define then an order $\preceq$ on $G_{|I \cap K}$ by setting $g \preceq h$ if and only if there exists a point x_0 of $I \cap K$ such that $g(x_0) \leq h(x_0)$. The above remark proves that this defines a total order on the group $G_{|I \cap K}$. With a proof similar to the proof of Lemma 4.13 (and even easier!), we can show that this defines a biinvariant Archimedean order on the group $G_{|I \cap K}$. By Lemma 4.15, the group $G_{|I \cap K}$ is abelian. □

By Lemma 4.23, the group $G_{2|(p_1,p_2)}$ is abelian. As this lemma is true for any such connected component (p_1, p_2), we deduce that

$$\forall x \in [p', p] \cap K, g_1 g_2(x) = g_2 g_1(x).$$

Hence $\xi_1 \xi_2 = \xi_2 \xi_1$.

Second case: There exists an element g in the group G_2 and a point p_0 in $(K \setminus K_2) \cap [p', p]$ with the following properties.

1. $g(p_0) = p_0$.
2. If we denote by (p_1, p_2) the connected component of $[p', p] \setminus K_2$ which contains the point p_0,

$$g_{|(p_1,p_2) \cap K} \neq Id_{(p_1,p_2) \cap K}.$$

Without loss of generality, we can suppose that the point p_0 is one of the endpoints of a connected component of $[p', p] \setminus \mathrm{Fix}(g)$ which meets the Cantor set K. We will find a contradiction, namely we will construct a free subsemigroup on two generators of the group G_1. Hence the first case always holds.

As the point p_0 does not belong to the set $K_2 = \mathrm{Fix}(G_2)$, there exists a diffeomorphism h in G_2 such that $h(p_0) > p_0$.

Let (a, b) be the connected component of $\mathbb{R} \setminus \mathrm{Fix}(G'_1)$ which contains the points p_0 and $h(p_0)$. Take $N' > 0$ sufficiently large so that $f^{N'}(a) > b$ and let $f_1 = f^{N'}$. Observe that the sets $f_1^n([a, b] \cap K)$, for $n \geq 0$, are pairwise disjoint. Moreover, as the set $\mathrm{Fix}(G'_1)$ is invariant under the action of the group G_1, for any $n \geq 0$, the points $f_1^n(a)$ and $f_1^n(b)$ are fixed under the elements of $G_2 < G'_1$.

The proof uses the following lemma which relies on distortion estimates.

Lemma 4.24. *There exists an integer $N \geq 0$ such that, for any $n \geq N$,*

$$h(f_1^n(p_0)) = f_1^n(p_0).$$

Before proving this lemma, let us see how we can obtain a contradiction from this lemma. More precisely, we want to prove that the semigroup generated by f_1 and h is free, a contradiction. We will use the following lemma to do so.

Lemma 4.25. *Let $p' < p$ be points of K. Suppose that the point p is accumulated on the left by points of K. Let f and h be diffeomorphisms in $\mathfrak{Diff}^1(K)$ such that*

1. *The diffeomorphism f is increasing on $[p', p]$ and*
$$\begin{cases} f(p) = p \\ \forall x \in [p', p) \cap K, \ f(x) > x \end{cases}.$$
2. *There exists a point $p_* \in (p', p) \cap K$ such that*
$$\forall n \geq 0, h(f^n(p_*)) = f^n(p_*).$$
3. *There exists a point p_0 in $[p', p_*) \cap K$ and an integer $N > 0$ such that*
$$\begin{cases} h(p_0) \neq p_0 \\ \forall n \geq N, h(f^n(p_0)) = f^n(p_0). \end{cases}$$

Then there exists $N' > 0$ such that the semigroup generated by $f^{N'}$ and $f^{N'}h$ is free.

To obtain the wanted contradiction, use the above lemma with f_1, h and $p_* = b$. The last hypothesis is satisfied thanks to Lemma 4.24. To complete the proof of Proposition 4.22, it suffices to prove those two lemmas.

Proof of Lemma 4.24. Recall that the point p_0 is the endpoint of a connected component of $\mathbb{R} \setminus \mathrm{Fix}(g)$ which meets K. Suppose that this connected component is of the form (p_0, p_1) (the case where it is of the form (p_1, p_0) is analogous). Observe that the interval $(f_1^n(p_0), f_1^n(p_1))$ is a connected component of $\mathbb{R} \setminus \mathrm{Fix}(f_1^n g f_1^{-n})$. Then, as the elements h and $f_1^n g f_1^{-n}$, as well as the elements h^{-1} and $f_1^n g f_1^{-n}$, are not crossed by Lemma 4.18, either $h(f_1^n(p_0)) = f_1^n(p_0)$ or $h(f_1^n(p_0)) \geq f_1^n(p_1)$ or $h(f_1^n(p_1)) \leq f_1^n(p_0)$: if none of those statements occur,

1. either $h(f_1^n(p_0)) \in (f_1^n(p_0), f_1^n(p_1))$ and the elements h and $f_1^n g f_1^{-n}$ are crossed,
2. or $h(f_1^n(p_1)) \in (f_1^n(p_0), f_1^n(p_1))$ and the elements h and $f_1^n g f_1^{-n}$ are crossed,
3. or $h(f_1^n(p_0)) \leq f_1^n(p_0)$ and $h(f_1^n(p_1)) \geq f_1^n(p_1)$. In this case, $h^{-1}(f_1^n(p_0)) \geq f_1^n(p_0)$ and $h^{-1}(f_1^n(p_1)) \leq f_1^n(p_1)$ and the elements h^{-1} and $f_1^n g f_1^{-n}$ are crossed.

Suppose for a contradiction that there exists a sequence of integer $n_k \to +\infty$ as $k \to +\infty$ such that, for any $k \geq 0$, $h(f_1^{n_k}(p_0)) \geq f_1^{n_k}(p_1)$. Then, for any $k \geq 0$,

$$\frac{h(f_1^{n_k}(p_0)) - h(f_1^{n_k}(a))}{f_1^{n_k}(p_0) - f_1^{n_k}(a)} = \frac{h(f_1^{n_k}(p_0)) - f_1^{n_k}(a)}{f_1^{n_k}(p_0) - f_1^{n_k}(a)}$$

as the point $f_1^{n_k}(a)$ belongs to $\mathrm{Fix}(G_1')$ and the diffeomorphism h belongs to G_1'. Hence

$$\begin{aligned} \frac{h(f_1^{n_k}(p_0)) - h(f_1^{n_k}(a))}{f_1^{n_k}(p_0) - f_1^{n_k}(a)} &\geq \frac{f_1^{n_k}(p_1) - f_1^{n_k}(a)}{f_1^{n_k}(p_0) - f_1^{n_k}(a)} \\ &\geq \frac{f_1^{n_k}(p_1) - f_1^{n_k}(p_0)}{f_1^{n_k}(p_0) - f_1^{n_k}(a)} + 1. \end{aligned}$$

Denote by $\tilde{f}_1 : [a, p] \to [a, p]$ a C^{1+Lip}-diffeomorphism such that $\tilde{f}_{1|[a,p] \cap K} = f_{1|[a,p] \cap K}$. By the mean value theorem, there exist points c_1 and c_2 of the interval (a, b) such that

$$\begin{cases} \tilde{f}_1^{n_k}(p_1) - \tilde{f}_1^{n_k}(p_0) = (\tilde{f}_1^{n_k})'(c_1)(p_1 - p_0) \\ \tilde{f}_1^{n_k}(p_0) - \tilde{f}_1^{n_k}(a) = (\tilde{f}_1^{n_k})'(c_2)(p_0 - a). \end{cases}$$

Moreover, if we denote by K the Lipschitz constant of $\log(f')$ then

$$\begin{aligned} \left| \log((\tilde{f}_1^{n_k})'(c_1)) - \log((\tilde{f}_1^{n_k})'(c_2)) \right| &\leq K \sum_{i=0}^{n_k - 1} \left| \tilde{f}_1^i([a, b]) \right| \\ &\leq K \left| p - a \right| = M, \end{aligned}$$

because the intervals $\tilde{f}_1^i([a,b])$ are pairwise disjoint and contained in the interval (a,p). Hence

$$\frac{h(f_1^{n_k}(p_0)) - h(f_1^{n_k}(a))}{f_1^{n_k}(p_0) - f_1^{n_k}(a)} \geq e^{-M} \frac{p_1 - p_0}{p_0 - a} + 1 > 1.$$

However, recall that fixed points of h accumulate to p because $h \in G_1'$. Hence $h'(p) = 1$, in contradiction with the above inequality and the continuity of h'.

In the case where there exists a sequence of integers n_k which tends to $+\infty$ as $k \to +\infty$ such that, for any k, $h(f^{n_k}(p_1)) \leq f^{n_k}(p_0)$ (or $f^{n_k}(p_1) \leq h^{-1}(f^{n_k}(p_0))$), we find a similar contradiction by using h^{-1} instead of h. □

Proof of Lemma 4.25. Let $N' - 1$ be the largest integer n such that $h(f^n(p_0)) \neq f^n(p_0)$. Let $f_1 = f^{N'}$. Hence, for any integer $n > 0$, the point $f_1^n(p_0)$ is fixed under h (but the point p_0 is not). We want to prove that the semigroup generated by f_1 and $g = f_1 h$ is free. Let

$$w_1 = f_1^{n_k} g^{m_k} \ldots f_1^{n_2} g^{m_2} f_1^{n_1} g^{m_1},$$

with $n_k \geq 0$, $m_1 \geq 0$ and $n_i > 0$, $m_i > 0$ otherwise, and

$$w_2 = f_1^{n'_{k'}} g^{m'_{k'}} \ldots f_1^{n'_2} g^{m'_2} f_1^{n'_1} g^{m'_1},$$

with $n'_{k'} \geq 0$, $m'_1 \geq 0$ and $n'_i > 0$, $m'_i > 0$ otherwise, be two distinct words on f_1 and g. We see each of those words as a diffeomorphism in $\mathfrak{Diff}^1(K)$. Suppose for a contradiction that, as elements of the group $\mathfrak{Diff}^1(K)$, $w_1 = w_2$. Then, simplifying these words on the right and exchanging the roles of w_1 and w_2 if necessary, we can suppose that $m_1 > 0$ and $m'_1 = 0$. Now, let us look at the image of the point p_* under those two diffeomorphisms. We have

$$\begin{cases} w_1(p_*) = f_1^{n_1+m_1+n_2+m_2+\ldots+n_k+m_k}(p_*) \\ w_2(p_*) = f_1^{n'_1+m'_1+n'_2+m'_2+\ldots+n'_{k'}+m'_{k'}}(p_*) \end{cases}.$$

Hence $\sum_{i=1}^{k}(n_i + m_i) = \sum_{i=1}^{k'}(n'_i + m'_i)$. We denote by l the common value of these sums. Now, we will prove that $w_1(p_0) \neq w_2(p_0)$, in contradiction with the equality $w_1 = w_2$ as elements of the group $\mathfrak{Diff}^1(K)$.

We write

$$w_1 = w_3 h$$

where

$$w_3 = f_1^{n_k} g^{m_k} \ldots f_1^{n_2} g^{m_2} f_1^{n_1} g^{m_1 - 1} f_1.$$

Observe that $w_2(p_0) = w_3(p_0) = f_1^l(p_0)$. As we supposed that $w_1 = w_2$ as elements of $\mathfrak{Diff}^1(K)$, we have $w_3 h(p_0) = w_1(p_0) = w_2(p_0)$ hence $h(p_0) = p_0$, a contradiction. □

□

Now, let us deduce Proposition 4.19 from Proposition 4.22.

Proof of Proposition 4.19. By Proposition 4.22, the group $\mathcal{G}_1$ is metabelian, meaning that its derived subgroup is abelian. A theorem by Rosenblatt (see [10]) states that any metabelian group without free subsemigroups on two generators is nilpotent. Hence the group $\mathcal{G}_1$ is nilpotent.

Suppose for a contradiction that the group $\mathcal{G}_1$ is not abelian and take a nontrivial element ξ in the center of the group $\mathcal{G}_1$ which belongs to the derived subgroup $\mathcal{G}_1'$.

Recall that, by Corollary 4.12, the group $\mathcal{G}_1$ contains an element whose representative has no fixed point in a left-neighbouhood of the point p. Hence, by Lemma 4.21, the element ξ has a representative g_ξ in G_1 with no fixed point on a left-neighbourhood of the point p.

We want to apply Lemma 4.13 to prove that the group $\mathcal{G}_1$ is abelian and finish the proof of Proposition 4.19. In the rest of this proof, we make sure that the hypothesis of Lemma 4.13 are satisfied.

Take $\alpha_0 > 0$ such that the diffeomorphisms g_ξ and g_ξ^{-1} are increasing on $[p - \alpha_0, p]$ and has no fixed point in $[p - \alpha_0, p) \cap K$. Taking ξ^{-1} instead of ξ if necessary, we can further suppose that, for any point x of $[p - \alpha_0, p] \cap K$,

$$g_\xi(x) > x.$$

Suppose for a contradiction that there exist a real number $\alpha \in (0, \alpha_0)$ and a diffeomorphism $g \in G_1$ with the following properties.

1. The diffeomorphism g is increasing on $[p - \alpha, p]$.
2. The diffeomorphism g has a fixed point p_1 in $[p - \alpha, p) \cap K$.
3. There exists a point x_0 in $[p_1, p] \cap K$ such that $g(x_0) \neq x_0$.

As $g([p_1, p]) = [p_1, p]$ and $g_\xi([p_1, p]) \subset [p_1, p]$, we have

$$g^{-1} g_\xi g([p_1, p]) \subset [p_1, p] \subset (p - \alpha_0, p)$$

and the diffeomorphism $h = [g_\xi, g] = g_\xi^{-1} g^{-1} g_\xi g$ is increasing on $[p_1, p]$. Take $\alpha' > 0$ small enough so that $p - \alpha_0 < p_1 - \alpha'$ and the diffeomorphism h is increasing on $(p_1 - \alpha', p]$.

Lemma 4.26. *For any point x in $(p_1 - \alpha', p] \cap K$, $h(x) = x$*

Proof. Suppose that there exists a point p_0 in $(p_1 - \alpha', p] \cap K$ such that $h(p_0) \neq p_0$. As the element ξ of $\mathcal{G}_1$ lies in the center of $\mathcal{G}_1$, the diffeomorphism h pointwise fixes a left-neighbourhood of p in K. Let

$$p_* = \inf \{x \in [p_0, p], \forall y \in [x, p] \cap K, h(y) = y\}.$$

Then apply Lemma 4.25 to h and $f = g_\xi$ to find a free subsemigroup of G_1 on two generators, a contradiction. $\square$

By Lemma 4.26, for any point x in $(p_1 - \alpha', p] \cap K$,

$$g g_\xi(x) = g_\xi g(x).$$

Hence, by Lemma 4.21, the diffeomorphism g has no fixed point in $(p_1 - \alpha', p]$, a contradiction.

Therefore, we can apply Lemma 4.13 and the group $\mathcal{G}_1$ is abelian. $\square$

Proof of Proposition 4.20. Fix a finite generating set S of G_1. As the point p is accumulated on the left by points of $\mathrm{Fix}(G_1)$, for any diffeomorphism s in S, there exists a point $p_s < p$ in $\mathrm{Fix}(G_1)$ such that s is increasing on the interval $[p_s, p]$. Let

$$p' = \max \{p_s \mid s \in S\}.$$

Then any element of S is increasing on $L_p = [p', p]$ and preserves $L_p \cap K$. Hence any element of $G_1 =< S >$ is increasing on L_p and preserves $L_p \cap K$.

Now, let us prove by contradiction that the group G'_1 acts trivially on $L_p \cap K$. Suppose there exists a point $x_0 \in (p', p) \cap K$ which is displaced by some element of G'_1. Let (p_1, p_2) be the connected component of $(p', p) \setminus \mathrm{Fix}(G_1)$ which contains the point x_0. Then the points p_1 and p_2 belong to $\mathrm{Fix}(G_1)$. Moreover, the point p_1 is accumulated on the right by points of K: otherwise, if I is connected component of $\mathbb{R} \setminus K$ whose left-end is the point p_1, then its right-end is a fixed point of G_1, in contradiction with $(p_1, p_2) \cap \mathrm{Fix}(G_1) = \emptyset$. Likewise, the point p_2 is accumulated on the left by points of K. By Corollary 4.12, there exists a point $p'_2 < p_2$ of K and an element f of G_1 such that

$$\forall x \in [p'_2, p_2) \cap K, \ f(x) > x.$$

We then need the following lemma.

Lemma 4.27.

$$[p'_2, p_2] \cap K \subset \mathrm{Fix}(G'_1).$$

Of course, we can likewise prove that there exists a point $p'_1 > p_1$ of K such that $[p_1, p'_1] \cap K \subset \mathrm{Fix}(G'_1)$.

Before proving this lemma, let us see why it gives us the wanted contradiction. The set

$$A = \overline{\{x \in [p_1, p_2] \cap K \mid \exists g_1 \in G'_1, \ g_1(x) \neq x\}}$$

is a closed G_1-invariant set as the group G'_1 is a normal subgroup of G_1 and any element of G_1 preserves $[p_1, p_2] \cap K$. Moreover, by Lemma 4.27, this set A is contained in $[p'_1, p'_2]$. Also, this set contains a minimal invariant set for the action of G_1 on K, hence a fixed point for G_1 by Proposition 4.4. Hence

$$\emptyset \neq A \cap \mathrm{Fix}(G_1) \subset [p'_1, p'_2] \cap \mathrm{Fix}(G_1) \subset (p_1, p_2) \cap \mathrm{Fix}(G_1) = \emptyset,$$

a contradiction. □

Proof of Lemma 4.27. Suppose for a contradiction that there exists a point $x_0 \in [p'_2, p_2] \cap K$ and an element h in G'_1 such that $h(x_0) \neq x_0$. Then let

$$p_{max} = \sup \{x \in [p'_2, p_2] \cap K, h(x) \neq x\}.$$

By Proposition 4.19, $p_{max} < p_2$. Observe that this point p_{max} is fixed under the diffeomorphism h.

Finally, take a point $p_0 < p_{max}$ of K such that $h(p_0) \neq p_0$ and $f(p_0) > p_{max}$. Taking h^{-1} instead of h if necessary, we can suppose that $h(p_0) > p_0$. Then, by Lemma 4.25, the group generated by h and f contains a free semigroup on two generators, a contradiction. □

4.4 End of the proof of Theorem 1.5

Now, we finish the proof of Theorem 1.5, namely we prove the following Proposition.

Proposition 4.28. *The group G'_1 is trivial.*

Proof of Proposition 4.28. For any point x in $\mathrm{Fix}(G_1)$, we want to define a left neighbourhood L_x and a right-neighbourhood R_x of the point x in K which will be useful for the proof. To define those, we have to distinguish cases.

1. If the point x is accumulated on the left (respectively the right) by points of $\mathrm{Fix}(G_1)$, take a left-neighbourhood L_x (resp. a right-neighbourhood R_x) of x such that the set L_x (resp. R_x) is pointwise fixed under the elements of G_1' (such a neighbourhood exists by Proposition 4.20).
2. If the point x is accumulated on the left (resp. the right) by points of K but isolated on the left (resp. the right) in $\mathrm{Fix}(G_1)$ then take a left-neighbourhood L_x (resp. a right-neighbourhood R_x) of x such that there exists a diffeomorphism f in G_1 such that
   (a) The diffeomorphism f is increasing on L_x (resp. R_x).
   (b) For any point y in $L_x \setminus \{x\}$, $f(y) > y$ (resp. for any point y in $R_x \setminus \{x\}$, $f(y) < y$).

   Such a diffeomorphism f exists by Corollary 4.12.
3. If the point x is isolated on the left (resp. on the right) in K, take $L_x = \{x\}$ (resp. $R_x = \{x\}$).

Let

$$U = \bigcup_{x \in \mathrm{Fix}(G_1)} (L_x \cup R_x)$$

and choose the neighbourhoods L_x and R_x in such a way that the set U is open in K. Suppose for a contradiction that the group G_1' contains anontrivial element h. Let A be a subset of K consisting of points x of K which are displaced under some element of the group G_1', *i.e.* there exists an element h of G_1' such that $h(x) \neq x$. Of course, this set is disjoint from the set $\mathrm{Fix}(G_1)$. We can say even more by the following lemma.

Lemma 4.29. *$A \cap U = \emptyset$.*

Before proving the lemma, let us see how we can finish the proof of Proposition 4.28. We denote by $\overline{A}$ the closure of the set A. As the set U is open, $\overline{A} \cap U = \emptyset$.

As the group G_1' is a normal subgroup of the group G_1, the set $\overline{A}$ is a closed G_1-invariant subset. Hence there exists a minimal set $M \subset \overline{A}$ for the action of G_1 on $\overline{A}$. By Proposition 4.4, $M \subset \mathrm{Fix}(G_1) \subset U$, a contradiction with Lemma 4.29. □

Proof of Lemma 4.29. Suppose for a contradiction that $A \cap U \neq \emptyset$ and take a point p_0 in the intersection $A \cap U$. By definition of the set A, there exists an element h in G_1' such that $h(p_0) \neq p_0$.

By definition of U, there exists a point p in $\mathrm{Fix}(G_1)$ such that either $p_0 \in L_p$ or $p_0 \in R_p$. Suppose for instance that the point p_0 belongs to the left-neighbourhood L_p of p. Necessarily, the point p is isolated on the left in $\mathrm{Fix}(G_1)$: otherwise, the diffeomorphisms in G_1' pointwise fix L_p. Moreover, by construction of L_p there exists an element f in G_1 such that

1. For any point y in $L_p \setminus \{p\}$, $f(y) > y$.
2. The diffeomorphism f is increasing on L_p.

Finally, we use Lemma 4.25 (recall that the diffeomorphism h pointwise fixes a neighbourhood of p to find the point p_*). By this lemma, the group G_1 contains a free semigroup on two generators, a contradiction. □

References

[1] C. Bleak, H. Bowman, A. Gordon Lynch, G. Graham, J. Hughes, F. Matucci, E. Sapir, *Centralizers in the R. Thompson group Vn*, Groups Geom. Dyn. 7 (2013), no. 4, 821-865.

[2] W. Burnside, *On an unsettled question in the theory of discontinuous groups*, Quart. J. Math., vol. 33, 1902, 230-238.

[3] L. Funar, Y. Neretin, *Diffeomorphism groups of tame Cantor sets and Thompson-like groups*, Compos. Math. 154 (2018), no. 5, 1066-1110.

[4] E. Golod, I. R. Shafarevich, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat. 28, 1964, 261-272.

[5] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[6] M. W. Hirsch, *Differential topology*, corrected reprint of the 1976 original, Graduate Texts in Mathematics, 33, Springer-Verlag, New York, 1994.

[7] N. Kopell, *Commuting diffeomorphisms*, 1970, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), 165-184, Amer. Math. Soc., Providence, R.I.

[8] A. Navas, *Groups of circle diffeomorphisms*, translation of the 2007 Spanish edition, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2011.

[9] A. Navas, *Growth of groups and diffeomorphisms of the interval*, Geom. and Functional analysis 18(2008), 988-1028.

[10] J. M. Rosenblatt, *Invariant measures and growth conditions*, Trans. Amer. Math. Soc. 193 (1974), 33-53.

[11] C.E. Rover, *Constructing finitely presented simple groups that contain Grigorchuk groups*, J. Algebra 220 (1999), no. 1, 284-313.

[12] Schachermayer, W., *Addendum: Une modification standard de la démonstration non standard de Reeb et Schweitzer Un théorème de Thurston établi au moyen de l'analyse non standard*, Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontificia Univ. Católica, Rio de Janeiro, 1976), pp. 139-140. Lecture Notes in Math., Vol. 652, Springer, Berlin, 1978.

[13] Thurston, William P., *A generalization of the Reeb stability theorem*, Topology 13 (1974), 347-352.