DISTORTION AND TITS ALTERNATIVE IN SMOOTH MAPPING CLASS GROUPS

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Abstract. In this article, we study the smooth mapping class group of a surface S relative to a given Cantor set, that is the group of isotopy classes of orientation-preserving smooth diffeomorphisms of S which preserve this Cantor set. When the Cantor set is the standard ternary Cantor set, we prove that the subgroup consisting of diffeomorphisms which are isotopic to the identity on S does not contain any distorted elements. Moreover, we prove a weak Tits alternative for these groups.

1. Introduction

Definition 1.1. Let S be a surface of finite type and let K be a closed subset contained in S. Let $\mathrm{Diff}(S,K)$ be the group of orientation-preserving C^∞-diffeomorphisms of S that leave K invariant (*i.e.* $f(K) = K$) and let $\mathrm{Diff}_0(S,K)$ be the identity component of $\mathrm{Diff}(S,K)$. We define the "smooth" mapping class group $\mathcal{M}^\infty(S,K)$ of S relative to K as the quotient:

$$\mathcal{M}^\infty(S,K) = \mathrm{Diff}(S,K)/\mathrm{Diff}_0(S,K)$$

If K is a finite set of points, $\mathcal{M}^\infty(S,K)$ coincides with the braid group with $|K|$ points in S.

The groups $\mathcal{M}^\infty(S,K)$ appear very naturally when studying group actions on surfaces, as given a smooth group action on S preserves a non-trivial closed set K, one obtains a group homomorphism into $\mathcal{M}^\infty(S,K)$. These groups were first studied by Funar and Neretin in [11]. The aim of this paper is to contribute to the study of $\mathcal{M}^\infty(S,K)$ for a Cantor set $K \subset S$. Our results are aiming towards the understanding of two basic properties of these groups, namely, the distortion of the cyclic subgroups and the Tits alternative.

A recent result related to mapping class groups of infinite type that deserves to be mentioned (but which we will not make use of here)

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is J.Bavard's proof that the mapping class group of $\mathbb{R}^2$ relative to a Cantor set K acts faithfully on a Gromov hyperbolic space. This hyperbolic space is similar to the curve complex for finite mapping class groups and was suggested by Calegari (see [2] and [4]). Possible lines of research and developments after Bavard's article are suggested by Calegari in his blog (see [4]). These developments partly inspired this work.

We now introduce some notation in order to state our results and explain the ideas involved in the proofs. Denote by $\mathcal{PM}^\infty(S,K) \subset \mathcal{M}^\infty(S,K)$ (the "pure" smooth mapping class group of K in S) the subgroup of the mapping class group $\mathcal{M}^\infty(S,K)$ consisting of the elements which fix K pointwise.

We also define the group $\mathfrak{diff}_S(K)$ as the group of homeomorphisms of K which are induced by orientation-preserving diffeomorphisms of S preserving K. In other words, a homeomorphism $f : K \to K$ belongs to $\mathfrak{diff}_S(K)$ if there exists $\bar{f} \in \mathrm{Diff}(S)$ such that $\bar{f}|_K = f$. There is a natural exact sequence of groups given by:

$$\mathcal{PM}^\infty(S,K) \to \mathcal{M}^\infty(S,K) \to \mathfrak{diff}_S(K). \tag{1}$$

The exact sequence (1) was studied by Funar and Neretin in [11], where it is proven that $\mathcal{PM}^\infty(S,K)$ is always a countable group (this follows from Lemma 2.2) and where for certain "affine" Cantor sets K, the group $\mathfrak{diff}_S(K)$ is shown to be countable and to have a "Thompson group" kind of structure.

We can now proceed to state our results. We begin with our results about distortion in $\mathcal{M}^\infty(S,K)$.

1.1. **Distortion.** We recall the concept of distortion which comes from geometric group theory and is due to Gromov (see [15]).

Definition 1.2. Let G be a group and let $\mathcal{G} \subset G$ be a finite set which is symmetric (*i.e.* $\mathcal{G} = \mathcal{G}^{-1}$). For any element $f \in G$ contained in the group generated by $\mathcal{G}$, the word length $l_{\mathcal{G}}(f)$ of f is defined as the minimal integer n such that f can be expressed as a product

$$f = s_1 s_2 ... s_n$$

where each $s_i \in \mathcal{G}$. An element f of a group G is called distorted if it has infinite order and there exists a finite symmetric set $\mathcal{G} \subset G$ such that

(1) $f \in \langle \mathcal{G} \rangle$.
(2) $\lim_{n\to\infty} \frac{l_{\mathcal{G}}(f^n)}{n} = 0$.

By a theorem of Farb, Lubotzky and Minsky, the mapping class group of a surface of finite type does not have distorted elements (see [12]). We believe that such a result should extend to the groups $\mathcal{M}^{\infty}(S,K)$ for every Cantor set K.

Let $\mathcal{M}_0^{\infty}(S,K)$ be the subgroup of $\mathcal{M}^{\infty}(S,K)$ consisting of elements whose representatives in $\mathrm{Diff}^{\infty}(S)$ are isotopic to the identity. Also let $\mathcal{PM}_0^{\infty}(S,K)$ be the subgroup of $\mathcal{M}_0^{\infty}(S,K)$ consisting of elements which fix K pointwise. Our first result is the following. Recall that a Cantor set is a totally disconnected topological space such that any point of K is an accumulation point.

Theorem 1.1. *Let S be a closed surface and K be a closed subset of S which is a Cantor set. The elements of $\mathcal{PM}_0^{\infty}(S,K)$ are undistorted in $\mathcal{M}_0^{\infty}(S,K)$.*

It is important to point out that any element in $\mathcal{PM}^{\infty}(S,K)$ can be thought as mapping class group of a surface of finite type (see Corollary 2.3) and is therefore much easier to deal with compared to other elements of $\mathcal{M}^{\infty}(S,K)$.

Theorem 1.1 is proven using some of the techniques developed by Franks and Handel in [9] to show that there are no distorted element in the groups of area-preserving diffeomorphisms of surfaces.

Denote by $\mathcal{M}^0(S,K)$ the quotient of the group of orientation-preserving homeomorphisms of S which preserve K by the subgroup consisting of homeomorphisms of S which are isotopic to the identity relative to K. It is worth pointing out that if the smoothness assumption is dropped and if K is a Cantor set, one can construct distorted elements in $\mathcal{M}^0(S,K)$. One can even construct elements which fix the set K pointwise and which are distorted in the subgroup $\mathcal{M}_0^0(S,K)$ of $\mathcal{M}^0(S,K)$ consisting of homeomorphisms isotopic to the identity in S. In particular, Theorem 1.2 below implies that the group $\mathcal{M}^0(S,K)$ is not isomorphic to $\mathcal{M}^{\infty}(S,K)$: the behavior of these groups is different from the behavior of classical mapping class groups (*i.e.* when K is a finite set).

In the smooth case, we have not been able so far to produce any distorted element for our groups $\mathcal{M}^{\infty}(S,K)$ (even when $S = \mathbb{S}^1$ is the circle and $K \subset \mathbb{S}^1$).

We will then focus on one of the simplest examples of Cantor sets in surfaces: the standard ternary Cantor set C_λ contained in an embedded segment $l \subset S$ (see Section 4 for a precise definition of C_λ). It is shown in [11] that the group $\mathfrak{Diff}_S(C_\lambda)$ consists of piecewise affine homeomorphisms of C_λ and therefore $\mathfrak{Diff}_S(K)$ is very"similar" to Thompson's group V_2, see Lemma 4.2.

Using the fact that there are no distorted elements in Thompson's groups V_n (see [1]) and the exact sequence (1), we are then able to show the following:

Theorem 1.2. *Let S be a closed surface and $0 < \lambda < 1/2$. Then, there are no distorted elements in the group $\mathcal{M}_0^\infty(S, C_\lambda)$, where C_λ is an embedding of the standard ternary Cantor set with parameter λ in S.*

1.2. **Tits alternative.** It is known that mapping class groups of finite type satisfy the Tits alternative, i.e., any subgroup $\Gamma \subset \mathcal{M}^\infty(S)$ either contains a free subgroup on two generators or is virtually solvable. In [18], Margulis proved that the group $Homeo(\mathbb{S}^1)$ of homeomorphisms of the circle satisfies a similar alternative. More precisely, he proved that a group $\Gamma \subset \mathrm{Homeo}(\mathbb{S}^1)$ either preserves a measure on $\mathbb{S}^1$ or contains a free subgroup on two generators, see [19].

Ghys asked whether the same statement holds for Diff(S) for a surface S (see [14]). We believe that some kind of similar statement should hold for our groups $\mathcal{M}_0^\infty(S,K)$. Here, we obtain the following result in the case where the Cantor set K is the standard ternary Cantor set C_λ.

Theorem 1.3. *Let Γ be a finitely generated subgroup of $\mathcal{M}^\infty(S, C_\alpha)$, then one of the following holds:*

1. *Γ contains a free subgroup on two generators.*
2. *Γ has a finite orbit, i.e. there exists $p \in C_\alpha$ such that the set $\Gamma(p) := \{g(p) \,|g \in \Gamma\}$ is finite.*

We will deduce the previous theorem as an immediate corollary of the following statement about Thompson's group V_n, which could be of independent interest:

Theorem 1.4. *For any finitely generated subgroup $\Gamma \subset V_n$, either Γ has a finite orbit or Γ contains a free subgroup.*

The proof of this result involves the study of the dynamics of elements of V_n on the Cantor set K_n where it acts naturally. These dynamics are known to be of contracting-repelling type and the spirit of our proof is similar to the proof of Margulis for $\mathrm{Homeo}(\mathbb{S}^1)$. The proof of this theorem uses the following lemma, which might be useful to prove similar results.

Lemma 1.5. *Let Γ be a countable group acting on a compact space K by homeomorphisms and a finite subset $F \subset K$. Then there is finite orbit of Γ on K or there is an element $g \in \Gamma$ sending F disjoint from itself (i.e. $g(F) \cap F = \emptyset$).*

The proof of the previous lemma is based on Horbez's recent proof of the Tits alternative for mapping class groups and related automorphisms groups (see [6] and [7]).

1.3. **Outline of the article.** In Section 2, we prove Theorem 1.1. In Section 3, we show that the group $\mathfrak{Diff}_S(K)$ is independent of the surface S where K is embedded. Then, we will focus on the study of the standard ternary Cantor set, and, in Section 4, we prove Theorem 1.2. Finally, in Section 5, we prove Theorems 1.3, 1.4 and Lemma 1.5. Section 5 is independent of Section 2.

2. Distorted elements in smooth mapping class groups

In this section, we prove Theorem 1.1, which we restate now:

Theorem 2.1. *Let S be a closed surface and K be a closed subset of S which is a Cantor set. The elements of $\mathcal{PM}_0^{\infty}(S,K)$ are undistorted in $\mathcal{M}_0^{\infty}(S,K)$.*

The proof and main ideas of Theorem 2.1 come from the work of Franks and Handel about distorted elements in surface diffeomorphism groups (see [9]).

The reason why the hypothesis $f|_K = \mathrm{Id}$ makes things more simple is the following observation.

Lemma 2.2 (Handel)**.** *Let S be a surface. Suppose f is a diffeomorphism fixing a compact set K pointwise. Suppose that K contains an accumulation point p. Then, there exists a neighborhood U of p and an isotopy f_t fixing K pointwise such that $f_0 = f$ and $f_1|_U = Id$.*

Proof. We consider the homotopy $h_t = tf + (1-t)\mathrm{Id}$ in a coordinate chart around p. Take a sequence of points $p_n \in K$ converging to p. As $f(p_n) = p_n$, there exists $v \in T_p(S)$ such that $D_pf(v) = v$. Observe that the equation $D_ph_t(w) = 0$ implies that $D_pf(w) = -\alpha w$, for some $\alpha > 0$. As $D_pf(v) = v$ and f is orientation preserving, this is not possible unless $w = 0$. Hence D_ph_t is invertible.

This implies that, for every t, Dh_t is invertible in a neighborhood U of p: there is a neighborhood U of p where h_t is invertible and is therefore an actual isotopy between the inclusion $i : U \to S$ and $f|_U$.

Now, using the Isotopy Extension Theorem (Lemma 5.8 in Milnor h-cobordism book) we can extend the isotopy $h_t|_U$ to an actual isotopy g_t of S such that $g_0 = \mathrm{Id}$, g_t fixes K pointwise and that coincides with h_t on U. Therefore the isotopy $f_t = g_t^{-1}f$ gives us the desired result. □

If the closed set K is perfect, we can take an appropriate finite cover of K by coordinate charts and use the previous lemma to prove that every element $f \in \mathcal{PM}^\infty(S,K)$ is isotopic to a diffeomorphism f' which is the identity in a small neighborhood of K. Hence it can be considered as an element of a mapping class group of a surface of finite type.

Corollary 2.3. *Let S be a surface. Suppose f is a diffeomorphism fixing a compact perfect set K pointwise. Then there exists a diffeomorphism g isotopic to f and a finite collection of smooth closed disjoint disks $\{D_i\}$ covering K such that $g|_{D_i} = Id$.*

As we will show next, this corollary implies that if f is distorted in $\mathcal{M}^\infty(S,K)$, then f must be isotopic to a composition of Dehn twists about disjoint closed curves β_i which do not meet K.

2.1. **Representatives of mapping classes.** We use the following theorem due to Franks and Handel (see Theorem 1.2, Definition 6.1 and Lemma 6.3 in [10]). This theorem is a consequence of Lemma 2.2 and of classical Nielsen-Thurston theory (see [13] on Nielsen-Thurston theory). Given a diffeomorphism f of the surface S, we denote by $\mathrm{Fix}(f)$ the set of points of S which are fixed under f.

Theorem 2.4 (Franks-Handel)**.** *Let f be a diffeomorphism in* $\mathrm{Diff}_0^\infty(S)$ *and $M = S - \mathrm{Fix}(f)$. There exists a finite set R of disjoint simple closed curves of M which are pairwise non-isotopic and a diffeomorphism φ of S which is isotopic to f relative to* $\mathrm{Fix}(f)$ *such that:*

(1) *For any curve γ in R, the homeomorphism φ preserves an open annulus neighbourhood A_γ of the curve γ.*
(2) *For any connected component S_i of $S - \cup_\gamma A_\gamma$,*
   (a) *if $\mathrm{Fix}(f) \cap S_i$ is infinite, then $\varphi_{|S_i} = Id_{|S_i}$.*
   (b) *if $\mathrm{Fix}(f) \cap S_i$ is finite, then either $\varphi_{|M_i} = Id_{|M_i}$ or $\varphi_{|M_i}$ is pseudo-Anosov, where $M_i = S_i - \mathrm{Fix}(f)$.*

We need more precisely the following corollary of this theorem.

Corollary 2.5. *Let ξ be an element in $\mathcal{M}_0^\infty(S,K)$ which fixes a closed set K pointwise. There exists a finite set R of disjoint simple closed curves of M which are pairwise non-isotopic and a diffeomorphism ψ of S which is a representative of ξ such that:*

(1) *For any curve γ in R, the diffeomorphism ψ preserves an open annulus neighbourhood A_γ of the curve γ.*
(2) *For any connected component S_i of $S - \cup_\gamma A_\gamma$,*
   (a) *if $K \cap S_i$ is infinite, then $\psi_{|S_i} = Id_{|S_i}$.*
   (b) *if $K \cap S_i$ is finite, then either $\psi_{|M_i} = Id_{|M_i}$ or $\psi_{|M_i}$ is pseudo-Anosov, where $M_i = S_i - K$.*

Proof. Apply Theorem 2.4 to a representative of ξ. This provides a diffeomorphism φ with the properties given by the theorem. When $K \cap S_i$ is infinite, the theorem states that $\varphi_{|S_i} = Id_{|S_i}$ and we take $\psi_{|S_i} = Id_{|S_i}$. If the set $K \cap S_i$ is finite, we can apply the classical Nielsen-Thurston theory (see Theorem 5 p.12 in [13]) to $\varphi_{|S_i}$ to obtain a decomposition of S_i and a diffeomorphism $\psi_{|S_i}$ whose restriction to each component of this decomposition satisfies (b). $\square$

Let ξ be an element in $\mathcal{M}_0^\infty(S,K)$ and denote by ψ its representative in the group $\mathrm{Diff}_0^\infty(S)$ given by the above corollary. Here, we distinguish two cases to prove Theorem 2.1.

(1) Either there exists a connected component S_i for which Case (b) occurs and $\psi_{|M_i}$ is pseudo-Anosov.
(2) Or this never occurs, *i.e.* the diffeomorphism ψ is a composition of Dehn twists about curves of R.

The first case is adressed in Subsection 2.3 whereas the second one is handled in Subsection 2.4. The following subsection is devoted to finding different obstructions to being distorted.

2.2. **Obstructions to distortion.** The obstructions we will provide here are analogous to those given in Franks and Handel's article [9]. However, as we are working in an isotopy invariant setting, Franks and Handel's invariants have to be changed a little to suit our needs.

2.2.1. *Length of curves.* Endow the surface S with a Riemannian metric g. For a loop α of S, we denote by $l_g(\alpha)$ its length with respect to the chosen Riemannian metric and, for an isotopy class of loops $[\alpha]$ of $S - K$, we denote by $l_g([\alpha])$ the infimum of the lengths of curves representing $[\alpha]$.

Lemma 2.6. *Let ξ be an element of $\mathcal{M}_0^{\infty}(S, K)$ and α be a smooth loop of $S - K$. Suppose that ξ is distorted in $\mathcal{M}_0^{\infty}(S, K)$. Then*

$$\lim_{n \to +\infty} \frac{\log(l_g(\xi^n([\alpha])))}{n} = 0.$$

Proof. By definition of a distorted element, there exists a finite set $\mathcal{G} \subset \mathcal{M}_0^{\infty}(S, K)$ and a sequence $(l_n)_n$ of integers such that

(1) for any n, $\xi^n = \eta_{1,n}\eta_{2,n}\eta_{3,n} \dots \eta_{l_n,n}$.
(2) $\lim_{n \to +\infty} \frac{l_n}{n} = 0$.

For any element η in $\mathcal{G}$, choose a representative g_η of this element in $\mathrm{Diff}_0^{\infty}(S)$. Also take a representative α of the isotopy class $[\alpha]$. Denote $M = \max_{\eta \in \mathcal{G}, x \in S} \|Dg_\eta(x)\|$ where $\|.\|$ is the norm associated to the chosen Riemannian metric on S. Then, for any n,

$$\begin{array}{rcl} l(\xi^n([\alpha])) & \leq & l(\eta_{1,n}\eta_{2,n}\eta_{3,n} \dots \eta_{l_n,n}(\alpha)) \\ & \leq & M^{l_n} l(\alpha). \end{array}$$

Hence $\frac{\log(l(\xi^n([\alpha])))}{n} \leq \frac{l_n \log(M) + \log(l(\alpha))}{n}$. This inequality yields the conclusion of the lemma. □

2.2.2. *Linear displacement.* Let S be a closed surface. Denote by $\tilde{S}$ its universal cover. We endow S with the spherical, a euclidean or a hyperbolic metric depending on whether S is a sphere, a torus or a higher genus surface. We denote by d the distance on $\tilde{S}$ which is induced by this metric. For any diffeomorphism f in $\mathrm{Diff}_0^{\infty}(S)$, we denote by $\tilde{f} : \tilde{S} \to \tilde{S}$ an identity lift of f, *i.e.* a lift which is obtained as time one of a lift starting from $Id_{\tilde{S}}$ of an isotopy between the identity I_S and f. Recall that such an isotopy lift is unique when the genus of S is greater than or equal to two.

Lemma 2.7. *Let S be a closed surface with $g(S) \geq 2$. Let ξ be an element of $\mathcal{M}_0^{\infty}(S, K)$, f be a representative of ξ in $\mathrm{Diff}_0^{\infty}(S)$ and $\tilde{f}$ be the identity lift of f to the universal cover $\mathbb{H}^2$ of S. Suppose that the element ξ is distorted in the group $\mathcal{M}_0^{\infty}(S, K)$. Then, for any point $\tilde{x}$ of $\mathbb{H}^2$ whose projection belongs to K, we have:*

$$\lim_{n \to +\infty} \frac{d(\tilde{f}^n(\tilde{x}), \tilde{x})}{n} = 0.$$

Proof. By definition of a distorted element, there exist elements $\eta_1, \eta_2, \dots, \eta_k$ in $\mathcal{M}_0^{\infty}(S,K)$ such that ξ belongs to the group G generated by these elements and such that, for any integer n, there exist indices $1 \leq i_{1,n}, \dots, i_{l_n,n} \leq k$ with the following properties.

(1) $\xi^n = \eta_{i_{1,n}} \dots \eta_{i_{l_n,n}}$.
(2) $\lim_{n\to+\infty} \frac{l_n}{n} = 0$.

For any i, fix a representative g_i of η_i in $\mathrm{Diff}_0^{\infty}(S)$. For any n, there exists a diffeomorphism h_n in $\mathrm{Diff}_0^{\infty}(S,K)$ such that

$$f^n = g_{i_{1,n}} \dots g_{i_{l_n,n}} h_n.$$

The diffeomorphism $\tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}} \tilde{h}_n$ is the identity lift of f^n. Hence $\tilde{f}^n = \tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}} \tilde{h}_n$. Observe that, as h_n is isotopic to the identity relative to K, $\tilde{h}_n(\tilde{x}) = \tilde{x}$. Therefore, for any n,

$$\tilde{f}^n(\tilde{x}) = \tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}}(\tilde{x})$$

and

$$d(\tilde{f}^n(\tilde{x}), \tilde{x}) \leq \sum_{k=1}^{l_n} d(\tilde{g}_{i_k} \dots \tilde{g}_{i_{l_n}}(\tilde{x}), \tilde{g}_{i_{k+1}} \dots \tilde{g}_{i_{l_n}}(\tilde{x})) \leq l_n M,$$

where $M = \max_{1\leq i\leq k,\ \tilde{y}\in\mathbb{H}^2} d(\tilde{g}_i(\tilde{y}), \tilde{y})$ (observe that the quantities $d(\tilde{g}_i(\tilde{y}), \tilde{y})$ are bounded on $\mathbb{H}^2$ as the diffeomorphisms $\tilde{g}_i$ commute with the deck transformations). As $\lim_{n\to+\infty} \frac{l_n}{n} = 0$, the lemma follows. □

Lemma 2.8. *Let S be a closed surface with $g(S) \geq 1$. Let $\xi \in \mathcal{M}_0^{\infty}(S,K)$. Denote by f a representative of ξ in $\mathrm{Diff}_0^{\infty}(S)$. Suppose that the element ξ is distorted in $\mathcal{M}_0^{\infty}(S,K)$. Then, for any two points $\tilde{x}$ and $\tilde{y}$ of $\tilde{S}$ whose projections belong to K,*

$$\lim_{n\to+\infty} \frac{d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y}))}{n} = 0.$$

Proof. As $d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) \leq d(\tilde{f}^n(\tilde{x}), \tilde{x}) + d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{f}^n(\tilde{y}))$, this lemma is a consequence of Lemma 2.7 if $g(S) \geq 2$.

Suppose now that S is the 2-torus. We use the same notation as in the proof of Lemma 2.7, meaning that there exist elements $\eta_1, \eta_2, \dots, \eta_k$ in $\mathcal{M}_0^{\infty}(S,K)$ such that ξ belongs to the group G generated by these elements and such that, for any integer n,

(1) $\xi^n = \eta_{i_{1,n}} \dots \eta_{i_{l_n,n}}$.
(2) $\lim_{n\to+\infty} \frac{l_n}{n} = 0$.

As before, for any i, fix a representative g_i of η_i in $\mathrm{Diff}_0^\infty(S)$. For any n, there exists a diffeomorphism h_n in $\mathrm{Diff}_0^\infty(S,K)$ such that

$$f^n = g_{i_{1,n}} \dots g_{i_{l_n,n}} h_n.$$

Now, there is a little difference as $\tilde{f}^n$ is not necessarily equal to $\tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}} \tilde{h}_n$. Instead, we can just say that, for any n, there exists an integral translation T_n of the universal cover $\mathbb{R}^2$ of the torus $\mathbb{R}^2/\mathbb{Z}^2$ such that

$$\tilde{f}^n = T_n \tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}} \tilde{h}_n.$$

Therefore, for any points $\tilde{x}$ and $\tilde{y}$ in $\mathbb{R}^2$,

$$\begin{array}{rcl} d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) & = & d(\tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}}(\tilde{x}), \tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}}(\tilde{y})) \\ & \leq & d(\tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}}(\tilde{x}), \tilde{x}) + d(\tilde{x}, \tilde{y}) + d(\tilde{g}_{i_{1,n}} \dots \tilde{g}_{i_{l_n,n}}(\tilde{y}), \tilde{y}) \\ & \leq & 2 l_n M + d(\tilde{x}, \tilde{y}), \end{array}$$

where $M = \max_{1 \leq i \leq k,\ \tilde{y} \in \mathbb{R}^2} d(\tilde{g}_i(\tilde{y}), \tilde{y})$. Hence the left-hand side of this inequality divided by n tends to 0 as n tends to $+\infty$. □

2.2.3. *Spread.* We now introduce the concept of spread.

Let γ be a smooth curve with endpoints p, q (smooth at the endpoints) and β be a simple closed curve on S separating p and q. For any curve, the spread $L_{\beta,\gamma}(\alpha)$ is going to measure how many times α rotates around β with respect to γ.

More formally $L_{\beta,\gamma}(\alpha)$ is defined as the maximal number k, such that there exist subarcs $\alpha_0 \subset \alpha$, $\gamma_0 \subset \gamma$ such that $\overline{\gamma_0\alpha_0}$ is a closed curve isotopic to β^k in $S \setminus \{p,q\}$.

Example 2.9. *In the example depicted in Figure 1, we have a thrice-punctured sphere S together with the curves α, β and γ. In this case we have that $L_{\beta,\gamma}(\alpha) = 2$.*

Let us denote by $\mathcal{C}_{S,K}$ the set of simple smooth curves $[0,1] \to S$ whose endpoints belong to K and whose interior is contained in $S - K$. Two such curves are said to be isotopic if there exists a diffeomorphism in $\mathrm{Diff}_0^\infty(S,K)$ which sends one of these curves to the other one.

Take an isotopy class $[\alpha]$ of curves in $\mathcal{C}_{S,K}$. We denote by $\overline{\mathcal{C}}_{S,K}$ the set of isotopy classes of curves in $\mathcal{C}_{S,K}$. We define $\overline{L}_{\beta,\gamma}([\alpha])$ as the infimum of $L_{\beta,\gamma}(\alpha)$ over all the representatives α of the class $[\alpha]$.

The lemma below is stronger than Lemma 6.8 in [9] but it follows from the proof of Lemma 6.8. Notice that we state it only in the case where the curve β bounds a disk as we believe that the proof given

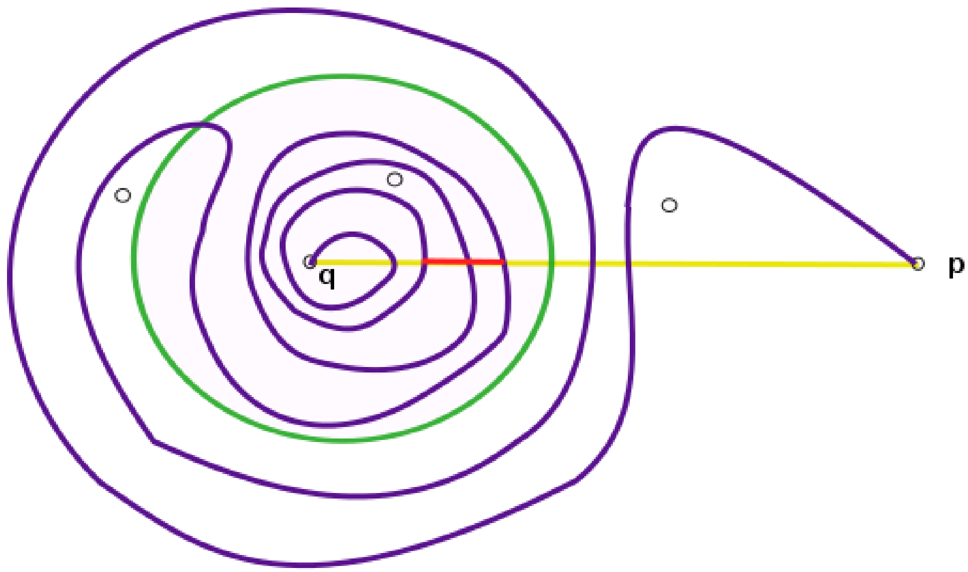

FIGURE 1. α(purple), β(green), γ(yellow), γ_0(red)

in [9] covers only this case (which is sufficient for the purposes of the article [9]).

Lemma 2.10 (Franks-Handel)**.** *Let $\mathcal{G} = \{g_i,\ 1 \leq i \leq k\}$ be a finite set of elements of $\mathrm{Diff}_0(S)$. Then there exists a constant $C > 0$ depending only on $\mathcal{G}$ such that the following property holds. Let f be any diffeomorphism which fixes K pointwise and which belongs to the group generated by the g_i's. Let α, γ be curves in $\mathcal{C}_{S,K}$. Finally, let β be a closed simple loop contained in $S - K$ which bounds a disk in the surface S and such that the two endpoints of γ do not belong to the same connected component of $S - \beta$. Then*

$$L_{\beta,\gamma}(f(\alpha)) \leq L_{\beta,\gamma}(\alpha) + Cl_{\mathcal{G}}(f).$$

Corollary 2.11. *Let $\overline{\mathcal{G}} = \{\xi_i,\ 1 \leq i \leq k\}$ be a finite set of elements in $\mathcal{M}_0^\infty(S,K)$. Let η be an element of the group generated by the ξ_i's which fixes K pointwise. Then there exists a constant $C > 0$ such that, for any $n > 0$, $[\alpha]$ in $\overline{\mathcal{C}}_{S,K}$, β closed essential simple loop and γ in $\mathcal{C}_{S,K}$ which crosses β in only one point, $\overline{L}_{\beta,\gamma}(\eta^n([\alpha])) \leq \overline{L}_{\beta,\gamma}([\alpha]) + Cl_{\overline{\mathcal{G}}}(\eta^n)$.*

Proof. Let $l_n = l_{\overline{\mathcal{G}}}(\eta^n)$. Take a representative f in $\mathrm{Diff}_0^\infty(S)$ of η and, for each i, choose a representative g_i of ξ_i. For any curve α representing a class $[\alpha]$ in $\mathcal{C}_{S,K}$, the curve $f^n(\alpha)$ represents the class $\eta^n([\alpha])$. Additionally, by hypothesis, we can write $f^n = g_{i_1} g_{i_2} \dots g_{i_{l_n}} h'$, where $1 \leq i_j \leq k$ and h' is a diffeomorphism of S isotopic to the identity relative to K. Franks and Handel's lemma implies that there exists a constant $C>0$ independent of n, α, β and γ such that

$$L_{\beta,\gamma}(f^n(\alpha)) \leq L_{\beta,\gamma}(h'(\alpha)) + Cl_n.$$

Hence

$$\overline{L}_{\beta,\gamma}(\eta^n([\alpha])) \leq L_{\beta,\gamma}(h'(\alpha)) + Cl_n.$$

As α is any curve in the isotopy class of $[\alpha]$,

$$\overline{L}_{\beta,\gamma}(\eta^n([\alpha])) \leq \overline{L}_{\beta,\gamma}([\alpha]) + Cl_n.$$

□

The above corollary immediately yields the result below.

Corollary 2.12. *Let η be a distorted element in $\mathcal{PM}_0^{\infty}(S,K)$. Then, for any $[\alpha]$ in $\overline{\mathcal{C}}_{S,K}$, γ in $\mathcal{C}_{S,K}$ and β closed simple loop which separates the endpoints of γ and which bounds a disk,*

$$\lim_{n\to+\infty} \frac{\overline{L}_{\beta,\gamma}(\eta^n([\alpha]))}{n} = 0.$$

We are now ready to start the proof of Theorem 2.1.

2.3. **The pseudo-Anosov case.** Suppose that there exists a connected component S_i for which Case (b) in Corollary 2.5 occurs. In this case, it is a classical fact that there exists an isotopy class of essential loop $[\alpha]$ such that $\lim_n \frac{\log(l(\xi^n([\alpha])))}{n} > 0$ (see Proposition 19 p.178 in [13]). Hence, by Lemma 2.6, the element ξ is undistorted in $\mathcal{M}_0^{\infty}(S,K)$.

2.4. **The Dehn twist case.** The following proposition completes the proof of Theorem 2.1.

Proposition 2.13. *Let ξ be an element of $\mathcal{M}_0^{\infty}(S,K)$. Suppose that ξ fixes K pointwise and is equal to a finite composition of Dehn twists about disjoint simple loops of $S-K$ which are pairwise non-isotopic relative to K and which are not homotopic to a point relative to K. Then ξ is not distorted in $\mathcal{M}_0^{\infty}(S,K)$.*

Proof. Denote by f a representative of ξ in $\mathrm{Diff}_0^{\infty}(S)$ which is equal to the identity outside small tubular neighbourhoods of the loops appearing in the decomposition of ξ. Also denote by $\mathcal{C}(f)$ the set of loops appearing in this decomposition. We will distinguish two cases: the case where one of these curves is essential in S (*i.e.* does not bound a disk in S) and the case where none is essential.

First case. Assume that at least one of the curves of $\mathcal{C}(f)$ is essential in S. Observe that the surface S is necessarily different from the sphere in this case. We denote by $\mathcal{C}_e(f)$ the subset of $\mathcal{C}(f)$ consisting of curves which are essential in S.

In the case where the surface S has genus greater than or equal to 2, we call *good ray* a one-to-one map $\alpha : [0,+\infty) \to S$ with a lift $\tilde{\alpha} : [0,+\infty) \to \tilde{S}$ such that $\tilde{\alpha}(t)$ converges to a point of the circle at infinity when $t \to +\infty$. Observe that, in this case, any lift of α will converge to a point of the boundary at infinity.

Lemma 2.14. *One of the two following possibilities occur:*

1. *There exist two points x_1 and x_2 in K and a simple curve α joining x_1 and x_2 which meets the union of the curves in $\mathcal{C}_e(f)$ in only one point. Moreover, the intersection is transverse.*
2. *The surface S is different from the torus and there exists a good ray $\alpha : [0,+\infty) \to S$ starting from a point x_1 in K which meets the union of curves in $\mathcal{C}_e(f)$ in only one point. Moreover, the intersection is transverse.*

Proof. Denote by $\mathcal{S}$ the set of connected components of the complement in S of the union of curves in $\mathcal{C}_e(f)$. Two such components are said to be adjacent if their closures share a curve of $\mathcal{C}_e(f)$ in common (or equivalently if the intersection of their closures is nonempty). If two adjacent components in $\mathcal{S}$ contain points of K or if there is only one component in $\mathcal{S}$, the first possibility occurs.

Hence suppose that there are at least two elements in $\mathcal{S}$ and that, for any two adjacent components in $\mathcal{S}$, one of them does not contain any point of K. Fix a component U_0 in $\mathcal{S}$ which contains at least a point of K and a component $U_1 \neq U_0$ which is adjacent to U_0. The component U_1 has to be different from an annulus: otherwise two distinct curves of $\mathcal{C}_e(f)$ would be isotopic. Hence the surface S is different from the torus. To conclude the proof, take any good ray $\alpha : [0,+\infty) \to S$ starting from a point of $K \cap U_0$, meeting the union of curves in $\mathcal{C}_e(f)$ in only one point and such that the image under α of a neighbourhood of $+\infty$ is contained in U_1. □

First subcase. Item (1) in Lemma 2.14 occurs. Denote by β the curve in Lemma 2.14 of $\mathcal{C}_e(f)$ which is met by α. Take a lift $\tilde{\beta}$ of the curve β to the universal cover $\tilde{S}$ of the surface S. Denote by $T : \tilde{S} \to \tilde{S}$ the deck transformation corresponding to $\tilde{\beta}$. Finally, take a lift $\tilde{\alpha}$ of α which meets $\tilde{\beta}$ and denote by $\tilde{x}_1$ and $\tilde{x}_2$ its endpoints.

Assume first that the surface S has genus greater than or equal to 2 and that there exists a non-trivial deck transformation γ of $\tilde{S}$ such that $\tilde{f}(\tilde{x}_1) = \gamma(\tilde{x}_1)$. In this case, for any integer $n \geq 0$, $\tilde{f}^n(\tilde{x}_1) = \gamma^n(\tilde{x}_1)$, as the homeomorphism $\tilde{f}$ commutes with deck transformations. Hence, by Lemma 2.7, the element ξ is undistorted in $\mathcal{M}_0^{\infty}(S,K)$.

Suppose now that the homeomorphism $\tilde{f}$ fixes the point $\tilde{x}_1$ (in the case of the torus, we can choose an identity lift $\tilde{f}$ which satisfies this property). Then, by definition of a Dehn twist, there exists $k \neq 0$ such that, for any n,

$$\tilde{f}^n(\tilde{x}_2) = T^{kn}(\tilde{x}_2).$$

See Figure 2 for an illustration. Hence the element ξ is undistorted in $\mathcal{M}_0^{\infty}(S,K)$ by Lemma 2.8.

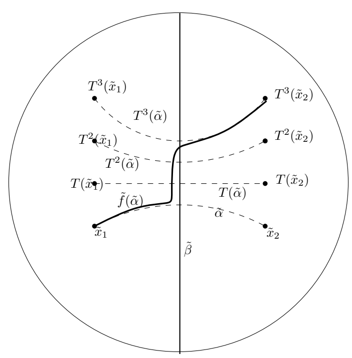

FIGURE 2. $\tilde{f}(\tilde{\alpha})$ in the case $k = 3$ (First subcase)

<u>Second subcase.</u> Let us suppose that item (2) in Lemma 2.14 occurs. Denote by β the loop in $\mathcal{C}_e(f)$ which meets α and fix a lift $\tilde{\beta} : \mathbb{R} \to \mathbb{H}^2$ of β. Denote by $T : \mathbb{H}^2 \to \mathbb{H}^2$ the deck transformation corresponding to $\tilde{\beta}$. Let $\tilde{\alpha} : [0, +\infty) \to \mathbb{H}^2$ be a lift of α which meets $\tilde{\beta}$ and p be the point in the boundary at infinity of $\mathbb{H}^2$ defined by $\tilde{\alpha}$: if we see $\mathbb{H}^2$ as the Poincaré disk, the unit disk in the plane, then $p = \lim_{t \to +\infty} \tilde{\alpha}(t)$. As the homeomorphism f is isotopic to the identity, the extension of the homeomorphism $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ to the disk at infinity fixes the point p. Then, by definition of a Dehn twist, there exists $k \neq 0$ such that, for any n, $\tilde{f}^n(\tilde{\alpha}(0)) = T^{kn}(\tilde{\alpha}(0))$ (see Figure 3). By Lemma 2.7, the element ξ is undistorted in $\mathcal{M}_0^{\infty}(S,K)$.

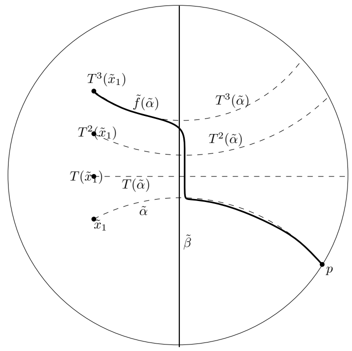

FIGURE 3. $\tilde{f}(\tilde{\alpha})$ in the case $k = 3$ (Second subcase)

Second case. Let us suppose that any curve in $\mathcal{C}(f)$ bounds a disk in S.

First subcase: Suppose that at least two connected components of the complement of the curves in $\mathcal{C}(f)$ contain points in K.

Lemma 2.15. *There exists a simple smooth curve $\alpha : [0, 1] \to S$ such that its endpoints $\alpha(0) = x_1$ and $\alpha(1) = x_2$ belong to the Cantor set K and with one of the following properties.*

(1) *The curve α meets exactly one loop β of $\mathcal{C}(f)$ transversely in only one point. Moreover, this loop β is isotopic to a small loop around x_1 relative to $\{x_1, x_2\}$.*
(2) *The curve α meets exactly two loops β and γ of $\mathcal{C}(f)$ transversally and in one point. Moreover, the loop β is isotopic to a small loop around x_1 relative to $\{x_1, x_2\}$ and the loop γ is isotopic to a small loop around x_2 relative to $\{x_1, x_2\}$.*

Proof. In order to carry out this proof, we have to introduce some notation. Suppose that the surface S is different from the sphere. Then, for any loop γ in $\mathcal{C}(f)$, there exists a unique connected component of $S - \gamma$ which is homeomorphic to an open disk. We call this connected component the interior of γ. In the case where S is a sphere, we fix a point on this surface which does not belong to any curve of $\mathcal{C}(f)$. Then, for any loop γ of $\mathcal{C}(f)$, we call interior of γ the connected component of $S - \gamma$ which does not contain this point.

To each curve γ in $\mathcal{C}(f)$, we will associate a number $l(\gamma) \in \mathbb{N}$ which we call its level. For any curve γ of $\mathcal{C}(f)$ which is not contained in the interior of a loop of $\mathcal{C}(f)$, we set $l(\gamma) = 0$. Denote by $\mathcal{C}_i(f)$ the set of curves of $\mathcal{C}(f)$ whose level is i. The following statement defines inductively the level of any curve in $\mathcal{C}(f)$: for any $i \geq 1$, a curve γ in $\mathcal{C}(f) - \mathcal{C}_{i-1}(f)$ satisfies $l(\gamma) = i$ if it is not contained in the interior of a curve in $\mathcal{C}(f) - \mathcal{C}_{i-1}(f)$. See Figure 4 for an example. The represented loops are the elements of $\mathcal{C}(f)$ in this example and the numbers beside them are their levels.

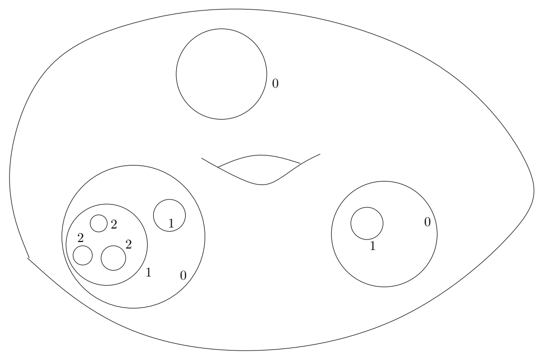

FIGURE 4. Level of curves of $\mathcal{C}(f)$.

Now, we are ready for the proof. Take a loop β in $\mathcal{C}(f)$ whose level is maximal. As the level of this curve is maximal, its interior does not contain any curve of $\mathcal{C}(f)$. However, its interior has to contain a point x_1 of K: otherwise, the loop β would be homotopically trivial.

If $l(\beta) = 0$, denote by U the complement of the interiors of the curves of level 0 in $\mathcal{C}(f)$. Otherwise, let U be the complement of the interiors of the curves of level $l(\beta)$ in the interior of δ, where δ is the unique loop of level $l(\beta) - 1$ whose interior contains β.

If the open set U contains some point x_2 of K, it is not difficult to find a curve α which satisfies the first property of the lemma. Otherwise, there exists a loop γ of level $l(\beta)$ which is different from β. If $l(\beta) = 0$, this is a consequence of the hypothesis made for the first subcase. If $l(\beta) \neq 0$, we can further require that this loop is contained in the interior of δ: if this was not the case, β would be isotopic to δ, a contradiction. In this case, take a path α going from x_1 to U crossing β once and from U to a point x_2 of K contained in the interior of γ crossing γ once. This path satisfies the second property of the lemma. $\square$

In this subcase, we will prove Proposition 2.13 only in the case where the surface S is different from the sphere. The case of the sphere is similar and is left to the reader. Denote by S' the surface obtained from S by blowing up the points x_1 and x_2 (replacing these points with circles). Denote by $C_1 \subset S'$ the circle which projects to the point x_1 in S and by $C_2 \subset S'$ the circle which projects to the point x_2. Denote by $\tilde{S}'$ the universal cover of S'. The space $\tilde{S}'$ can be seen as a subset of the universal cover $\mathbb{H}^2$ of the double of S', which is the closed surface obtained by gluing S' with itself by identifying their boundaries. The curve α lifts to a smooth curve α' of S' whose endpoints are $x_1' \in C_1$ and $x_2' \in C_2$. By abuse of notation, we denote by β the lift of the curve β to S'.

As the diffeomorphism f fixes both points x_1 an x_2, it induces a smooth diffeomorphism f' of S', acting by the differential of f on the circles C_1 and C_2. Observe that the diffeomorphism f' preserves C_1 and C_2 and is not a priori isotopic to the identity. As the compact set K is perfect, the point x_1 is accumulated by points of K which are fixed under f. Denote by z a point of C_1 which corresponds to a direction in $T_{x_1}(S)$ accumulated by points of K. Then the diffeomorphism f' fixes the point z.

Denote by $\tilde{C}_1$ a lift of C_1, *i.e.* a connected component of $p^{-1}(C_1)$, where $p : \tilde{S}' \to S'$ is the projection, and by $\tilde{z}$ a lift of the point z contained in $\tilde{C}_1$. Denote by $\tilde{\alpha}'$ a lift of α' which meets $\tilde{C}_1$ and by $\tilde{\beta}$ a lift of β which meets $\tilde{\alpha}'$. The curve $\tilde{\alpha}'$ joins the points $\tilde{x}_1' \in \tilde{C}_1$ and $\tilde{x}_2' \in \tilde{C}_2$. Denote by $\tilde{f}' : \tilde{S}' \to \tilde{S}'$ the lift of f' which fixes the point $\tilde{z}$. This lift $\tilde{f}'$ preserves $\tilde{C}_1$. Denote by T the deck transformation corresponding to $\tilde{\beta}$ (or equivalently $\tilde{C}_1$). Then the diffeomorphism $\tilde{f}'$ fixes all the points of the orbit of $\tilde{z}$ under T. Hence the orbit of $\tilde{x}_1'$ under $\tilde{f}'$ lies on $\tilde{C}_1$ between two consecutive points of the orbit of $\tilde{z}$ under T. Up to translating $\tilde{z}$, we can suppose that these points are $\tilde{z}$ and $T(\tilde{z})$.

First suppose that the only loop in $\mathcal{C}(f)$ met by the curve α is β. In this case, by definition of a Dehn twist, there exists $k \neq 0$ such that, for any integer n, $\tilde{f}'^n(\tilde{x}_2') \in T^{kn}(\tilde{C}_2)$ (see Figure 5). For any $1 \leq j \leq kn-1$, the curve $T^j(\tilde{\alpha}')$ separates $\tilde{S}'$ into two connected components. Observe that the point $\tilde{f}^n(\tilde{x}_1')$ belongs to one of them and that the component $T^{kn}(C_2)$ belongs to the other one. Hence the curve $\tilde{f}'^n(\tilde{\alpha}')$ has to meet each of the curves $T^j(\tilde{\alpha}')$, for $1 \leq j \leq kn-1$. Denote by $\tilde{y}_1$ the first intersection point of the curve $\tilde{f}'^n(\tilde{\alpha})$ with the curve $T(\tilde{\alpha}')$ and by $\tilde{y}_2$ the last intersection point of $\tilde{f}'^n(\tilde{\alpha}')$ with the curve $T^{kn-1}(\tilde{\alpha}')$.

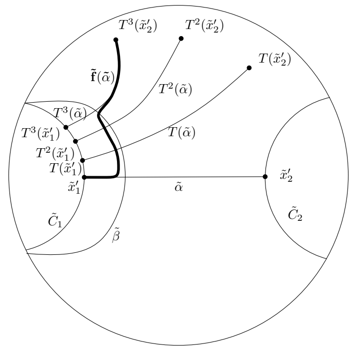

FIGURE 5. $\tilde{f}(\tilde{\alpha})$ in the case $k=3$

As the curve $T(\tilde{\alpha}')$ belongs to the same component of the complement of $T^{kn-1}(\tilde{\alpha}')$ as the point $\tilde{x}'_1$, the curve $\tilde{\alpha}'$ meets the point $\tilde{y}_1$ before meeting the point $\tilde{y}_2$. Consider the curve $\tilde{\gamma}$ which is the concatenation of the segment of $\tilde{f}'^n(\tilde{\alpha}')$ between $\tilde{y}_1$ and $\tilde{y}_2$ and the segment of the curve $T^{kn-1}(\tilde{\alpha}')$ between $\tilde{y}_2$ and $T^{kn}(\tilde{y}_1)$ The projection on S of $\tilde{\gamma}$ is isotopic to β^{kn-2} relative to x_1 and x_2. Hence $L_{\beta,\alpha}(f^n(\alpha)) \geq kn-2$.

In the above proof, we just used the fact that the curve $\tilde{\alpha}'$ joined a point of the boundary component $\tilde{C}_1$ which is between $\tilde{z}$ and $T(\tilde{z})$ to the boundary component $\tilde{C}_2$. We will prove the following claim.

Claim: *Any curve δ of $\mathcal{C}_{S,K}$ which represents the same class as α in $\overline{\mathcal{C}}_{S,K}$ has a lift $\tilde{\delta}'$ to $\tilde{S}'$ with the following property. The curve $\tilde{\delta}'$ joins a point of the boundary component $\tilde{C}_1$ which is between $\tilde{z}$ and $T(\tilde{z})$ to the boundary component $\tilde{C}_2$.*

Hence, for any curve δ isotopic to α relative to K, we obtain that, for any n, $L_{\beta,\alpha}(f^n(\delta)) \geq kn-2$ and $\overline{L}_{\beta,\alpha}(\xi^n([\alpha])) \geq kn-2$. By Lemma 2.12, the element ξ is undistorted in the group $\mathcal{M}_0^{\infty}(S,K)$.

Proof of the claim. By definition, there exists a continuous path $(g_t)_{t\in[0,1]}$ of diffeomorphisms in $\mathrm{Diff}_0^{\infty}(S,K)$ such that $g_0 = Id_S$ and $g_1(\alpha) = \delta$. This path of diffeomorphisms lifts (by acting by the differential on C_1 and C_2) to a path of diffeomorphisms $(g'_t)_{t\in[0,1]}$ of S' which in turns lifts to a path $(\tilde{g}'_t)_{t\in[0,1]}$ of diffeomorphisms such that $\tilde{g}'_0 = Id_{\tilde{S}'}$.

As the direction in S corresponding to $z \in C_1$ is accumulated by points of K, we have that for any t, $g'_t(z) = z$, then, for any t, $\tilde{g}'_t(\tilde{z}) = \tilde{z}$ and $\tilde{g}'_t(T(\tilde{z})) = T(\tilde{z})$. Moreover, for any t, the diffeomorphism $\tilde{g}'_t$ preserves $\tilde{C}_1$ and $\tilde{C}_2$. Therefore, for any t, the point $\tilde{g}'_t(\tilde{\alpha}'(0))$ lies between the points $\tilde{z}$ and $T(\tilde{z})$ and the point $\tilde{g}'_t(\tilde{\alpha}'(1))$ belongs to $\tilde{C}_2$. In particular, the curve $\tilde{g}'_1(\tilde{\alpha}')$ is a lift of δ which satisfies the required properties. □

Now, suppose that the curve α meets a loop of $\mathcal{C}(f)$ which is different from β, *i.e.* the second case in Lemma 2.15 occurs. Denote by $\tilde{\gamma}$ a lift of γ to $\tilde{S}'$. Observe that there exists $k \neq 0$ such that, for any n, the curve $\tilde{f}'^n(\tilde{\alpha}')$ meets $T^{kn}(\tilde{\gamma})$. As the curves $T^j(\tilde{\alpha})$ separate the point $\tilde{x}_1$ from the curve $T^{kn}(\tilde{\gamma})$ for any $1 \leq j \leq kn-1$, this case can be handled in the same way as the case where the curve α meets only one loop in $\mathcal{C}(f)$.

<u>Second subcase:</u> Only one component of the complement of the curves of $\mathcal{C}(f)$ meets K. The idea in this subcase is to lift our element to a two-fold cover where we can apply the first subcase. We need the following lemma. Let $p : \hat{S} \to S$ be a twofold cover of S. Let $\hat{K} = p^{-1}(K)$. Observe that any element ξ in $\mathcal{M}^{\infty}_0(S,K)$ can be lifted to an element $\hat{\xi}$ in $\mathcal{M}^{\infty}_0(\hat{S},\hat{K})$ (such a lift might not be unique).

Lemma 2.16. *Let ξ be an element in $\mathcal{PM}^{\infty}_0(S,K)$ which is distorted in this group. Then the lift $\hat{\xi}$ of ξ is distorted in $\mathcal{M}^{\infty}_0(\hat{S},\hat{K})$.*

Before proving Lemma 2.16, let's see how the lemma completes the proof of our proposition for this subcase. Observe that if $\hat{\xi}$ were distorted, then $\hat{\xi}^2$ would also be distorted and $\hat{\xi}^2$ also fixes the cantor set $\hat{K}$. Moreover, the element $\hat{\xi}^2$ is a composition of Dehn twists about disjoint loops and at least two connected components of the complement of these curves contain points of $\hat{K}$. This reduces the second subcase to the first subcase and so we are done.

Proof. (Lemma 2.16) By definition of distorted elements, there exists a finite subset $\mathcal{G} = \{s_1, \dots, s_k\}$ of $\mathcal{M}^{\infty}_0(S,K)$ such that the element ξ belongs to the group generated by this subset and, for any n, there exists indices $1 \leq i_1, \dots, i_{l_n} \leq k$ with

$$\xi^n = s_{i_1} \dots s_{i_{l_n}}$$

and

$$\lim_{n \to +\infty} \frac{l_n}{n} = 0$$

For any element s_i of $\mathcal{G}$, choose a lift $\hat{s}_i$ in $\mathcal{M}^{\infty}_0(\hat{S},\hat{K})$. Let G be the (finite) group of deck transformations isotopic to the identity of the covering $\hat{S} \to S$, seen as a subgroup of $\mathcal{M}^\infty(\hat{S},\hat{K})$ and let $\hat{\mathcal{G}} = \mathcal{G} \cup G$. Then, for any n, there exists $T_n \in G$ such that

$$\hat{\xi}^n = T_n \hat{s}_{i_1} \dots \hat{s}_{i_{l_n}}$$

Hence

$$l_{\hat{\mathcal{G}}}(\hat{\xi}^n) \leq l_n + 1$$

and the element $\hat{\xi}$ is distorted in $\mathcal{M}^\infty(\hat{S},\hat{K})$. □

□

3. Independence of the surface

Let S be a closed surface and K be any closed subset of S. Recall that $\mathfrak{diff}_S(K) = \mathcal{M}^\infty(S,K)/\mathcal{PM}^\infty(S,K)$. This group can also be seen as the quotient of the group $\mathrm{Diff}^\infty(S,K)$ consisting of diffeomorphisms that preserve K by the subgroup of $\mathrm{Diff}^\infty(S,K)$ consisting of elements which fix K point-wise.

In what follows, we call disk the image of the unit closed disk $\mathbb{D}^2$ under an embedding $\mathbb{D}^2 \hookrightarrow S$.

Suppose that the closed set K is contained in the interior of a disk D. The goal of this section is to prove the following proposition.

Proposition 3.1. *Any element of $\mathfrak{diff}_S(K)$ has a representative in $\mathrm{Diff}^\infty_0(S)$ which is supported in D.*

The proof of this proposition also implies that $\mathfrak{diff}_S(K)$ is isomorphic to $\mathcal{M}^\infty_0(S,K)/\mathcal{PM}^\infty_0(S,K)$.

The following corollary implies that, to prove something about groups of the form $\mathfrak{diff}_S(K)$, it suffices to prove it in the case where the surface S is the sphere.

Corollary 3.2. *Let S an S' be surfaces and $K \subset S$ and $K' \subset S'$ be closed subsets. Suppose that there exist disks $D_S \subset S$ and $D_{S'} \subset S'$ as well as a diffeomorphism $\varphi : D_S \to D_{S'}$ such that:*

1. *The closed set K is contained in the interior of the disk D_S.*
2. *The closed set K' is contained in the interior of the disk $D_{S'}$.*
3. *$\varphi(K) = K'$.*

Then, the group $\mathfrak{diff}_S(K)$ is isomorphic to $\mathfrak{diff}_{S'}(K')$.

Proof. Let $\psi : \mathbb{D}^2 \to D_S$ be a diffeomorphism. This diffeomorphism induces a morphism

$$\begin{array}{rrcl} \Psi : & \mathfrak{Diff}_{\mathbb{D}^2}(\psi^{-1}(K)) & \to & \mathfrak{Diff}_S(K) \\ & \xi & \mapsto & \psi\xi\psi^{-1} \end{array}.$$

By Proposition 3.1, this morphism is onto. Let us prove that it is into. Let ξ and ξ' be elements of $\mathfrak{Diff}_{\mathbb{D}^2}(\psi^{-1}(K))$ such that $\Psi(\xi) = \Psi(\xi')$. Take representatives f and f' of ξ and ξ' in $\mathrm{Diff}_0^{\infty}(\mathbb{D}^2)$. Then there exists a diffeomorphism g in $\mathrm{Diff}_0^{\infty}(S)$ which fixes K pointwise such that $\psi f\psi^{-1} = g\psi f'\psi^{-1}$. Then the support of g, which is also the support of $\psi f f'^{-1}\psi^{-1}$, is contained in D_S. Hence, there exists a diffeomorphism g' in $\mathrm{Diff}_0^{\infty}(\mathbb{D}^2)$ which fixes $\psi^{-1}(K)$ pointwise such that $g = \psi g'\psi^{-1}$. Therefore, $f = g'f'$ and $\xi = \xi'$.

For the same reason, the map

$$\begin{array}{rrcl} \Psi' : & \mathfrak{Diff}_{\mathbb{D}^2}(\psi^{-1}(K)) & \to & \mathfrak{Diff}_{S'}(K') \\ & \xi & \mapsto & \varphi\psi\xi\psi^{-1}\varphi^{-1} \end{array}$$

is an isomorphism. Hence the map $\Psi'\Psi^{-1}$ is an isomorphism between $\mathfrak{Diff}_S(K)$ and $\mathfrak{Diff}_{S'}(K')$. □

To prove Proposition 3.1, we need the following lemma (see [17] Theorem 3.1 p.185).

Lemma 3.3. *Let Σ be a surface and $e_1, e_2 : \mathbb{D}^2 \to \Sigma$ be orientation preserving smooth embeddings such that $e_1(\mathbb{D}^2) \cap \partial\Sigma = \emptyset$ and $e_2(\mathbb{D}^2) \cap \partial\Sigma = \emptyset$. Then there exists a diffeomorphism h in $\mathrm{Diff}_0^{\infty}(\Sigma)$ such that $h \circ e_1 = e_2$.*

Proof of Proposition 3.1. Let ξ be an element of $\mathfrak{Diff}_S(K)$ and take a representative f of ξ in $\mathrm{Diff}_0^{\infty}(S)$. In the case where the surface S is the sphere, choose a representative f which fixes a point p in $S - D$.

Let Σ be an embedded compact surface contained in the interior $\mathring{D}$ of D which is a small neighbourhood of K. More precisely, this embedded surface Σ is chosen close enough to K so that the sets Σ and $f(\Sigma)$ are contained in $\mathring{D}$. Observe that this surface Σ is not necessarily connected and denote by $\Sigma_1, \ldots, \Sigma_l$ its connected components. As these surfaces are embedded in a disk, each of these components is diffeomorphic to a disk with or without holes.

Fix $1 \leq i \leq l$. Denote by U_i the connected component of $S - \Sigma_i$ which contains ∂D. Finally, let $D_i = S - U_i$. Observe that the surface D_i is diffeomorphic to a disk : it is the surface Σ_i with "filled holes".

Claim 3.4. *For any* i*,* $D_i \cup f(D_i) \subset \mathring{D}$*.*

Proof. In the case where $S \neq \mathbb{S}^2$, observe that the connected component of $S - f(\Sigma_i)$ which contains ∂D (and hence $S - D$ as $f(\Sigma_i) \subset D$) is not homeomorphic to a disk. Therefore, this connected component has to be $f(U_i)$ and $S - \mathring{D} \subset f(U_i) \cap U_i$. Taking complements, we obtain the desired property.

The case of the sphere is similar: $f(U_i)$ is the only connected component of $S - f(\Sigma)$ which contains $p = f(p)$. □

Given two disks in the family $D_1, D_2, \ldots, D_l$, observe that either they are disjoint or one of them is contained in the other one. Indeed, for any i, the boundary of D_i is a boundary component of Σ_i and the surfaces Σ_i are pairwise disjoint. Hence, it is possible to find pairwise disjoint disks $D'_1, \ldots D'_m$ among the disks $D_1, \ldots, D_l$ such that $D_1 \cup D_2 \cup \ldots \cup D_l = D'_1 \cup D'_2 \cup \ldots \cup D'_m$.

We prove by induction on i that, for any $i \leq m$, there exists a diffeomorphism g_i supported in D such that

$$f_{|D'_1 \cup D'_2 \cup \ldots \cup D'_i} = g_{i|D'_1 \cup D'_2 \cup \ldots \cup D'_i}.$$

Then the diffeomorphism g_m is a representative of ξ supported in D.

In the case $i = 1$, use Lemma 3.3 to find a diffeomorphism g_1 supported in D such that $f_{|D'_1} = g_{|D'_1}$.

Suppose that we have built the diffeomorphism g_i for some $i < m$. Observe that the diffeomorphism $g_i^{-1} f$ fixes $D'_1 \cup D'_2 \cup \ldots \cup D'_m$ pointwise and satisfies $g_i^{-1} f(D'_{i+1}) \cup D'_{i+1} \subset \mathring{D}$. Apply Lemma 3.3 to find a diffeomorphism h supported in $D - (D'_1 \cup D'_2 \cup \ldots \cup D'_i)$ such that $g_i^{-1} f_{|D'_{i+1}} = h_{|D'_{i+1}}$ and take $g_{i+1} = g_i h$. □

4. Standard Cantor set

Fix a parameter $0 < \lambda < 1/2$. The central ternary Cantor set C_λ in $\mathbb{R} \times \{0\} \subset \mathbb{R}^2 \subset \mathbb{S}^2$ is obtained from the interval $[0,1]$ as the limit of the following inductive process: At the first step, take out the central subinterval of length $(1 - 2\lambda)$ from the interval $[0,1]$ to obtain a collection $\mathcal{I}_1$ of two intervals of size λ. At the n-th step of the process, we obtain a collection $\mathcal{I}_n$ of 2^n intervals of size λ^n by removing from each interval I in $\mathcal{I}_{n-1}$, the middle subinterval of size $(1 - 2\lambda)|I|$. Our

Cantor set is given by the formula:

$$C_\lambda := \cap_{n\geq 0}(\cup \mathcal{I}_n)$$

In this section we will prove Theorem 1.2, which we restate as follows:

Theorem 4.1. *Let S be a closed surface. For any $\lambda > 0$, there are no distorted elements in the group $\mathcal{M}_0^\infty(S, C_\lambda)$, where C_λ is a smooth embedding in S of the standard ternary Cantor set with affine parameter λ.*

Definition 4.1. We call **elementary interval** in our Cantor set C_λ a set of the form $C_\lambda \cap I'$, where I' is an interval in the collection $\mathcal{I}_n$ for some n.

It was proven in Funar-Neretin ([11], see Theorem 6) that any element $\phi \in \mathfrak{diff}_{\mathbb{R}^2}(C_\lambda)$ is piecewise affine, i.e. there exists a finite collection of elementary intervals $\{I_k\}$ covering C_λ such that ϕ sends I_k into another elementary interval $\phi(I_k)$ and such that $\phi|_{I_k} = \pm\lambda^{n_k}x + c_k$, for some $n_k \in \mathbb{Z}$ and an appropriate constant c_k.

One consequence of this fact is that the group $\mathfrak{diff}_{\mathbb{R}^2}(C_\lambda)$ does not depend on λ. Therefore we define the group $\overline{V_2} := \mathfrak{diff}_{\mathbb{R}^2}(C_\lambda)$. The group $\overline{V_2}$ contains Thompson's group V_2, which we define as the group of homeomorphisms of C_λ which are piecewise affine, where the affine maps are of the form $x \to \lambda^n x + c$, see 5.1 for another description of V_2. (See also [1], [5] and [11]).

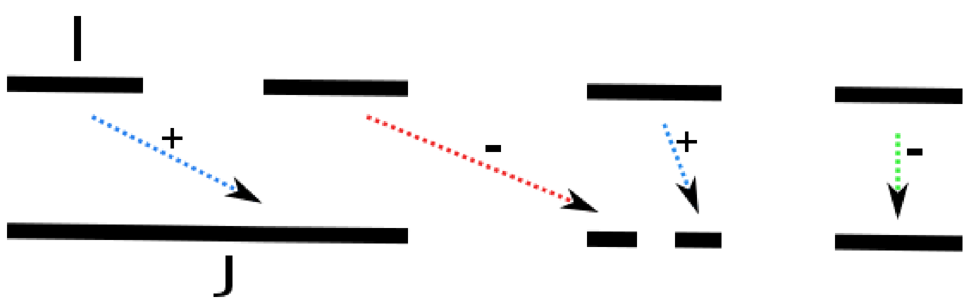

FIGURE 6. An element $f \in \overline{V_2}$

Even though the groups V_2 and $\overline{V_2}$ are different and $V_2 \subset \overline{V_2}$, there is an embedding of the group $\overline{V_2}$ into V_2 as we will show in the following proposition:

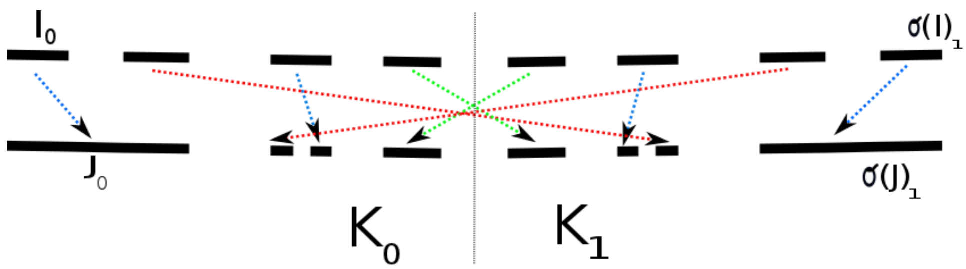

FIGURE 7. The corresponding element $\phi(f) \in V_2$.

Proposition 4.2. *There is an injective homomorphism* $\phi : \overline{V_2} \to V_2$.

Proof. Let us define $K := C_\lambda$ for some $0 < \lambda < 1$. Consider the Cantor set $K' := K_0 \cup K_1$, where $K_0 = K$ and K_1 is the Cantor set obtained by reflecting K_0 in the vertical line $x = 3/2$ as in Figure 7.

There is an involution $\sigma : K \to K$ given by $x \to 1 - x$. We will use the following notation: for any interval $I \subset K$, I_i is the corresponding interval of I in the copy K_i.

For a homeomorphism $f \in \overline{V_2}$ of the ternary Cantor set C_λ, we define $\phi(f) \in V_2$ as the piecewise affine homeomorphism of K' which satisfies the following property. Let $I \subset K$ be any elementary interval such that our homeomorphism f sends I affinely into the elementary interval $J := f(I)$. In this situation, we will define $\phi(f)$ on $I_0 \cup \sigma(I)_1$ by the following rules:

(1) If f preserves the orientation of I, we define:
$$\phi(f)(I_0) = J_0 \text{ and } \phi(f)(\sigma(I)_1) = \sigma(J)_1.$$
(2) If f reverses the orientation of I, we define:
$$\phi(f)(I_0) = \sigma(J)_1 \text{ and } \phi(f)(\sigma(I)_1) = J_0.$$
(3) In (1) and (2), the maps $\phi(f)|_{I_0}$ and $\phi(f)|_{\sigma(I)_1}$ are affine orientation preserving maps.

One can easily check that $\phi(f)$ is well defined on K' (If $I \subset J$ and $f|_I$ and $f|_J$ are affine maps, then the definition of $\phi(f)$ on $I_0 \cup \sigma(I)_1$ and on $J_0 \cup \sigma(J)_1$ should coincide). See Figure 7 for an illustration of this construction.

We still need to prove that ϕ is a group homomorphism, but that can be shown easily as follows: Suppose f and g are two elements of $\overline{V_2}$. If we partition our Cantor set K into a collection $\{I_n\}$ of sufficiently small intervals, we can suppose that each interval I_n is mapped affinely by f and also that each interval $f(I_n)$ is mapped affinely by g. One can then do a case by case check (whether f, g are preserving orientation or not in I_n and $f(I_n)$ respectively) to show that, restricted to the intervals $(I_n)_0$ and $(\sigma(I_n))_1$ in K', we have $\phi(gf) = \phi(g)\phi(f)$.

□

Example: Consider the element f described in Figure 6. The arrows in the picture indicate where f is mapping each of the 4 elementary intervals affinely. The symbols $-, +$ denote whether the interval is mapped by f preserving orientation or not. The corresponding element $\phi(f)$ is depicted in Figure 7.

The previous construction is useful for our purposes because it is known there are no distorted elements in Thompson's group V_n (see Bleak-Collin-et al. [1], Sec.8). By Proposition 4.2, this implies there are no distorted elements in $\overline{V_2}$. Having that fact in mind and in view of Theorem 2.1 we can easily finish the proof of Theorem 4.1:

Proof of Theorem 4.1. Suppose $f \in \mathcal{M}_0^\infty(S, C_\lambda)$ is distorted. Observe that, by Corollary 3.2, the group $\mathfrak{Diff}_S(C_\lambda)$ is isomorphic to $\mathfrak{Diff}_{\mathbb{R}^2}(C_\lambda)$. From the exact sequence:

$$\mathcal{PM}_0^\infty(S, C_\lambda) \to \mathcal{M}_0^\infty(S, C_\lambda) \underset{\pi}{\to} \mathfrak{Diff}_S(C_\lambda) = \overline{V_2},$$

we obtain that $\pi(f)$ is distorted. Now, by Lemma 4.2, $\overline{V_2}$ embeds in V_2 and as there are no infinite order distorted elements in V_2 (See [1], Sec.8), the element $\pi(f)$ has finite order. Thus, we obtained that $f^k \in \mathcal{PM}_0^\infty(S, C_\lambda)$ for some $k \geq 1$, and the element f^k is as well distorted. By Theorem 2.1, $f^k = \mathrm{Id}$ and so f has finite order. □

5. Tits alternative

The "Tits alternative" states that a finitely generated group Γ which is linear (isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{R})$ for some n) either contains a copy of the free subgroup on two generators $\mathbb{F}_2$ or it is virtually solvable. In [18], Margulis proved a similar statement for $\mathrm{Homeo}(\mathbb{S}^1)$: Any subgroup $\Gamma \subset \mathrm{Homeo}(\mathbb{S}^1)$ either contains a free subgroup or preserves a measure in $\mathbb{S}^1$. As the derived subgroup $[F_n, F_n]$ of Thompson's group F_n is a simple subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ (see [5], Theorem

4.5) and does not contain free subgroups on two generators by Brin-Squier's Theorem (see [14], Theorem 4.6 p.344), the actual statement of the Tits alternative cannot hold in Homeo($\mathbb{S}^1$) (see also [19]). In this section, we prove Theorem 1.3. By Corollary 3.2, Theorem 1.3 reduces to the following theorem.

Theorem 5.1. *Let Γ be a finitely generated subgroup of $\mathcal{M}^\infty(\mathbb{R}^2, C_\alpha)$, then one of the following holds:*

1. *Γ contains a free subgroup on two generators $\mathbb{F}_2$*
2. *Γ has a finite orbit, i.e. there exists $p \in C_\alpha$ such that the set $\Gamma(p) := \{g(p) \,|g \in \Gamma\}$ is finite.*

Using the description of $\mathfrak{diff}_{\mathbb{R}^2}(C_\lambda)$ explained at the beginning of Section 4 and Proposition 4.2, we deduce the previous theorem as an immediate corollary of the following statement about Thompson's group V_n, which could be of independent interest:

Theorem 5.2. *For any finitely generated subgroup $\Gamma \subset V_n$, either the action of Γ on the Cantor set K_n has a finite orbit or Γ contains a free subgroup.*

The finite generation condition is indeed necessary, as the following example shows:

The finite group of permutations S_{2^n} is a subgroup of V_2 as it acts on C_λ by permutations of the elementary intervals of the collection $\mathcal{I}_n$ described at the beginning of Section 4. Defining the group $S_\infty := \bigcup_n S_{2^n}$, we easily see that S_∞ has no finite orbit (the action is in fact minimal) and there is no free subgroups in S_∞ as any element has finite order.

5.1. **Elements of V_n and tree pair diagrams.** We need to describe the action of the elements of V_n on the Cantor set K_n in detail. In order to do this, we will use as a tool the description of the elements of V_n as tree pair diagrams as described for example in [1], [3], [5], [20]. We will follow very closely the description given in [1] and we refer to it for a more detailed explanation of the material introduced in this subsection. The main tool for us is the existence of "revealing tree pair diagrams" which were first introduced by Brin in [3]. These "revealing diagrams" allow us to read the dynamics of elements of V_n easily.

Indeed, we will show that for each element $g \in V_n$, there are two $g-$invariant clopen sets V_g and U_g such that $K_n = U_g \cup V_g$, where $g|_{U_g}$

has finite order and $g|_{V_g}$ has "repelling-contracting" dynamics. The reader that decide to skip this introductory subsection, should look at Lemma 5.5, where all the properties of the dynamics of elements of V_n in K_n that we will use are described.

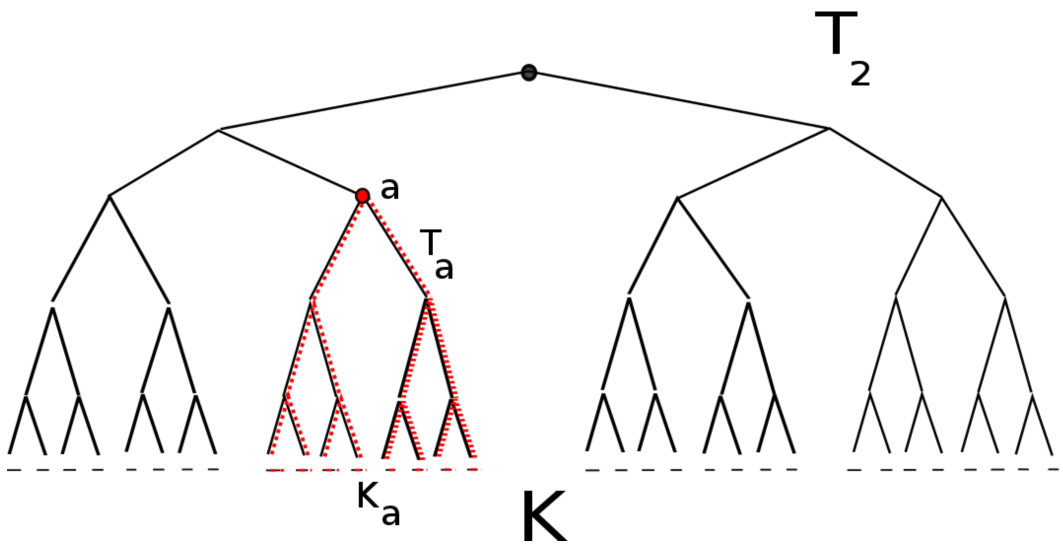

FIGURE 8. The rooted tree $\mathcal{T}_2$, for a vertex $a \in \mathcal{T}_2$, the tree $\mathcal{T}_a$ and the elementary interval K_a are depicted

5.1.1. *Notation.* From now on, K_n denotes the Cantor set that we identify with the ends of the infinite rooted $n-$tree $\mathcal{T}_n$ (see Figure 8, where $\mathcal{T}_2$ and K are depicted). For a vertex $a \in \mathcal{T}_n$, we define T_a as the infinite n-ary rooted tree descending from a. We define the clopen set $K_a \subset K$ as the ends of T_a (see Figure 8). Any subset of K_n of the form K_a is an **elementary interval** as in Definition 4.1.

The group V_n is a subgroup of the group Homeo(K_n). Our next task is to describe which kind of homeomorphisms of K_n belong to V_n, looking at the example depicted in Figure 9 might be instructive to understand what an element of V_n can be.

An element g of V_n is described by a triple (A, B, σ) where A and B are n-ary rooted trees (connected subtrees of $\mathcal{T}_n$) with the same number of endpoints (A, B tell us a way of partitioning our Cantor set K_n

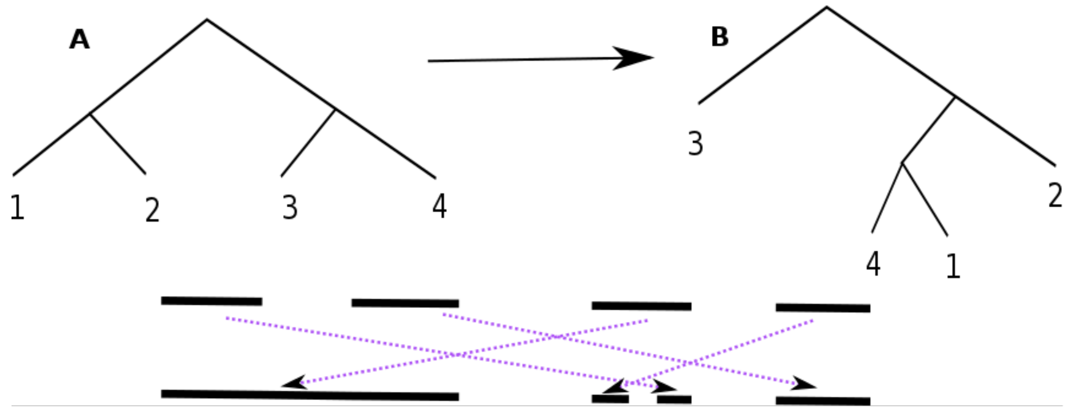

FIGURE 9. An element f of V_2

into elementary intervals) together with a bijection σ between the endpoints of A and the endpoints of B that tell us how an interval is going to be mapped by g to another interval. More formally, for an endpoint $a \in A$, g maps K_a into $K_{\sigma(a)}$ by mapping T_a into $T_{\sigma(a)}$ in the obvious way (see Figure 9).

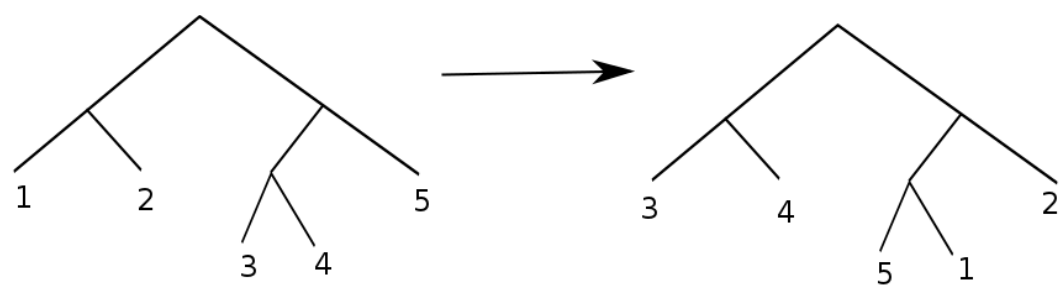

FIGURE 10. A revealing pair diagram for the element in Figure 9.

5.2. **Revealing pairs.** It should be noted that an element $g \in V_n$ is not described by a unique tree pair diagram (A, B, σ). Some tree pair diagrams describe the dynamics of an element V_n better than others. As an example, consider the element f with the diagrams depicted in Figures 9 and 10. In the tree pair diagram in Figure 10, the trees A

and B coincide and so f must have finite order.

Before defining what a revealing tree pair diagram is, we will need to set up some notation. Let $g \in V_n$ be an element described by a tree pair (A, B, σ).

Consider the sets $X = \overline{A - B}$ and $Y = \overline{B - A}$. Notice that each connected component of X (respectively Y) is a rooted tree whose root is an endpoint of B but not of A (respectively A but not B). In the example depicted in Figure 11, X is blue and Y is red.

Let $L_{(A,B,\sigma)}$ denote the set of vertices of $\mathcal{T}_n$ which are endpoints of either A or B. A vertex in $L_{(A,B,\sigma)}$ is called neutral if x is an endpoint of both A and B. Observe that if $\lambda \in L_{(A,B,\sigma)}$ and if $g^i(\lambda)$ is a neutral vertex of $L_{(A,B,\sigma)}$ for every $i \geq 0$, then the vertex λ must be periodic for g. Let t be the period of λ, *i.e.* the minimal $t > 0$ such that $g^t(\lambda) = \lambda$. Observe that in this case $g^t|_{K_\lambda} = \mathrm{Id}$. If λ is not periodic we can find the largest integers $s \geq 0$ and $r \geq 0$, such that for any $-r < i < s$, the vertex $g^i(\lambda)$ is a neutral vertex of $L_{(A,B,\sigma)}$. In this case, we define the iterated augmentation chain as

$$IAC(\lambda) := \left(g^i(\lambda)\right)_{i=-r}^{s}.$$

Observe that the vertex $g^{-r}(\lambda)$ is an endpoint of A but not of B and the vertex $g^s(\lambda)$ is an endpoint of B but not of A.

An *attractor* in $L_{(A,B,\sigma)}$ is defined as an endpoint of A such that $g^s(\lambda)$ belongs to $B \setminus A$ and such that $g^s(\lambda)$ is strictly contained in T_λ ($g^s(\lambda)$ is under λ). In this case we see that $g^s|_{K_\lambda}$ has attracting dynamics, and there is a unique attracting point p for g^s inside K_λ. In a similar way, one defines a "*repeller*" as a vertex λ in B, such that $g^{-r}(\lambda)$ is strictly contained in T_λ. In Figure 11, the red vertices are attractors and the blue one is a repeller. Observe that attractors are always roots of components of Y and repellers are always roots of components of X.

Definition 5.1. Let (A, B, σ) be a tree pair diagram for an element $g \in V_n$. The set $X = \overline{A \setminus B}$ (respectively $Y = \overline{B \setminus A}$) consists of a union of rooted trees, whose roots are endpoints of B (respectively A). If all these vertices are repellers (respectively attractors), then (A, B, σ) is said to be a *revealing tree pair diagram*.

Theorem 5.3 (Brin [3])**.** *For every $g \in V_n$, there exists a revealing tree pair diagram (A, B, σ), even more, there is an algorithm to extend any tree pair diagram into a revealing tree pair diagram.*

One easy consequence of Theorem 5.3 is that every periodic element of V_n has a tree pair diagram (A, B, σ) where $A = B$, as it is illustrated in Figure 10.

If (A, B, σ) is a revealing pair, the dynamics of each interval under a vertex of $L_{(A,B,\sigma)}$ can be easily described as we will show next.

Let λ be a vertex of $L_{(A,B,\sigma)}$ and suppose that λ is not a periodic vertex. Let $\mathrm{IAC}(\lambda) = \big(g^i(\lambda)\big)_{i=-r}^{s}$ be its iterated augmented chain. In this case, $g^{-r}(\lambda)$ is an endpoint of A but not of B, and $g^s(\lambda)$ is an endpoint of B but not of A. Hence, there are two possibilities: either $g^s(\lambda)$ is a root of a component of X, or $g^s(\lambda)$ is a vertex of a tree in Y.

If $g^s(\lambda)$ is a root of a component of X, then, as (A, B, σ) is a revealing tree pair diagram, $g^s(\lambda)$ is a repeller. The vertex $g^{-r}(\lambda)$ is then strictly under $g^s(\lambda)$ and there is a unique fixed point p for g^{-r-s} in $K_{g^s(\alpha)}$. This point p is a repelling periodic point of order $s + r$. In that case, the elementary intervals $\{g^i(K_\lambda)\}_{i=-r}^{s-1}$ are disjoint and each of them contains a unique repelling periodic point in the orbit of p.

If $g^s(\lambda)$ is a vertex of a tree in Y and the vertex $g^{-r}(\lambda)$ is a root of a component of Y, then $g^{-r}(\lambda)$ is an attractor. In this case, there is a unique periodic attracting point q of order $s+r$ in $K_{g^{-r}(\alpha)}$. Again, the intervals $\{g^i(K_\lambda)\}_{i=-r}^{s-1}$ are disjoint and each of them contains a unique attracting periodic point in the orbit of p.

If the vertex $g^{-r}(\lambda)$ is a vertex of a tree in Y under a repeller α and the vertex $g^s(\lambda)$ is a vertex of a tree in X under an attarctor ω, then we see that the forward orbit of K_λ is getting attracted toward the periodic orbit p_ω corresponding to ω and the backward orbit of K_λ gets attracted toward the periodic repeller p_α corresponding to α.

Example (see Figure 11). We use the following notation, for a number j, we denote by j_A the vertex in $\mathcal{T}_2$ numbered by j in the tree A. We define similarly the vertices j_B. Observe that for our particular example we have $1_A = 3_B$ and $2_A = 9_B$.

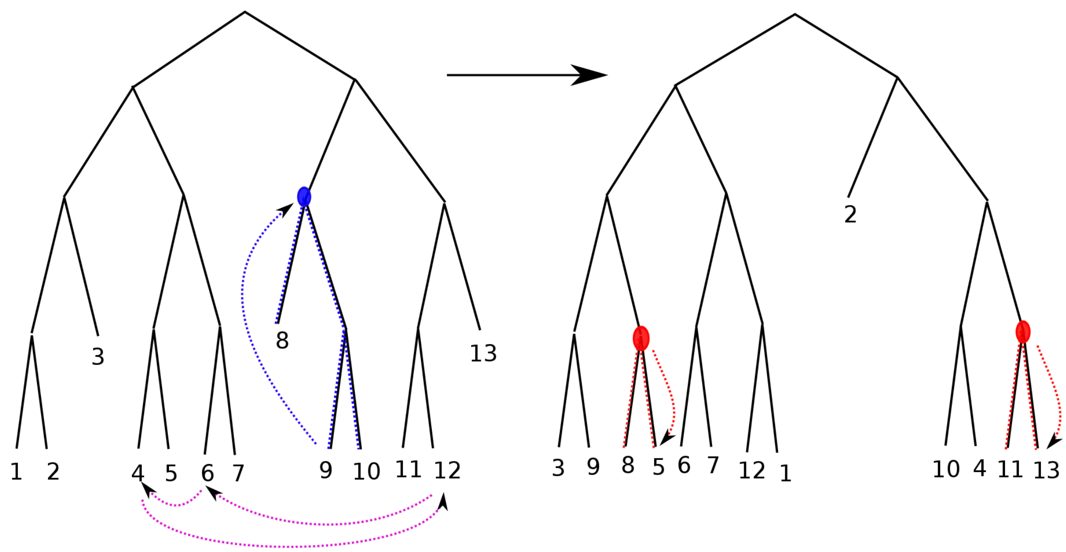

FIGURE 11. A revealing pair diagram for an element of V_2

In Figure 11, X is depicted blue. It consists of one rooted tree with root 2_B. Observe that, for this element, we have

$$9_A \to 9_B = 2_A \to 2_B$$

and the vertex 9_A is under 2_B, and so 2_B is a repeller. There is a unique repelling periodic point under 2_B of period 2.

The set Y consists of two trees, one tree with root 3_A and the other one with root 13_A, we have

$$3_A \to 1_A \to 7_A \to 5_A \to 5_B$$

and 5_B is under 3_A and so 3_A is an attractor, there is an attracting periodic point of period 4 under 3_A . We also notice that 13_B is under 13_A and so there is an attracting fixed point under 13_A.

Observe also that $10_A \to 11_A \to 11_B$, the vertex 10_A is under the repeller 2_B and 11_B under the attractor 13_A, this means there are points arbitrarily close to the repelling periodic point that converge toward the attracting fixed point under 13_A and we also have $8_A \to 8_B$, where 8_A is under the repeller 2_B and 8_B under the attractor at 3_A, and so there are orbits going from the repelling periodic orbit to the attracting periodic orbit corresponding to 3_A.

As a consequence of the previous discussion, we obtain the following lemma. For a more detailed discussion, see [1] and [20].

Lemma 5.4. *Let (A,B,σ) be a revealing pair for an element $g \in V_n$. Let λ be a vertex in $L_{A,B,\sigma}$, then exactly one of the following holds.*

1. *λ is periodic, in which case there is $t > 0$ such that $g^t\lambda = \lambda$ and $g^t|_{K_\lambda} = Id$.*
2. *K_λ contains a unique contracting periodic point p and there is $t > 0$ such that $g^t(K_\lambda) \subset K_\lambda$ and $g^t|_{K_\lambda}$ is contracting affinely (i.e g^t sends the interval K_λ into the interval $g^t(K_\lambda)$ in the obvious way).*
3. *K_λ contains a unique repelling periodic point p in λ and there is $r > 0$ such that $g^{-r}(K_\lambda) \subset K_\lambda$ and $g^{-r}|_{K_\lambda}$ is contracting affinely.*
4. *There exist $s \geq 0, r \geq 0$ such that $g^s(\lambda)$ and $g^r(\lambda)$ are vertices but not roots of components of Y and X respectively. In this case, the following property holds. As $n \to \infty$, $g^n(K_\lambda)$ gets contracted affinely converging towards an attracting periodic orbit of g and $g^{-n}(K_\lambda)$ gets attracted towards a repelling periodic orbit of g.*

In the proof of Theorem 5.1, we use the following notation which makes the proof easier to digest.

Definition 5.2 (Neighborhoods)**.** In the Cantor set K_n, let us consider the metric coming from the standard embedding of K_n in the interval $[0,1]$. For a point $p \in K_n$ and $\epsilon > 0$, we define the neighborhood $N_\epsilon(p)$ as the maximal elementary interval I such that $p \in I$ and $\text{length}(I) < \epsilon$. Similarly, if S is a finite set, we define $N_\epsilon(S) := \cup_{s\in S} N_\epsilon(s)$.

Definition 5.3. For any element $g \in V_n$, we define the following:

$$\text{Att}(g) := \{\, p \in K_n \text{ such that } p \text{ is periodic and attracting}\}$$

$$\text{Rep}(g) := \{\, p \in K_n \text{ such that } p \text{ is periodic and repelling}\}$$

$$\text{Per}_0(g) := \text{Att}(g) \cup \text{Rep}(g)$$

By Lemma 5.4, both sets $\text{Att}(g)$ and $\text{Rep}(g)$ are finite. Hence the set $\text{Per}_0(g)$ is also finite.

As a conclusion of Lemma 5.4 we obtain the following lemma that enumerates all the dynamical properties of the action of elements V_n

on K_n that we will use.

Lemma 5.5. *Given an element $g \in V_n$ there exist two $g-$invariant clopen sets U_g, V_g (i.e. finite union of elementary intervals in K_n) such that:*

(1) $K_n = U_g \cup V_g$.

(2) $g|_{U_g}$ *has finite order.*

(3) *There are finitely many periodic points of g contained in V_g(the set $Per_0(g)$) and the dynamics of $g|_{V_g}$ are "attracting-repelling" i.e. for every $\epsilon > 0$, there exists m_0 such that, for $m \geq m_0$, we have:*
$$g^m(V_g \setminus N_\epsilon(Rep(g))) \subset N_\epsilon(Att(g))$$
$$g^{-m}(V_g \setminus N_\epsilon(Att(g))) \subset N_\epsilon(Rep(g)).$$

(4) *If ϵ is small enough, for any point $p \in Att(g)$, there exists s (the period of p) such that $g^s(N_\epsilon(p)) \subset N_\epsilon(p)$ and $g^s|_{N_\epsilon(p)}$ is an affine contraction. The analogous condition also holds for points in $Rep(g)$.*

5.3. **Proof of Theorem 5.1.** The idea of the proof of Theorem 5.1 is to use the "attracting-repelling" dynamics of elements of V_n and the "ping-pong" lemma to obtain a free subgroup $\mathbb{F}_2$ contained in Γ (this strategy was the one used by Margulis to prove his "alternative" for Homeo($\mathbb{S}^1$), see [18]). To illustrate the idea of the proof, suppose our group G contains two elements f and h such that h sends $\mathrm{Per}_0(f) \cup U_f$ disjoint from itself. In that case, if we consider the element $g = hfh^{-1}$, the sets $\mathrm{Per}_0(f) \cup U_f$ and $\mathrm{Per}_0(g) \cup U_g$ are disjoint.

Under this last condition, one can apply the ping-pong lemma (see [16], Ch. 2) as follows to show that $\langle f^n, g^n \rangle$ generate a free group if n is large enough. Let us take small disjoint neighborhoods $N_\epsilon(\mathrm{Per}_0(f))$ and $N_\epsilon(\mathrm{Per}_0(g))$ and consider the set
$$X = K \setminus (U_f \cup U_g \cup N_\epsilon(\mathrm{Per}_0(f)) \cup N_\epsilon(\mathrm{Per}_0(g))).$$
If we take ϵ small enough, then $X \neq \emptyset$, and, by Lemma 5.5, if n is large enough, we have
$$f^n(X) \subset N_\epsilon(\mathrm{Att}(f))$$
$$f^{-n}(X) \subset N_\epsilon(\mathrm{Rep}(f))$$
and we also have
$$f^n(N_\epsilon(\mathrm{Per}_0(g))) \subset N_\epsilon(\mathrm{Att}(f))$$

$$f^{-n}(N_\epsilon(\mathrm{Per}_0(g))) \subset N_\epsilon(\mathrm{Rep}(f)).$$

The corresponding statement for g^n and g^{-n} are also true. This implies that f^n, g^n generate a free group, as for any nontrivial word w on the elements f^n, g^n, we have by the "ping-pong" argument that $w(X) \subset N_\epsilon(\mathrm{Per}_0(f) \cup \mathrm{Per}_0(g))$ and therefore $w \neq \mathrm{Id}$.

As a conclusion, we have proved the following lemma:

Lemma 5.6. *Let $\Gamma \subset V_n$. If there are two elements $f, h \in \Gamma$ such that the sets $Per_0(f) \cup U_f$ and $h(Per_0(f) \cup U_f)$ are disjoint, then Γ contains a free subgroup on two generators.*

To prove Theorem 5.1, we will show that either a pair of elements f, h of Γ as in Lemma 5.6 exists or that Γ has a finite orbit in K_n. The following result is the key lemma for proving the existence of such an element h sending $\mathrm{Per}_0(f) \cup U_f$ disjoint from itself. It is based on a recent proof by Camille Horbez (See [6], Sec.3, [7]) of the Tits alternative for mapping class groups, outer automorphisms of free groups and other related groups.

Lemma 5.7. *Let Γ be a countable group acting on a compact space K by homeomorphisms and let $F \subset K$ be a finite subset. Then either there is finite orbit of Γ on K or there exists an element $g \in \Gamma$ sending F disjoint from itself (i.e. $g(F) \cap F = \emptyset$).*

Before beginning the proof of Lemma 5.7, we recall the following basic notions of random walks on groups and harmonic measures.

For a discrete group Γ, let us take a probability measure μ on Γ and suppose that $\langle \mathrm{supp}(\mu) \rangle = \Gamma$. Suppose our group Γ acts continuously on a compact space X. A harmonic measure in X for (Γ, μ) is a Borel probability measure ν on X such that $\mu * \nu = \nu$, where "$*$" denotes the convolution operator. This means that, for every ν-measurable set $A \subseteq X$,

$$\nu(A) = \sum_{g \in \Gamma} \nu(g^{-1}(A))\mu(g) \tag{2}$$

A harmonic measure always exists (see the proof below) and one can think of it as a measure on X that is invariant under the action of Γ on average (with respect to μ).

Proof of Lemma 5.7. Suppose that there is no element of Γ sending F disjoint from itself. Let $n = |F|$. If $n = 1$, the theorem is obvious and so we assume $n > 1$. Consider the diagonal action of Γ on K^n. Let $\vec{p} = (p_1, p_2, ..., p_n)$ be an n-tuple consisting of the n different elements of F in some order. We take a probability measure μ supported in our group Γ such that $\langle \text{supp}\mu \rangle = \Gamma$ and take a harmonic probability measure ν on K^n supported in $\overline{\Gamma \vec{p}}$. Such a harmonic measure ν can be obtained as follows: Take the Dirac probability measure $\delta_{\vec{p}}$ in K^n supported in $\{\vec{p}\}$ and consider the averages of convolutions $\nu_l := \frac{1}{l} \sum_{i=1}^{l} \mu^i * \delta_{\vec{p}}$ (μ^i is the measure obtained by convoluting μ i times with itself). Then ν can be taken as any accumulation point of ν_l in the space of probability measures in K^n.

Observe that, by our assumption, for each $g \in \Gamma$, the element $g(\vec{p})$ is contained in a set of the form $K^l \times \{p_i\} \times K^m$ for some integers i, l and m such that $l + m = n - 1$ and therefore:

$$\overline{\Gamma \vec{p}} \subset \bigcup_{0 \leq i \leq n,\ l+m=n-1} K^l \times \{p_i\} \times K^m.$$

As $\nu(\overline{\Gamma \vec{p}}) = 1$, we can conclude that there exist integers i, l and m such that $\nu(K^l \times \{p_i\} \times K^m) > 0$.

Let us take $q \in K$ such that $\nu(K^l \times \{q\} \times K^m)$ is maximal. We will show that q has a finite Γ-orbit. Observe that, for $g \in \Gamma$, $g(K^l \times \{q\} \times K^m) = K^l \times \{g(q)\} \times K^m$ and, therefore,

$$\nu(K^l \times \{q\} \times K^m) = \sum_i \nu(K^l \times \{g_i^{-1}(q)\} \times K^m)\mu(g_i).$$

So we obtain by our maximality assumption that $\nu(K^l \times \{q\} \times K^m) = \nu(K^l \times \{g^{-1}(q)\} \times K^m)$ for every g in the support of μ. Hence this also holds for every $g \in \Gamma$. But being ν a probability measure this can only happen if the orbit $\Gamma(q)$ is finite and so we are done. □

Remark 5.1. One can also conclude with a little bit more of extra work that, for some i, $\Gamma(p_i)$ is a finite orbit. We will not make use of this fact.

The following proposition is our main tool to construct free subgroups of a subgroup Γ of V_n.

Proposition 5.8. *Suppose $f, g \in \Gamma \subset V_n$ are such that U_f and U_g are disjoint and suppose there is no periodic orbit for Γ in K_n. Then, there exists a free subgroup on two generators contained in Γ.*

Proof. Taking powers of f and g we can suppose that $f|_{U_f} = \mathrm{Id}$, $g|_{U_g} = \mathrm{Id}$ and that the repelling and attracting periodic points of f and g are fixed by f and g respectively. We will prove that, given any $\epsilon > 0$, there exists an element $w_\epsilon \in G$ such that $\mathrm{Per}_0(w_\epsilon) \cup U_{w_\epsilon}$ is contained in $N_\epsilon(\mathrm{Per}_0(f) \cup \mathrm{Per}_0(g))$. First, let us show this implies Proposition 5.8.

By Lemma 5.7, we can find an element $h \in \Gamma$ sending $\mathrm{Per}_0(f) \cup \mathrm{Per}_0(g)$ disjoint from itself. Hence, if ϵ is small enough, h sends $N_\epsilon(\mathrm{Per}_0(f)) \cup N_\epsilon(\mathrm{Per}_0(g))$ disjoint from itself, which implies that the sets $\mathrm{Per}_0(w_\epsilon) \cup U_{w_\epsilon}$ and $h(\mathrm{Per}_0(w_\epsilon) \cup U_{w_\epsilon})$ are disjoint. By Lemma 5.6, Γ contains a free subgroup.

We will prove that our desired element w_ϵ can be taken of the form $w_\epsilon := g^{m_1} f^{m_2}$. To illustrate the idea of the proof of this fact, suppose first that $\mathrm{Per}_0(f)$ and $\mathrm{Per}_0(g)$ are disjoint. In this case, let us define the set $V := V_f - N_\epsilon(\mathrm{Rep}(f))$. Take ϵ small enough so that, for a point $p \in \mathrm{Att}(f)$ either $N_\epsilon(p) \subset U_g$ or $N_\epsilon(p)$ is contained in $V_g \setminus N_\epsilon(\mathrm{Rep}(g))$. Also, take ϵ small enough so that the sets $N_\epsilon(\mathrm{Att}(f))$, $N_\epsilon(\mathrm{Rep}(f))$, $N_\epsilon(\mathrm{Att}(g))$ and $N_\epsilon(\mathrm{Rep}(g))$ are pairwise disjoint. By Lemma 5.5, we can take m large enough so that $f^m(V) \subset N_\epsilon(\mathrm{Att}(f))$ and $g^m(N_\epsilon(\mathrm{Att}(f))) \subset N_\epsilon(\mathrm{Att}(f) \cup \mathrm{Att}(g))$. As a conclusion, we obtain that:

$$g^m f^m(V) \subset N_\epsilon(\mathrm{Att}(f)) \cup N_\epsilon(\mathrm{Att}(g)). \tag{3}$$

Also, if we consider the set $U := U_f \setminus N_\epsilon(\mathrm{Rep}(g))$, taking m larger if necessary, we have

$$g^m f^m(U) = g^m(U) \subset g^m(V_g \setminus N_\epsilon(\mathrm{Rep}(g))) \subset N_\epsilon(\mathrm{Att}(g)). \tag{4}$$

As we are assuming for the moment that the sets $\mathrm{Per}_0(f)$ and $\mathrm{Per}_0(g)$ are disjoint, we also obtain that

$$g^m f^m(N_\epsilon(\mathrm{Att}(f)) \cup N_\epsilon(\mathrm{Att}(g))) \subset N_\epsilon(\mathrm{Att}(f)) \cup N_\epsilon(\mathrm{Att}(g)) \tag{5}$$

Inclusions 3, 4 and 5 imply that, for ϵ small enough and m sufficiently large, all the periodic points of $w_\epsilon := g^m f^m$ in $K \setminus N_\epsilon(\mathrm{Rep}(f) \cup \mathrm{Rep}(g)) \subset U \cup V$ must be contained in $N_\epsilon(\mathrm{Att}(f)) \cup N_\epsilon(\mathrm{Att}(g))$ and therefore the periodic points of w_ϵ must be contained in $N_\epsilon(\mathrm{Per}_0(f)) \cup N_\epsilon(\mathrm{Per}_0(g))$ as we wanted.

To finish the proof of Proposition 5.8, we need to deal with the case where $\mathrm{Per}_0(f)$ and $\mathrm{Per}_0(g)$ have points in common. This case is significantly trickier but the proof is similar to the one above. We include this case as an independent lemma:

Lemma 5.9. *Let f, g be elements of V_n such that U_f and U_g are disjoint. For every $\epsilon > 0$, we can find an element $w_\epsilon \in V_n$ of the form $w_\epsilon = g^{m_1} f^{m_2}$ such that all the periodic points of w_ϵ (*i.e. *$U_{w_\epsilon} \cup Per_0(w_\epsilon)$) are contained in $N_\epsilon(Per_0(f) \cup Per_0(g))$.*

Proof. We can suppose all the periodic points of f and g are fixed and, taking $\epsilon > 0$ small enough, we can suppose that for $p \in \mathrm{Per}_0(f) \cup \mathrm{Per}_0(g)$, the sets $N_\epsilon(p)$ are pairwise disjoint and entirely contained in the sets U_f, V_f, U_g, V_g if p intersects such a set.

Let $\epsilon_0 := \epsilon$ and take an integer n large enough such that

$$g^n(V_g \setminus N_{\epsilon_0}(\mathrm{Rep}(g))) \subset N_{\epsilon_0}(\mathrm{Att}(g)).$$

Choose $0 < \epsilon_1 < \epsilon_0$ small enough so that

$$N_{\epsilon_1}(\mathrm{Att}(g) \cap \mathrm{Rep}(f)) \subset g^n(N_{\epsilon_0}(\mathrm{Att}(g) \cap \mathrm{Rep}(f)))$$

and

$$g^n(N_{\epsilon_1}(\mathrm{Rep}(g) \cap \mathrm{Att}(f))) \subset N_{\epsilon_0}(\mathrm{Rep}(g) \cap \mathrm{Att}(f)).$$

Finally choose an integer m large enough so that

$$f^m(V_f \setminus N_{\epsilon_1}(\mathrm{Rep}(f))) \subset N_{\epsilon_1}(\mathrm{Att}(f)).$$

We can now define the sets

$$W_g := V_g \setminus N_{\epsilon_0}(\mathrm{Rep}(g) \cup (\mathrm{Att}(g) \cap \mathrm{Rep}(f)))$$

and

$$A_{\epsilon_0,\epsilon_1} := N_{\epsilon_0}(\mathrm{Att}(g) \cap \mathrm{Rep}(f)) \setminus N_{\epsilon_1}(\mathrm{Att}(g) \cap \mathrm{Rep}(f)).$$

We observe that by our choices of ϵ_0, ϵ_1 and n, we have:

$$g^n(W_g) \subset N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f)) \cup A_{\epsilon_0,\epsilon_1}. \tag{6}$$

We will show that the element $w_\epsilon := g^n f^m$ has the desired properties. One should have in mind that m is chosen much bigger than n in order to guarantee that all the points in $\mathrm{Att}(f) \cap \mathrm{Rep}(g)$ are attractors for w_ϵ.

We define the set:

$$X := N_{\epsilon_0}(\mathrm{Att}(f)) \cup N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f)) \cup A_{\epsilon_0,\epsilon_1}.$$

First, we show that X is attracting most of K towards itself. More concretely, we show the following:

Lemma 5.10. *For the set X defined above, the following properties hold:*

(1) $g^n f^m(X) \subset X$ *(Invariance)*
(2) $g^n f^m(K \setminus N_{\epsilon_0}(Rep(f) \cup Rep(g))) \subset X$. *(Contractivity)*

Proof. We start by proving item (1). X was defined as the union of the sets $N_{\epsilon_0}(\mathrm{Att}(f)), N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f))$ and $A_{\epsilon_0,\epsilon_1}$, we will show that when we apply $g^n f^m$ to each of these sets, the resulting set is still contained in X. Let us start with $N_{\epsilon_0}(\mathrm{Att}(f))$. We have:

$$g^n f^m(N_{\epsilon_0}(\mathrm{Att}(f))) \subset g^n(N_{\epsilon_1}(\mathrm{Att}(f))).$$

To understand $g^n(N_{\epsilon_1}(\mathrm{Att}(f)))$, we consider each of the cases whether $\mathrm{Att}(f)$ intersects the sets $\mathrm{Per}_0(g)$, U_g or none of them. Observe the following:

- By our choice of ϵ_1, we have:
$$g^n(N_{\epsilon_1}(\mathrm{Att}(f) \cap \mathrm{Rep}(g))) \subset N_{\epsilon_0}(\mathrm{Att}(f) \cap \mathrm{Rep}(g)) \subset X.$$
- As g is attracting in $N_{\epsilon_1}(\mathrm{Att}(g))$, we have:
$$g^n(N_{\epsilon_1}(\mathrm{Att}(f) \cap \mathrm{Att}(g))) \subset N_{\epsilon_1}(\mathrm{Att}(f) \cap \mathrm{Att}(g)) \subset X.$$
- As $g|_{U_g} = \mathrm{Id}$, we have:
$$g^n(N_{\epsilon_1}(\mathrm{Att}(f)) \cap U_g) = N_{\epsilon_1}(\mathrm{Att}(f)) \cap U_g \subset X$$
- As $V_g \setminus N_\epsilon(\mathrm{Per}_0(g)) \subset W_g$ and by Inclusion 6 we know that $g^n(W_g) \subset N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f)) \cup A_{\epsilon_0,\epsilon_1} \subset X$, we have:
$$g^n(N_{\epsilon_1}(\mathrm{Att}(f)) \cap (V_g \setminus N_\epsilon(\mathrm{Per}_0(g)))) \subset g^n(W_g) \subset X.$$

As s consequence, we obtain that $g^n(N_{\epsilon_1}(\mathrm{Att}(f))) \subset X$ and therefore $g^n f^m(N_{\epsilon_0}(\mathrm{Att}(f))) \subset X$ as we wanted.

We now consider the set $N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f))$. We distinguish two cases, whether $N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f))$ intersects U_f, or V_f. Let us consider the former case first. Observe that:

$$f^m(N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f)) \cap U_f) = N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f)) \cap U_f. \tag{7}$$

We clearly have:

$$g^n(N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f))) \subset N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f)). \tag{8}$$

From Inclusions 7 and 8 we obtain:

$$g^n f^m(N_{\epsilon_0}(\mathrm{Att}(g) \setminus \mathrm{Rep}(f)) \cap U_f) \subset X. \tag{9}$$

Now we consider the set $N_{\epsilon_0}(\mathrm{Att}(g)\setminus \mathrm{Rep}(f))\cap V_f$. We have:

$$g^nf^m(N_{\epsilon_0}(\mathrm{Att}(g)\setminus \mathrm{Rep}(f))\cap V_f)\subset g^nf^m(V_f\setminus N_{\epsilon_0}(\mathrm{Rep}(f))).$$

Observe that $f^m(V_f\setminus N_{\epsilon_0}(\mathrm{Rep}(f)))\subset N_{\epsilon_1}(\mathrm{Att}(f))$. We have already proved that $g^n(N_{\epsilon_1}(\mathrm{Att}(f)))\subset X$ and so together with Inclusion 9 we have

$$g^nf^m(N_{\epsilon_0}(\mathrm{Att}(g)\setminus \mathrm{Rep}(f)))\subset X$$

as we wanted.

It remains to show that $g^nf^m(A_{\epsilon_0,\epsilon_1})\subset X$. Observe that $A_{\epsilon_0,\epsilon_1}\subset V_f\setminus N_{\epsilon_1}(\mathrm{Rep}(f))$ and also that $f^m(V_f\setminus N_{\epsilon_1}(\mathrm{Rep}(f)))\subset N_{\epsilon_1}(\mathrm{Att}(f))$. Using the fact that $g^n(N_{\epsilon_1}(\mathrm{Att}(f)))\subset X$ as we proved before, we obtain:

$$g^nf^m(A_{\epsilon_0,\epsilon_1})\subset g^n(N_{\epsilon_1}(\mathrm{Att}(f)))\subset X.$$

We have shown so far that $g^nf^m(X)\subset X$. Along the way we also proved that $g^nf^m(V_f\setminus N_{\epsilon_0}(\mathrm{Rep}(f)))\subset X$. To conclude the proof of Lemma 5.10, we only need to show that $g^nf^m(U_f\setminus N_{\epsilon_0}(\mathrm{Rep}(g)))\subset X$.

As the set U_f is contained in the set V_g (because U_f and U_g are disjoint) and so the inclusion $U_f\setminus N_{\epsilon_0}(\mathrm{Rep}(g))\subset W_g$ holds, we obtain:

$$g^nf^m(U_f\setminus N_{\epsilon_0}(\mathrm{Rep}(g)))=g^n(U_f\setminus N_{\epsilon_0}(\mathrm{Rep}(g)))\subset g^n(W_g)$$

and by Inclusion 6 we have:

$$g^n(W_g)\subset N_{\epsilon_0}(\mathrm{Att}(g)\setminus \mathrm{Rep}(f))\cup A_{\epsilon_0,\epsilon_1}\subset X$$

and so we are done with the proof of Lemma 5.10. □

Now, to finish the proof of Lemma 5.9 observe that as $X\subset K\setminus N_\epsilon(\mathrm{Rep}(f)\cup\mathrm{Rep}(g))$ and $w_\epsilon(K\setminus N_\epsilon(\mathrm{Rep}(f)\cup\mathrm{Rep}(g)))\subset X$, we have that all the periodic points of w_ϵ contained in $K\setminus N_\epsilon(\mathrm{Rep}(f)\cup\mathrm{Rep}(g))$ are actually contained in X, which is a subset of $N_\epsilon(\mathrm{Att}(f)\cup\mathrm{Att}(g))$. Therefore the periodic points of w_ϵ (namely the set $U_{w_\epsilon}\cup\mathrm{Per}_0(w_\epsilon)$) must be contained in $N_\epsilon(\mathrm{Per}_0(f)\cup\mathrm{Per}_0(g))$ as we wanted. □

□

As a consequence of Proposition 5.8, for any group $\Gamma\subset V_n$, either there is a finite orbit, a free subgroup, or for every pair of elements f,g in Γ, we have $U_f\cap U_g\neq\emptyset$. We will generalize Proposition 5.8 to an arbitrary number of group elements of Γ. For any finite set $F\subset\Gamma$, we define the set $K_F:=\cap_{g\in F}U_g$.

Proposition 5.11. *Suppose the action of $\Gamma \subset V_n$ on the Cantor set K_n does not have a finite orbit. Then, for every finite set $F \subset \Gamma$, there exists a finite set $S_F \subset K$ such that, for any $\epsilon > 0$, there exists an element $h_\epsilon \in \Gamma$ with the following properties:*

1. *the set $U_{h_\epsilon} \cup Per_0(h_\epsilon)$ (the set of periodic points of h_ϵ) is contained in $K_F \cup N_\epsilon(S_F)$.*
2. *h_ϵ fixes point-wise K_F.*

Proof. The proof is by induction on the size of the set F. The proposition is clearly true if the set F contains only one element. Suppose the result is true for a set F and let $F' = F \cup \{g\}$. By induction hypothesis, we can find a finite set S_F for F with the desired properties. Consider the set $S_{F'} := S_F \cup \mathrm{Per}_0(g)$. Given $\epsilon > 0$ we can find h_ϵ that fixes K_F whose periodic points are contained in $K_F \cup N_{\epsilon/2}(S_F)$. By taking powers of h_ϵ, we can suppose that h_ϵ fixes point-wise the clopen set U_{h_ϵ}. We can also suppose that U_g is fixed by g by replacing g with a power g^k. Observe that both h_ϵ and g fix the set $K' := U_{h_\epsilon} \cap U_g$, and so we have two elements g and h_ϵ preserving the clopen set $C := K \backslash K'$.

We now consider the actions of g and h_ϵ on our new Cantor set C. Restricted to C, we have $U_{h_\epsilon} \cap U_g = \emptyset$ and so we are in position to apply Proposition 5.9 (see Remark 5.2 below) to find an element h'_ϵ in the subgroup $\langle h_\epsilon, g\rangle \subseteq \Gamma$ such that all the periodic points of h'_ϵ in C are contained in the set $N_{\epsilon/2}(\mathrm{Per}_0(h_\epsilon) \cup \mathrm{Per}_0(g))$. As $\mathrm{Per}_0(h_\epsilon) \subset N_{\epsilon/2}(S_F)$, we obtain that $\mathrm{Per}_0(h'_\epsilon) \subset N_\epsilon(S_{F'})$ and that $U_{h'_\epsilon}$ is contained in $(U_{h_\epsilon} \cap U_g) \cup N_\epsilon(S_{F'})$, which is a subset of $K_{F'} \cup N_\epsilon(S_{F'})$ and so we are done. □

Remark 5.2. Even though Proposition 5.9 is stated for our original Cantor set K_n, it works equally well for actions on clopen sets $C \subset K_n$.

By applying Lemma 5.7, we have the following corollary generalizing Proposition 5.8:

Corollary 5.12. *For every subgroup $\Gamma \subset V_n$ one of the following holds:*

1. *The action of Γ on K_n has a finite orbit.*
2. *Γ contains a free subgroup on two generators.*
3. *The set $K_\Gamma := \cap_{g\in\Gamma} U_g$ is non-empty.*

Proof. Suppose Γ does not have a finite orbit and $K_\Gamma = \emptyset$. By the finite intersection property for compact sets, there is a finite set $F \subset \Gamma$ such that $K_F = \emptyset$. By Proposition 5.11, we can find a finite set $S_F \subset K$ such that, for every $\epsilon > 0$, there is an element h_ϵ in Γ whose periodic points are contained in $N_\epsilon(S_F)$. By Lemma 5.7, we can find an element

t sending S_F disjoint from itself and therefore also sending $N_\epsilon(S_F)$ disjoint from itself for ϵ small enough, which implies by Lemma 5.6 that there is a free group on two generators contained in Γ and we are done.

□

We will finish the proof of Theorem 5.1 by proving the following lemma. It is important to point out that it is the only place where we use the finite generation condition on Γ.

Lemma 5.13. *If Γ is a finitely generated subgroup of V_n and $K_\Gamma := \cap_{g\in\Gamma} U_g \neq \emptyset$, then the action of Γ on K has a finite orbit.*

Proof. Observe that K_Γ is Γ-invariant and therefore there is a minimal closed set $\Lambda \subset K_\Gamma$ which is invariant under the action of Γ on K. If the minimal set Λ is a finite set, then Γ has a finite orbit and we are done. We will now show that if Λ were infinite, then there would exist an element g in Γ with an attracting fixed point in Λ, contradicting that $\Lambda \subset K_\Gamma \subset U_g$.

Let S be a finite generating set for Γ. We can take $\epsilon_0 > 0$ such that for any $g \in S$ and any elementary interval I of size less than ϵ_0, the element g maps affinely I into the elementary interval $g(I)$. We also take $\epsilon_1 < \epsilon_0$, such that for any elementary interval I of size less than ϵ_1 and $g \in S$, we have that $g(I)$ is an elementary interval of size less than ϵ_0.

Take $x \in \Lambda$. For every elementary interval I_n of length less than ϵ_1 containing x, we will show arguing by contradiction that there exists $g_n \in \Gamma$ such that $\epsilon_1 \leq |g_n(I_n)| \leq \epsilon_0$ and such that $g_n|_{I_n}$ is affine. Suppose that there is no such g_n in Γ. In this case, proceeding by induction on the word length of $g \in \Gamma$, we can see that for every $g \in \Gamma$, $g|_{I_n}$ is an affine map and that $|g(I_n)| < \epsilon_1$.

As the orbit $\Gamma(x)$ is dense in Λ and as Λ is infinite, the point x is a non-trivial accumulation point of $\Gamma(x)$. Hence, for every I_n containing x, there exists $h_n \in \Gamma$ such that $h_n(x) \in I_n$ and therefore the intersection $h_n(I_n) \cap I_n$ is non-trivial. Furthermore, we can suppose that $h_n(x) \neq x$. Remember that for any pair of elementary intervals I_n, I_m, either one is contained in the other one or they are disjoint.

Therefore as the elementary intervals I_n and $h_n(I_n)$ intersect, either $h_n(I_n) = I_n$ or one interval is contained strictly in the other one. If $I_n = h_n(I_n)$, then $h_n|_{I_n} = \mathrm{Id}$ because h_n is affine, contradicting that $h_n(x) \neq x$. We can then suppose that $h_n(I_n)$ is contained strictly into I_n. As the map $h_n|_{I_n}$ is affine, h_n is a contraction inside I_n and therefore h_n has exactly one contracting fixed point $y \in I_n$. This implies that y does not belong to U_{h_n}, contradicting the fact that $y = \lim_{l\to\infty} h_n^l(x) \in \Lambda \subset K_\Gamma$.

In conclusion, we found a contradiction for the non-existence of g_n and so for every elementary interval I_n of size less than ϵ_1 containing x, there exists $g_n \in \Gamma$ such that $g_n(I_n)$ is an elementary interval, $g_n|_{I_n}$ is affine and $\epsilon_1 \leq |g_n(I_n)| \leq \epsilon_0$. As there is a finite number of elementary intervals satisfying $\epsilon_1 \leq |I| \leq \epsilon_0$, there exist two intervals I_m, I_l, one contained strictly in the other (Let's say $I_l \subset I_m$) such that $g_m(I_m) = g_l(I_l)$. This implies the element $g := g_l^{-1} g_m$ is affine on I_m and hence $g|_{I_m}$ is a contraction, which implies that there is a unique contracting fixed point y for g in I_m. Hence $y \notin U_g$, but $y = \lim_{n\to\infty} g^n(x)$ and therefore y also belongs to Λ. This contradicts the inclusion $\Lambda \subset K_\Gamma \subset U_g$. Therefore, we obtain a contradiction to the fact that Λ is infinite. □

References

[1] Bleak, Collin, et al. *"Centralizers in R. Thompson's group V_n."* arXiv preprint arXiv:1107.0672 (2011).

[2] Bavard, Juliette. *"Hyperbolicite du graphe des rayons et quasi-morphismes sur un gros groupe modulaire."* arXiv preprint arXiv:1409.6566 (2014).

[3] Brin, Matthew G. *"Higher dimensional Thompson groups."* Geometriae Dedicata 108.1 (2004): 163-192.

[4] Calegari Danny, Blogpost: https://lamington.wordpress.com/2014/10/24/mapping-class-groups-the-next-generation/

[5] Cannon, James W., William J. Floyd, and Walter R. Parry. *"Introductory notes on Richard Thompson's groups."* Enseignement Mathmatique 42 (1996): 215-256.

[6] Horbez, Camille. *"A short proof of Handel and Mosher's alternative for subgroups of textOut(F_N)."* arXiv preprint arXiv:1404.4626 (2014).

[7] Horbez, Camille. *"The Tits alternative for the automorphism group of a free product."* arXiv preprint arXiv:1408.0546 (2014).

[8] Fisher, David. *"Groups acting on manifolds: around the Zimmer program."* arXiv preprint arXiv:0809.4849 (2008).

[9] J. Franks, M. Handel, *Distortion elements in group actions on surfaces*, Duke Math. J. 131 (2006), no 3, 441-468.

[10] J. Franks, M. Handel, *Periodic points of Hamiltonian surface diffeomorphisms*, Geom. Topol. 7 (2003), 713-756.

[11] Funar, Louis, and Yurii Neretin. *"Diffeomorphisms groups of Cantor sets and Thompson-type groups."* arXiv preprint arXiv:1411.4855 (2014).

[12] Farb, Benson, Alexander Lubotzky, and Yair Minsky. *"Rank-1 phenomena for mapping class groups."* Duke Mathematical Journal 106.3 (2001): 581-597.

[13] A. Fathi, F. Laudenbach, V. Poenaru, *Travaux de Thurston sur les surfaces*, Astrisque vol. 66, SMF, Paris, France.

[14] Ghys, Etienne. *"Groups acting on the circle."* Enseignement Mathematique 47.3/4 (2001): 329-408.

[15] M. Gromov *Asymptotic invariants of infinite groups. Geometric group theory. Volume 2* Cambridge Univ. Press, Cambridge(1993), 1-295

[16] de La Harpe, Pierre. *Topics in geometric group theory.* University of Chicago Press, 2000.

[17] M. W. Hirsch, *Differential Topology*, Graduate texts in Mathematics, Springer.

[18] Margulis, Gregory. *"Free subgroups of the homeomorphism group of the circle."* Comptes Rendus de l'Acadmie des Sciences-Series I-Mathematics 331.9 (2000): 669-674.

[19] Navas, Andres. *Groups of circle diffeomorphisms.* University of Chicago Press, 2011.

[20] Salazar-Daz, Olga Patricia. *"Thompsons group V from a dynamical viewpoint."* International Journal of Algebra and Computation 20.01 (2010): 39-70. APA

[21] Robert J. Zimmer, *Ergodic theory and semisimple groups*