Distortion elements of $\mathrm{Diff}_0^\infty(M)$

E. Militon

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1 Introduction

Distortion elements in non-conservative diffeomorphism groups have been studied since Franks and Handel asked the following question: *is a rotation of the circle (or of the sphere) distorted ?* Calegari and Freedman managed to give a first answer to this question. They proved that a rotation of the circle is distorted in the group $\mathrm{Diff}^1(\mathbb{S}^1)$ of C^1-diffeomorphisms. They also proved that a rotation of the sphere is distorted in the group of C^∞-diffeomorphisms of the sphere $\mathbb{S}^2$. Finally, they proved that, for any N, a homeomorphism of the N-dimensional sphere is distorted. Avila then proved that a rotation of the circle is distorted in the group $\mathrm{Diff}^1(\mathbb{S}^1)$. He actually proved that every recurrent orientation-preserving element in $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is distorted. In this article, we generalize Avila's result to any manifold.

2 Statement of the results

Let us start by introducing some definitions and notations which will be useful.

Définition 1 *Let G be a group. For a finite subset $S \subset G$ and for an element g of G is in the subgroup $< S >$ generated by S, we define:*

$$l_S(g) = \inf \left\{ n \in \mathbb{N}, \exists (\epsilon_i)_{i=1\dots n} \in \{\pm 1\}^n, \exists (s_i)_{i=1\dots n} \in S^n, g = \prod_{i=1}^{n} s_i^{\epsilon_i} \right\}.$$

An element g in G is said to be distorted *if there exists a finite subset $S \subset G$ such that:*

$$\left\{ \begin{array}{c} g \in < S > . \\ \lim_{n \to +\infty} \frac{l_S(g^n)}{n} = 0. \end{array} \right.$$

We notice that, as the sequence $(l_S(g^n))_{n\in\mathbb{N}}$ is subadditive, it suffices to show that:

$$\liminf_{n\to+\infty} \frac{l_S(g^n)}{n} = 0$$

in order to prove that the element g is distorted.

Given a differentiable manifold M, we denote by:

- $\mathrm{Homeo}_0(M)$ the group of compactly supported homeomorphisms of M compactly isotopic to the identity;
- for r in $\mathbb{N}\cup\{\infty\}$, $\mathrm{Diff}_0^r(M)$ the group of compactly supported C^r-diffeomorphisms of M compactly isotopic to the identity.

Notice that $\mathrm{Homeo}_0(M) = \mathrm{Diff}_0^0(M)$.

Here, the support of a homeomorphism f of M is the set:

$$supp(f) = \overline{\{x \in M,\ f(x) \neq x\}}.$$

We denote by d_r a distance which is compatible with the topology on $\mathrm{Diff}_0^r(M)$.

Définition 2 *A diffeomorphism f in $\mathrm{Diff}_0^\infty(M)$ is recurrent when:*

$$\liminf_{n\to+\infty} d_\infty(f^n, Id_M) = 0.$$

The purpose of this article is to show the following theorems. The first one generalizes Avila's result on diffeomorphisms of the circle (see [1]).

Theorem 1 *If M is a compact connected manifold, every recurrent element in $\mathrm{Diff}_0^\infty(M)$ is distorted.*

The method used to prove this theorem is the same as in [1]. With this method, we can indeed prove that every sequence $(f_n)_{n\in\mathbb{N}}$ of recurrent elements is simultaneously distorted which means that we can find a finite set S such that:

$$\forall n \in \mathbb{N},\ \liminf_{p\to+\infty} \frac{l_S(f_n^p)}{p} = 0.$$

With this method, we can also provide a new proof of Calegari and Freedman's result (see [4]).

Theorem 2 *(Calegari-Freedman) Every element in $\mathrm{Homeo}_0(\mathbb{S}^n)$ is distorted.*

Here also, it can be shown that every sequence of elements in $\mathrm{Homeo}_0(\mathbb{S}^n)$ is simultaneously distorted.

3 Proof of the theorems

The proof of these theorems relies on a generalisation of results by [1] to any manifold. The method and the notations are similar to those in [1] but the generalisation to higher dimensions make a crucial use of a local perfection result shown in the appendice of this paper. We will prove theorem 1 only when the dimension of M is greater than 1. The 1-dimensional case is proved in [1].

The theorems will be a direct consequence of the following lemmas.

Lemma 1 *Let M be a compact manifold with dimension greater than 1. There exist sequences $(\epsilon_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ of positive real numbers such that very sequence $(h_n)_{n\in\mathbb{N}}$ of diffeomorphisms in $\mathrm{Diff}_0^\infty(M)$ with:*

$$\forall n \in \mathbb{N},\ d_\infty(h_n, Id_M) < \epsilon_n,$$

satisfies the following property: there exists a finite set $S \subset \mathrm{Diff}_0^\infty(M)$ such that, for any integer n,

- *The diffeomorphism h_n belongs to the subgroup generated by S.*
- $l_S(h_n) \leq k_n$.

Lemma 2 *Let N be a positive integer. There exists a sequence $(k_n)_{n\in\mathbb{N}}$ of positive real numbers such that, for every sequence $(h_n)_{n\in\mathbb{N}}$ of homeomorphisms in $\mathrm{Homeo}_0(\mathbb{S}^N)$, there exists a finite set $S \subset \mathrm{Homeo}_0(\mathbb{S}^N)$ such that, for any integer n:*

- *the homeomorphism h_n belongs to the group generated by S.*
- $l_S(h_n) \leq k_n$.

Now, let us prove that these lemmas imply the theorems.

<u>Proof of the theorems.</u> Let f be a recurrent element in $\mathrm{Diff}_0^\infty(M)$. Let us consider a strictly increasing map $p : \mathbb{N} \to \mathbb{N}$ such that:

$$\begin{cases} \lim_{n\to+\infty} \frac{k_n}{p(n)} = 0 \\ d_\infty(f^{p(n)}, Id_M) \leq \epsilon_n \end{cases},$$

where the sequences $(\epsilon_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ are given by lemma 1. This lemma applied to the sequence $(f^{p(n)})_{n\in\mathbb{N}}$ shows that f is distorted. Theorem 2 can be shown the same way by using lemma 2.

4 Proof of lemmas 1 et 2

In order to prove these lemmas, we will need the following lemmas which will be shown in the next section. Lemmas 3 and 4 are analogous to lemmas 1 and 2 but deal with commutator of diffeomorphisms in $\mathbb{R}^N$. Let us denote by $B(0,2)$ the open ball of center 0 and radius 2. If f and g are diffeomorphisms of $\mathbb{R}^N$, we denote by $[f,g]$ the diffeomorphism $fgf^{-1}g^{-1}$.

Lemma 3 *Let N be a positive integer and r be an element in $\mathbb{N} \cup \{\infty\}$. There exist sequences $(\epsilon'_n)_{n\in\mathbb{N}}$ and $(k'_n)_{n\in\mathbb{N}}$ of positive real numbers which satisfy the following property. Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be sequences of diffeomorphisms in $\mathrm{Diff}^r_0(\mathbb{R}^N)$ with support included in $B(0,2)$ such that:*

$$\forall n \in \mathbb{N},\ d_r([f_n, g_n], Id_{\mathbb{R}^N}) < \epsilon'_n.$$

Then there exists a finite set $S' \subset \mathrm{Diff}^r_0(\mathbb{R}^N)$ such that, for any integer n, the diffeomorphism $[f_n, g_n]$ belongs to the group generated by S' and $l_{S'}([f_n, g_n]) \leq k'_n$.

When $r = 0$, we have a stronger lemma.

Lemma 4 *There exists a sequence $(k'_n)_{n\in\mathbb{N}}$ of positive real numbers such that, for any sequence $(\tilde{h}_n)_{n\in\mathbb{N}}$ of homeomorphisms in $\mathrm{Homeo}_0(\mathbb{R}^N)$ with support included in $B(0,2)$, there exists a finite set $S' \subset \mathrm{Homeo}_0(\mathbb{R}^N)$ such that, for any integer n, the homeomorphism $\tilde{h}_n$ belongs to the group generated by S' and $l_{S'}(\tilde{h}_n) \leq k'_n$.*

Proof of lemma 1 Denote by N the dimension of M ($N \geq 2$). We consider a finite open covering $(U_i)_{0\leq i\leq p}$ of M by open sets diffeomorphic to $\mathbb{R}^N$ whose closure is included in a chart. For any integer i between 0 and p, we chose a chart φ_i of M defined on a neighbourhood of the closure of U_i which satisfies:

$$\varphi_i(U_i) \subset B(0,2).$$

Denote by ψ a bijection from $\mathbb{N} \times \{1,...,p\} \times \{1,\ldots,4N\}$ onto $\mathbb{N}$.

We will now construct the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ using the sequence $(\epsilon'_l)_{l\in\mathbb{N}}$ given by lemma 3 in the case $r = \infty$.
By a local perfection lemma (see appendix), for every natural integer n, we can find ϵ_n so that, if a diffeomorphism h in $\mathrm{Diff}^\infty_0(M)$ satisfies $d(h, Id_M) < \epsilon_n$, then there exist two finite sequences $(f_{k,l})_{0\leq k\leq p, 1\leq l\leq 3N}$ and $(g_{k,l})_{0\leq k\leq p, 1\leq l\leq 3N}$ of diffeomorphisms in $\mathrm{Diff}^\infty_0(M)$ with support included in $B(0,2)$ such that:

$$\begin{cases} h = \prod_{k=0}^{p} \prod_{l=1}^{3N} [f_{k,l}, g_{k,l}] \\ \forall k,l \in [0,p] \times [1,3N],\ d_\infty(\varphi_k \circ f_{k,l} \circ \varphi_k^{-1}, Id_{\mathbb{R}^N}) < \epsilon'_{\psi(n,k,l)} \\ \forall k,l \in [0,p] \times [1,3N],\ d_\infty(\varphi_k \circ g_{k,l} \circ \varphi_k^{-1}, Id_{\mathbb{R}^N}) < \epsilon'_{\psi(n,k,l)} \end{cases}.$$

Let us take a sequence $(h_n)_{n\in\mathbb{N}}$ of diffeomorphisms in $\mathrm{Diff}_0^\infty(M)$ with:

$$\forall n \in \mathbb{N},\ d_\infty(h_n, Id_M) < \epsilon_n.$$

By construction of the sequence $(\epsilon_n)_{n\in\mathbb{N}}$, there exist two finite sequences $(\tilde{f}_{n,k,l})$ and$(\tilde{g}_{n,k,l})$ associated to h_n. We consider then:

$$f_{\psi(n,k,l)} = \varphi_k \circ \tilde{f}_{n,k,l} \circ \varphi_k^{-1}$$

$$g_{\psi(n,k,l)} = \varphi_k \circ \tilde{g}_{n,k,l} \circ \varphi_k^{-1}$$

and we apply lemma 3 to the sequences $(f_n)_{n\in\mathbb{N}}$ et $(g_n)_{n\in\mathbb{N}}$ to conclude by taking:

$$k_n = \sum_{k,l} k'_{\psi(n,k,l)}.$$

Proof of lemma 2 We use the following lemma which follows from a deep result by Kirby, Siebenmann and Quinn in the case of a dimension greater than 3 (see [4] lemma 6.10, [10] and [11]):

Lemma 5 *Consider two closed disks in S^n whose interiors cover S^n. Then every homeomorphism h isotopic to the identity can be written as a product of six diffeomorphisms in* $\mathrm{Homeo}_0(S^n)$ *with support included in one of those disks.*

Using this result, the same method as above with lemma 4 gives lemma 2.

5 Proof of lemmas 3 and 4

Here also, we follow the same method as in [1].

Proof of Lemma 3

Denote by F_1 a diffeomorphism in $\mathrm{Diff}_0^\infty(\mathbb{R}^N)$ which satisfies:

$$\forall x \in B(0,2),\ F_1(x) = \lambda x,$$

with $0 < \lambda < 1$.
Let F_2 be a diffeomorphism in $\mathrm{Diff}_0^\infty(\mathbb{R}^N)$ supported in $B(0,2)$ with:

$$\forall x \in B(0,1), F_2(x) = x + a,$$

with $\|a\| < 1$.
Let F_3 be a diffeomorphism in $\mathrm{Diff}_0^\infty(\mathbb{R}^N)$ supported in $B(0,2)$ which satisfies:

- the points $(F_3^n(0))_{n\in\mathbb{N}}$ are pairwise distinct;

- the sequence $(F_3^n(0))_{n\in\mathbb{N}}$ converges to $x_0 = (1, 0, \ldots, 0)$.

Consider a sequence of integers $(l_n)_{n\in\mathbb{N}}$ which increases sufficiently fast so that:

- $n \neq n' \Rightarrow F_3^n F_1^{l_n}(B(0,2)) \cap F_3^{n'} F_1^{l_{n'}}(B(0,2)) = \emptyset$.
- The diameter of the set $F_3^n F_1^{l_n}(B(0,2))$ converges to 0.

Consider $F_n = F_3^n F_1^{l_n}$, $U_n = F_3^n F_1^{l_n}(B(0,2))$ and $\hat{F}_n = F_n F_2 F_n^{-1}$.
For avery integer n, take an open set V_n with:

- $F_3^n(0) \in V_n \subset U_n$,
- $\hat{F}_n(V_n) \cap V_n = \emptyset$,

and a sequence of integers $(\tilde{l}_n)_{n\in\mathbb{N}}$ so that:

$$F_3^n F_1^{\tilde{l}_n}(B(0,2)) \subset V_n$$

Let us denote by $\tilde{F}_n$ the diffeomorphism $F_3^n F_1^{\tilde{l}_n}$.

Take two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ of siffeomorphisms in $\mathrm{Diff}_0^r(\mathbb{R}^N)$ supported in $B(0,2)$ and choose a sequence $(\epsilon'_n)_{n\in\mathbb{N}}$ which conserges sufficiently fast to 0 so as to satisfy the following property: if, for every integer n, $d_r(f_n, Id) < \epsilon'_n$ and $d_r(g_n, Id) < \epsilon'_n$, then the diffeomorphisms F_4 and F_5 defined by:

$$\begin{array}{c} \forall n \in \mathbb{N},\ F_{4|V_n} = \tilde{F}_n f_n \tilde{F}_n^{-1} \\ \forall n \in \mathbb{N},\ F_{5|V_n} = \tilde{F}_n g_n \tilde{F}_n^{-1} \\ F_{4|\mathbb{R}^N - \bigcup_{n\in\mathbb{N}} V_n} = F_{5|\mathbb{R}^N - \bigcup_{n\in\mathbb{N}} V_n} = Id_{\mathbb{R}^N - \bigcup_{n\in\mathbb{N}} V_n} \end{array}$$

are C^r in x_0.

Notice that

$$A_n = F_4 \hat{F}_n F_4^{-1} \hat{F}_n^{-1}$$

is supported in $V_n \cup \hat{F}_n(V_n)$, equals $\tilde{F}_n f_n \tilde{F}_n^{-1}$ on V_n and$\hat{F}_n \tilde{F}_n f_n^{-1} \tilde{F}_n^{-1} \hat{F}_n^{-1}$ on $\hat{F}_n(V_n)$. We can make an analogous statement for $B_n = F_5 \hat{F}_n F_5^{-1} \hat{F}_n^{-1}$ and $C_n = F_4^{-1} F_5^{-1} \hat{F}_n F_5 F_4 \hat{F}_n^{-1}$. Then:

$$A_n B_n C_n = \tilde{F}_n [f_n, g_n] \tilde{F}_n^{-1}.$$

That is why:

$$[f_n, g_n] = \tilde{F}_n^{-1} A_n B_n C_n \tilde{F}_n.$$

It suffices to take $S = \{F_1, F_2, F_3, F_4, F_5\}$ to end the proof.

Proof of lemma 4 Notice that, we need the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ to make the diffeomorphisms F_4 and F_5 regular. In the case of C^0 regularity, this problem doesn't occur anymore, which allows us to show lemma 4 in the case where each h_n is a commutator. Hence, to conclude the proof of lemma 4, it suffices to show the following lemma.

Lemma 6 *Every compactly supported homeomorphism of $\mathbb{R}^N$ isotopic to the identity is a commutator.*

Proof (taken from [3], proof of theorem 1.1.3) Denote by φ a homeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^n)$ whose restriction to $B(0,2)$ is:

$$\begin{array}{ccc} B(0,2) & \to & \mathbb{R}^N \\ x & \mapsto & \frac{x}{2} \end{array}.$$

For any integer n, consider the set:

$$A_n = \left\{ x \in \mathbb{R}^N, \frac{1}{2^{n+1}} \leq \|x\| \leq \frac{1}{2^n} \right\}.$$

Let h be a homeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^N)$. As any diffeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^N)$ is conjugated to a homeomorphism supported in the interior of A_0, we may suppose that h is supported in the interior of A_0. We define the homeomorphism g by:

- $g = Id$ outside $B(0,1)$.
- for any i, $g_{|A_i} = \varphi^i h_i \varphi^{-i}$.
- $g(0) = 0$.

Then we have:

$$h = [g, \varphi].$$

A Appendix: local perfection of $\mathrm{Diff}_0^\infty(M)$

In this appendix, we show the following result which was used during the proof of lemma 1:

Theorem 3 *Let M be a compact connected n-dimensional manifold. Fix a covering $(U_k)_{0\leq k\leq p}$ of M by open sets whose closure is diffeomorphic to the* *unit closed ball in $\mathbb{R}^n$. Then, for every neighbourhood Ω of the identity in $\mathrm{Diff}_0^\infty(M)$,, there exists a neighbourhood Ω' of the identity in $\mathrm{Diff}_0^\infty(M)$ such that, for any diffeomorphism f in Ω', there exist finite sequences of diffeomorphisms $(f_{k,l})_{0\leq k\leq p, 1\leq l\leq 3n}$ and $(g_{k,l})_{0\leq k\leq p, 1\leq l\leq 3n}$ in Ω such that:*

$$f = \prod_{k=0}^{p}\prod_{l=1}^{3n}[f_{k,l}, g_{k,l}]$$

and the diffeomorphisms $f_{k,l}$ and $g_{k,l}$ are supported in U_k.

This proof is an elementary realisation of Haller and Teichmann's idea to write a diffeomorphism as a product of diffeomorphisms which preserve some foliations (see [8]). This proof relies heavily on Herman's KAM theorem on diffeomorphisms of the circle. Notice that this property proves the perfectness of the group $\mathrm{Diff}_0^\infty(M)$, hence its simplicity: it gives a different proof of this fact from Thurston's proof and Mather's proof (see [3] and [2]). The proof given here also shows the local perfection of $\mathrm{Diff}_0^\infty(\mathbb{R}^n)$ for $n \geq 2$ but does not work for the 1-dimensional case.

By the fragmentation lemma (see [3] theorem 2.2.1 for a proof), for any $\eta > 0$, there exists $\alpha > 0$ such that, for every diffeomorphism h in $\mathrm{Diff}_0^\infty(M)$, if $d(h, Id_M) < \alpha$ then there exists a finite sequence $(h_i)_{0\leq i\leq p}$ of diffeomorphisms in $\mathrm{Diff}_0^\infty(M)$ such that:

$$\begin{cases} supp(h_i) \subset U_i \\ h = \prod_{i=0}^{p} h_i \\ \forall i \in [0,p]\cap\mathbb{N},\ d(h_i, Id_M) < \eta \end{cases}.$$

Take two open sets U, V in $\mathbb{R}^n$, where U is a cube whose closure is included in V and a neighbourhood Ω of the identity in $\mathrm{Diff}_0^\infty(V)$. We will show the following statement: there exists a neighbourhood Ω' of the identity in $\mathrm{Diff}_0^\infty(U)$ such that, for any diffeomorphism f in Ω' there exist diffeomorphisms $f_1, f_2, \ldots, f_{3n}$ in Ω such that:

$$f = [f_1, f_2] \circ [f_3, f_4] \circ \ldots \circ [f_{3n-1}, f_{3n}].$$

To show this last property, the startegy will be as follows: we will start by writing a diffeomorphism f close to the identity as a product of n diffeomorphisms which preserve each a foliation of U by straight lines. Each of these foliations can be extended as a foliation by circles of an annulus included in V. Then it will suffice to apply Herman's theorem on diffeomorphisms of the circle to conclude that each of the diffeomorphisms appearing in this decomposition can be written as a product of three commutators of small diffeomorphisms supported in V.

Let us detail this now. For any integer k between 1 and n, we denote by F_k the foliation by the straight lines parallel to the kth axis of coordinate. We denote by $\mathrm{Diff}_0^\infty(U,F_k)$ the group of compactly supported diffeomorphisms of U compactly isotopic to the identity which preserve the foliation F_k. Let us define by induction on k a map $\phi_k : \mathrm{Diff}_0^\infty(U) \to \mathrm{Diff}_0^\infty(U,F_k)$ defined and continuous on a neighbourhood of the identity such that $\phi_k(Id_U) = Id_U$ and such that, for a diffeomorphism f sufficiently close te the identity:

$$p_k \circ f \circ \phi_1(f)^{-1} \circ \phi_2(f)^{-1} \circ \ldots \circ \phi_k(f)^{-1} = p_k,$$

where p_k is the kth projection of $\mathbb{R}^n$. Suppose that $\phi_1, \phi_2, \ldots, \phi_k$ have been defined. For a diffeomorphism f in $\mathrm{Diff}_0^\infty(U)$ close to the identity and for x in U, consider:

$$f \circ \phi_1(f)^{-1} \circ \phi_2(f)^{-1} \circ \ldots \circ \phi_k(f)^{-1}(x) = (x_1, \ldots, x_k, f_{k+1}(x), f_{k+2}(x), \ldots, f_n(x)).$$

Then let us define ϕ_{k+1} the following way:

$$\forall x \in U, \phi_{k+1}(f)(x) = (x_1, \ldots, x_k, f_{k+1}(x), x_{k+2}, \ldots, x_n)$$

for f sufficiently close to the identity. This ends the induction. The maps ϕ_k are C^∞, map the identity to the identity and satisfy the following property:

$$f = \phi_n(f) \circ \phi_{n-1}(f) \circ \ldots \circ \phi_1(f).$$

Let us fix an integer k. As planned, we consider an embedding:

$$\psi_k : \mathbb{R}^{k-1} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{n-k} \to V$$

which maps $(-1,1)^{k-1} \times (-\frac{1}{4}, \frac{1}{4}) \times (-1,1)^{n-k}$ into U and the foliation by straight lines parallel to the kth axis of coordinate of $(-1,1)^{k-1} \times (-\frac{1}{4}, \frac{1}{4}) \times (-1,1)^{n-k}$ on the foliation by straight lines parallel to the kth axis of coordinate of U. The diffeomorphism $\psi_k^{-1} \circ \phi_k(f) \circ \psi_k$ can be identified with a family (with $n-1$ parameters) of diffeomorphisms of the circle. We write, for points (x,t,y) in $\mathbb{R}^{k-1} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{n-k}$:

$$\psi_k^{-1} \circ \phi_k(f) \circ \psi_k(x,t,y) = (x, g(x,y)(t), y),$$

where $g : \mathbb{R}^{n-1} \to \mathrm{Diff}_0^\infty(\mathbb{R}/\mathbb{Z})$ is a continuous map with:

$$g(x,y) = Id$$

when $(x,y) \notin [-1,1]^{n-1}$.

It remains to write $g(x,y)$ as a product of commutators of elements in $\mathrm{Diff}_0^\infty(\mathbb{R}/\mathbb{Z})$ close to the identity which depend in a C^∞ way on x and y, whose derivatives in the x and y directions are small and which are the identity when $(x,y) \notin (-2,2)^{n-1}$.

The following lemma was shown by Haller and Teichmann (see [9], example 3):

Lemma 7 *For any neighbourhood W' of the identity in $\mathrm{Diff}_0^\infty(\mathbb{R}/\mathbb{Z})$, there exists a neighbourhood W of the identity, diffeomorphisms A, B and C in W' and C^∞ maps*

$$a, b, c : W \to \mathrm{Diff}_0^\infty(\mathbb{R}/\mathbb{Z})$$

which map the identity to the identity and satisfy, for any diffeomorphism h in W:

$$h = [a(h), A][b(h), B][c(h), C].$$

Remark: the proof of this result relies on the KAM theorem by Herman on diffeomorphisms of the circle. Moreover, we may suppose that A, B and C are in $PSL_2(\mathbb{R}) \subset \mathrm{Diff}_0^\infty(\mathbb{R}/\mathbb{Z})$.

End of the proof of the theorem Denote by $\lambda : \mathbb{R}^{n-1} \to \mathbb{R}$ a C^∞ map supported in $(-2,2)^{n-1}$ which is equal to 1 on a neighbourhood of $[-1,1]^{n-1}$. Then we choose a neighbourhood W' of the identity in $\mathrm{Diff}_0^\infty(\mathbb{R}/\mathbb{Z})$ such that the following property is satisfied. For any diffeomorphism D in W', if we denote by $\tilde{D}$ the map defined by:

$$\forall (x,t,y) \in \mathbb{R}^{k-1} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{n-k},\ \tilde{D}(x,t,y) = (x, \lambda(x,y)D(t) + (1-\lambda(x,y))t, y),$$

then $\tilde{D}$ is a compactly supported diffeomorphism of $\mathbb{R}^{k-1} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{n-k}$ and $\psi_k \circ \tilde{D} \circ \psi_k^{-1}$ belongs to the neighbourhood Ω of the identity in $Diff_0^\infty(V)$.

When the diffeomorphism f is sufficiently close to the identity, $g(x,y)$ belongs to the neighbourhood W of the identity given by the preceding lemma for any x and y. Then, for any $(x,y) \in \mathbb{R}^{k-1} \times \mathbb{R}^{n-k}$:

$$g(x,y) = [a(g(x,y)), A][b(g(x,y)), B] \circ [c(g(x,y)), C]$$

and

$$g(x,y) = [a(g(x,y)), \tilde{A}(x,y)][b(g(x,y)), \tilde{B}(x,y)] \circ [c(g(x,y)), \tilde{C}(x,y)].$$

Indeed, on $[-1,1]^{n-1}$, we have $\tilde{A}(x,y) = A$, $\tilde{B}(x,y) = B$ et $\tilde{C}(x,y) = C$, and outside $[-1,1]^{n-1}$ both members of the equality equal the identity. That is why the diffeomorphism $\psi_k^{-1} \circ \phi_k(f) \circ \psi_k$ can be written as a product of 3 commutators of diffeomorphisms in Ω, when f is sufficiently close to the identity. Thus a diffeomorphism f close to the identity can be written as a product of $3n$ commutators of diffeomorphisms in Ω.

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