Decay rates for the damped wave equation on the torus

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The damped wave equation

Resolvent estimates and a priori bounds

Smooth damping

Rough damping
- $M$ a compact connected Riemannian manifold (or a bounded domain in $\mathbb{R}^n$), $\Delta$ the Laplace-Beltrami operator on $M$.

- Linear damped wave equation on $M$:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial^2}{\partial t^2} u - \Delta u + b(x) \partial_t u = 0 \quad \text{in } \mathbb{R}^+ \times M, \\
(u, \partial_t u)|_{t=0} = (u_0, u_1) \quad \in H^1(M) \times L^2(M).
\end{array} \right.
\]

(DWE)

- Damping coefficient $b(x) \geq 0$:
  - either $b \in C^0(M)$, and $\omega := \{b > 0\}$,
  - or $b = 1_\omega$, $\omega$ open.

- Energy of a solution:

\[
E(u, t) = \frac{1}{2}(\|\partial_t u(t)\|_{L^2(M)}^2 + \|\nabla u(t)\|_{L^2(M)}^2).
\]

- Dissipation identity

\[
\frac{d}{dt} E(u, t) = -\int_M b|\partial_t u|^2 \, dx \leq 0
\]

- If $\omega \neq \emptyset$, then $E(u, t) \to 0$ as $t \to +\infty$. 
• $M$ a compact connected Riemannian manifold (or a bounded domain in $\mathbb{R}^n$), $\Delta$ the Laplace-Beltrami operator on $M$.

• Linear damped wave equation on $M$:

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\begin{cases}
\partial_t^2 u - \Delta u + b(x)\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\
(u, \partial_t u)|_{t=0} = (u_0, u_1) & \in H^1(M) \times L^2(M).
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• If $\omega \neq \emptyset$, then $E(u, t) \to 0$ as $t \to +\infty$.

At which rate?
A first question: uniform decay

**Definition**
Uniform decay for (DWE) if \( \exists F(t) \xrightarrow{t \to \infty} 0 \) such that \( \forall (u_0, u_1) \in H^1 \times L^2, \)
\[
E(u, t) \leq F(t)E(u, 0).
\]

**Remark:** Uniform decay for (DWE) implies \( F(t) \leq Ce^{-\gamma t} \) for some \( C, \gamma > 0. \)

**Definition (Rauch-Taylor '74, Bardos-Lebeau-Rauch '92)**
\( \omega \) satisfies GCC in \( M \iff \) every geodesic (ray of geometric optics) traveling at speed 1 in \( M \) meets \( \omega \) in finite time.

**Theorem (Rauch Taylor '74, Bardos Lebeau Rauch '92, Burq Gérard '97)**
\( \omega \) satisfies GCC \( \iff \) uniform decay for (DWE) (general case)
\( \iff \) \( \omega \) satisfies GCC \( \iff \) uniform decay for (DWE) (if \( b \in C^0(M) \))
A first question: uniform decay

Definition
Uniform decay for (DWE) if $\exists F(t) \underset{t \to \infty}{\longrightarrow} 0$ such that $\forall (u_0, u_1) \in H^1 \times L^2,$

$$E(u, t) \leq F(t)E(u, 0).$$

Remark: Uniform decay for (DWE) implies $F(t) \leq Ce^{-\gamma t}$ for some $C, \gamma > 0.$

Definition (Rauch-Taylor '74, Bardos-Lebeau-Rauch '92)
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$\omega$ satisfies GCC $\iff$ uniform decay for (DWE) (general case) (if $b \in C^0(M)$)

What happens if GCC is not satisfied?
A first question: uniform decay (on the torus)

GCC is satisfied ($\Rightarrow$ uniform decay)
A first question: uniform decay (on the torus)

GCC is satisfied
\(\implies\) uniform decay

GCC is NOT satisfied
\(\implies\) NO uniform decay
A weaker notion: semi-uniform decay

**Definition**

(DWE) is (semi-uniformly) stable at rate $f(t)$, $f(t) \rightarrow_{t \rightarrow \infty} 0$ if $\exists C > 0$ such that $\forall (u_0, u_1) \in H^2 \times H^1$,

$$E(u, t) \leq C(f(t))^2 \left( \|u_0\|_{H^2(M)}^2 + \|u_1\|_{H^1(M)}^2 \right), \text{ for all } t > 0.$$  

**Theorem (Lebeau '96)**

- If $\omega \neq \emptyset$, then $f(t) = \frac{1}{\log(2+t)}$.
- This is optimal in general. Ex: $M = S^2$ and $\omega \cap N = \emptyset$, where $N$ is a neighborhood of an equator of $S^2$. 

A weaker notion: semi-uniform decay

Definition

(DWE) is (semi-uniformly) stable at rate $f(t)$, $f(t) \to 0$ if $\exists C > 0$ such that $\forall (u_0, u_1) \in H^2 \times H^1$,

$$E(u, t) \leq C(f(t))^2 \left( \|u_0\|_{H^2(M)}^2 + \|u_1\|_{H^1(M)}^2 \right), \text{ for all } t > 0.$$  

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- If $\omega \neq \emptyset$, then $f(t) = \frac{1}{\log(2+t)}$.
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Intermediate situations

Two extreme situations:

• Uniform decay ⇔ \( \omega \) satisfies GCC
• Decay at rate \( f(t) = \frac{1}{\log(2+t)} \) ⇔ \( \omega \neq \emptyset \).

Some intermediate situations:

• Liu-Rao '05: \( M \) is a square and \( \omega \) contains a vertical strip. trapped trajectories = family of parallel geodesics constituted by vertical lines.
  Energy decay at rate \( \left( \frac{\log(t)}{t} \right)^{\frac{1}{2}} \).

• Burq and Hitrik '07: \( M \) is a partially rectangular domain and \( \omega \) contains a neighborhood of the non-rectangular part. Energy decay at rate \( \left( \frac{\log(t)}{t} \right)^{\frac{1}{2}} \).

• Burq and Hitrik '07: if moreover
  • \( b \in C^\infty \),
  • \( b = b(x_1) \) (invariance in one direction),
  • + Assumption on the vanishing rate of \( b \).
  Then, the energy decays at rate \( 1/t^{1-\varepsilon} \).

\( \leadsto \) geodesic flow enjoys linear unstability properties around the trapped set
\( \leadsto \) On the torus, we expect similar polynomial decay.
Decay rates and resolvent estimates

\[(\text{DWE}) \iff \partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \mathcal{A} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & -b \end{pmatrix} \]
\[\iff \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{t\mathcal{A}} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.\]

**Lemma**

*The spectrum of \( \mathcal{A} \) contains only isolated eigenvalues and we have*

\[\text{Sp}(\mathcal{A}) \setminus \{0\} \subset (-\|b\|_{L^\infty(M)}, 0) + i\mathbb{R}.\]

We set \( P(s) = -\Delta - s^2 + isb, \ s \in \mathbb{R}.\)

**Theorem (Lebeau '96, Batty-Duyckaerts '08, Burq-Hitrik '07, Borichev-Tomilov '10)**

*For all \( \alpha > 0, \) following assertions are equivalent:*

\[\text{System (DWE) is stable at rate } \frac{1}{t^\alpha},\]
\[\|(is - \mathcal{A})^{-1}\|_{\mathcal{L}(H^1 \times L^2)} \leq C|s|^{\frac{1}{\alpha}}, \quad \forall s \in \mathbb{R}, |s| \geq s_0,\]
\[\|P(s)^{-1}\|_{\mathcal{L}(L^2)} \leq Cs^{\frac{1}{\alpha} - 1}, \quad \forall s \geq s_0.\]
A priori upper bound

Proposition

**Suppose that there exists** $T > 0$, $C > 0$ **such that**

$$
\|u_0\|_{L^2(M)}^2 \leq C \int_0^T \|\sqrt{b} e^{it\Delta} u_0\|_{L^2(M)}^2 \, dt, \quad \forall u_0 \in L^2(M),
$$

*(Observability for Schrödinger). Then System (DWE) is stable at rate $\frac{1}{\sqrt{t}}$. *

For instance on the torus $\mathbb{T}^2 := (\mathbb{R}/2\pi \mathbb{Z})^2$: $\omega \neq \emptyset \implies$ Observability for Schrödinger (Jaffard '90) $\implies$ always decay at rate $\frac{1}{\sqrt{t}}$ (at least).

**Figure:** Torus $\mathbb{T}^2 := (\mathbb{R}/2\pi \mathbb{Z})^2$ and damping region $\omega = \{b > 0\}$. 
A priori upper bound

Proof.

Observability for Schrödinger in some time $T > 0$

\[
\exists C > 0 \text{ s.t.} \quad \|u\|_{L^2}^2 \leq C \left( \|(-\Delta - s^2)u\|_{L^2}^2 + \|\sqrt{bu}\|_{L^2}^2 \right), \quad \forall s \in \mathbb{R}, u \in H^2
\]

\[
\leq C \left( \|(-\Delta - s^2 + isb - isb)u\|_{L^2}^2 + \|\sqrt{bu}\|_{L^2}^2 \right)
\]

\[
\leq C \left( \|P(s)u\|_{L^2}^2 + s^2 \|\sqrt{bu}\|_{L^2}^2 \right) \quad \forall s \geq s_0, u \in H^2.
\]

Skew-adjoint part of $P(s)$: $\text{Im} \left( (P(s)u, u)_{L^2} = s(bu, u)_{L^2} \right)$

\[
\implies s \|\sqrt{bu}\|_{L^2}^2 \leq \|P(s)u\|_{L^2}^2 \|u\|_{L^2}^2
\]

\[
\implies s^2 \|\sqrt{bu}\|_{L^2}^2 \leq \frac{C}{\varepsilon} s^2 \|P(s)u\|_{L^2}^2 + \varepsilon \|u\|_{L^2}^2.
\]

\[
\implies \|u\|_{L^2}^2 \leq Cs^2 \|P(s)u\|_{L^2}^2, \text{ i.e. polynomial stability at rate } \frac{1}{\sqrt{t}}. \quad \square
\]
A priori lower bound

Torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$, when GCC is “strongly violated”.

**Proposition**

*Suppose that there exists $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, such that*

$$\bar{\omega} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \emptyset.$$  

*Then there exist $C > 0$ and $(s_n)_{n \in \mathbb{N}}, s_n \to +\infty$ such that*

$$\|P(s_n)^{-1}\|_{\mathcal{L}(L^2)} \geq C.$$  

**NO GCC $\Rightarrow$ decay at rate at most $1/t$.**
A priori lower bound (simple quasimodes)

Proof (simple quasimodes).

We set \( \varphi_n(x_1, x_2) = \chi(x_1)e^{inx_2} \) and \( s_n = n \).

\[
P(s_n)\varphi_n = -\Delta(\chi(x_1)e^{inx_2}) - n^2\chi(x_1)e^{inx_2} + ib\chi(x_1)e^{inx_2} = \chi''(x_1)e^{inx_2}.
\]

Hence \( \|P(s_n)\varphi_n\|_{L^2} \sim cte \sim \|\varphi_n\|_{L^2} \) and \( \|P(s_n)^{-1}\|_{L(L^2)} \geq C \).
As soon as GCC is (strongly) not satisfied, we have

\[ 1 \lesssim \| P(s)^{-1} \|_{L^2} \lesssim s \]

Best decay rate \( \longrightarrow \) between \( 1/\sqrt{t} \) and \( 1/t \).
As soon as GCC is (strongly) not satisfied, we have

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Best decay rate \( \longrightarrow \) between \( 1/\sqrt{t} \) and \( 1/t \).

Depending on what?
Smooth damping coefficients

Theorem
Suppose that $\omega \neq \emptyset$, that $b \in C^\infty (\mathbb{T}^2)$, and that there exist $\varepsilon \in (0, \varepsilon_0)$ and $C > 0$ such that
\[ |\nabla b(x)| \leq C b^{1-\varepsilon}(x), \quad \text{for } x \in \mathbb{T}^2. \]  
(1)
Then, there exist $C > 0$ and $s_0 \geq 0$ such that for all $s \geq s_0$,
\[ \|P(s)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq Cs^\delta, \quad \text{with } \delta = 4\varepsilon \]
Hence, in this situation, (DWE) is stable at rate $\frac{1}{t^{1+\delta}}$. 

• generalizes Burq-Hitrik '07 in the case of non-invariant damping function $b$ with several trapped directions.
• (1) = local assumption in a neighborhood of $\partial \omega$.
• Ex: $b \sim e^{-x^\gamma}$, then $b' \sim \log(\frac{1}{b})^{\gamma+1}$
\[ \leq 1 b \varepsilon \forall \varepsilon > 0 \]
b on $\omega$; (1) is satisfied for all $\varepsilon > 0$.
• The a priori lower bound $\frac{1}{t}$ is sharp whatever the shape of $\omega$!
Smooth damping coefficients

Theorem
Suppose that $\omega \neq \emptyset$, that $b \in \mathcal{C}^\infty(\mathbb{T}^2)$, and that there exist $\varepsilon \in (0, \varepsilon_0)$ and $C > 0$ such that

$$|\nabla b(x)| \leq Cb^{1-\varepsilon}(x), \quad \text{for } x \in \mathbb{T}^2. \quad (1)$$

Then, there exist $C > 0$ and $s_0 \geq 0$ such that for all $s \geq s_0$,

$$\|P(s)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq Cs^\delta, \quad \text{with } \delta = 4\varepsilon$$

Hence, in this situation, (DWE) is stable at rate $\frac{1}{t^{1+\delta}}$.

- generalizes Burq-Hitrik '07 in the case of non-invariant damping function $b$ with several trapped directions.
- $(1) =$ local assumption in a neighborhood of $\partial \omega$.
- Ex: $b \sim e^{-\frac{1}{x^\gamma}}$, then $b' \sim \log\left(\frac{1}{b}\right)^{\frac{\gamma+1}{\gamma}} b$ on $\omega$; (1) is satisfied for all $\varepsilon > 0$.
- The $a \text{ priori}$ lower bound $1/t$ is sharp whatever the shape of $\omega$!
Smooth damping coefficients: idea of the proof

Prove $\|u\|_{L^2(T^2)} \leq Cs^\delta \|(-\Delta - s^2 + isb)u\|_{L^2(T^2)}$ for all $s \geq s_0$, $u \in H^2(T^2)$.

\[ \uparrow \text{ with } h = 1/s \]

Prove $\|u\|_{L^2(T^2)} \leq \frac{C}{h^{2+\delta}} \|(-h^2\Delta - 1 + ihb)u\|_{L^2(T^2)}$ for all $h \leq h_0$, $u \in H^2(T^2)$.

= $P_h$

Strategy of Lebeau '96: contradiction argument.
We suppose that this is false. There exists $0 < h_n \to 0$ and $u_n \in H^2(T^2)$ such that

\[
\left\{ \begin{array}{l}
\|u_n\|_{L^2(T^2)} = 1, \\
\|P_{h_n}u_n\|_{L^2(T^2)} = o(h_n^{2+\delta}).
\end{array} \right.
\]

Aim: prove that $\|u_n\|_{L^2(T^2)} \to 0$. 
Smooth damping coefficients: idea of the proof

- Skip the index \( n \)

\[
\begin{align*}
    h & \to 0^+, \\
    \|u_h\|_{L^2} & = 1, \\
    \|P_h u_h\|_{L^2} & = o(h^{2+\delta}), \\
    \|\sqrt{b}u_h\|_{L^2} & = o(h^{\frac{1+\delta}{2}}). \quad \text{(Bonus)}
\end{align*}
\]

\(\rightsquigarrow\) skew-adjoint part \( h\|\sqrt{b}u_h\|^2_{L^2} = \text{Im}(P_h u_h, u_h)_{L^2} = o(h^{2+\delta}). \)

- Semiclassical measure associated to \( (h, u_h) \): Up to a subsequence, there exists \( \mu \in \mathcal{M}^+(T^*\mathbb{T}^2) \) such that

\[
(\text{Op}_h(a)u_h, u_h)_{L^2(\mathbb{T}^2)} \to \langle \mu, a \rangle \quad \text{for all } a = a(x, \xi) \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2).
\]

- characterizes the defect of convergence to zero for \( (u_h) \).
Smooth damping coefficients: idea of the proof

Properties of the sequence:

\[
\begin{align*}
  h &\to 0^+, \\
  \|u_h\|_{L^2} &= 1, \\
  \|P_h u_h\|_{L^2} &= o(h^{2+\delta}), \\
  \|\sqrt{b} u_h\|_{L^2} &= o(h^{1+\delta}). \quad \text{(Bonus: skew-adjoint part of } P_h) \\
\end{align*}
\]

First properties of the semiclassical measure:

Lemma

We have

1. \( \text{supp}(\mu) \subset \mathbb{T}^2 \times \{|\xi|^2 = 1\} = S^* \mathbb{T}^2, \)
2. \( \mu(T^* \mathbb{T}^2) = 1, \)
3. \( "\mu(x + \tau \xi, \xi) = \mu(x, \xi)" \), for all \( \tau \in \mathbb{R}, \)
4. \( \langle \mu, b \rangle = 0. \)

Remark: Here, we only use \( \|P_h u_h\|_{L^2} = o(h^1)! \)
Smooth damping coefficients: idea of the proof

Properties of the sequence:

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\begin{aligned}
  & h \to 0^+, \\
  & \| u_h \|_{L^2} = 1, \\
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First properties of the semiclassical measure:

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1. \( \text{supp}(\mu) \subset \mathbb{T}^2 \times \{|\xi|^2 = 1\} = S^*\mathbb{T}^2, \)
2. \( \mu(T^*\mathbb{T}^2) = 1, \)
3. \( \mu(x + \tau \xi, \xi) = \mu(x, \xi)”, \) for all \( \tau \in \mathbb{R}, \)
4. \( \langle \mu, b \rangle = 0. \)

Remark: Here, we only use \( \| P_h u_h \|_{L^2} = o(h^1)! \)

**Goal:** prove that \( \mu \equiv 0 \sim \) obtain a contradiction with \( \mu(T^*\mathbb{T}^2) = 1. \)
Smooth damping coefficients: idea of the proof

Lemma

\[ \mu = \sum_{\Gamma \text{ rational direction}} \mu|_{T^2 \times \Gamma} \quad \text{where } \mu|_{T^2 \times \Gamma} \in \mathcal{M}^+(T^*\mathbb{T}^2) \text{ is invariant.} \]

- “\( \Gamma \) rational direction” if \( \Gamma = \mathbb{R}\xi_0 \) for \( \xi_0 \in \mathbb{R}^2 \setminus \{0\} \) such that \( k \cdot \xi_0 = 0 \) for some \( k \in \mathbb{Z}^2 \setminus \{0\} \).
  \[ \implies \Gamma = \mathbb{R}\xi_0 \text{ is periodic in } T^2. \]
- If \( \Gamma \) is an irrational direction
  \[ \implies \Gamma = \mathbb{R}\xi_0 \text{ is dense in } T^2. \]
  \[ \mu|_{T^2 \times \Gamma} \text{ is invariant and vanishes on } \omega \implies \mu|_{T^2 \times \Gamma} \equiv 0. \]
The damped wave equation  
Resolvent estimates and a priori bounds  
Smooth damping  
Rough damping

Smooth damping coefficients: idea of the proof

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\[ \mu = \sum_{\Gamma \text{ rational direction}} \mu|_{T^2 \times \Gamma} \quad \text{where } \mu|_{T^2 \times \Gamma} \in \mathcal{M}^+ \left( T^*T^2 \right) \text{ is invariant.} \]

- “\( \Gamma \) rational direction” if \( \Gamma = \mathbb{R} \xi_0 \) for \( \xi_0 \in \mathbb{R}^2 \setminus \{0\} \) such that \( k \cdot \xi_0 = 0 \) for some \( k \in \mathbb{Z}^2 \setminus \{0\} \).
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- If \( \Gamma \) is an irrational direction
  \[ \implies \Gamma = \mathbb{R} \xi_0 \text{ is dense in } T^2. \]
  \[ \mu|_{T^2 \times \Gamma} \text{ is invariant and vanishes on } \omega \implies \mu|_{T^2 \times \Gamma} \equiv 0. \]
Smooth damping coefficients: idea of the proof

We fix $\Gamma$, and want to prove that $\mu_\Gamma := \mu|_{T^2 \times \Gamma}$ vanishes. Take for instance $\Gamma = \{\xi_1 = 0\} = \mathbb{R}\xi_2$, $\xi_2 = (0,1)$.

THREE steps to prove that $\mu_\Gamma \equiv 0$. 
Smooth damping coefficients: idea of the proof

**STEP 1:** Understand the possible concentration rate of the sequence $u_h$ towards $\Gamma$:

**Lemma**

For all $0 < \alpha \leq \frac{1+\delta}{2}$, we have

$$\langle \mu_\Gamma, a \rangle = \lim_{h \to 0} \left( \text{Op}_h \left( a(x, \xi) \chi(\xi_1/h^\alpha) \right) u_h, u_h \right)_{L^2}$$

Idea of proof: consider 2-microlocal semiclassical measures (Miller '97, Fermanian-Kammerer '05) at scale $\alpha$: $\nu_\alpha$:

$$\langle \nu_\alpha, a(x_1, \xi, \xi_1) \rangle = \lim_{h \to 0} \left( \text{Op}_h \left( a(x_1, \xi, \frac{\xi_1}{h^\alpha}) \left( 1 - \chi(\xi_1/h^\alpha) \right) \right) u_h, u_h \right)_{L^2}$$

- $\langle \nu_\alpha, \langle b \rangle_\Gamma \rangle = 0$, ($\langle b \rangle_\Gamma$ average of $b$ in the direction $\Gamma$).
- Transverse propagation law: $\partial_{x_1} \nu_\alpha = 0$ if $0 < \alpha \leq \frac{1+\delta}{2}$.
- Hence $\nu_\alpha = 0$ for $0 < \alpha \leq \frac{1+\delta}{2}$. 
Smooth damping coefficients: idea of the proof

STEP 2: construction of a particular cutoff function:

Proposition

Set $w_h = \text{Op}_h (\chi(\xi_1/h^\alpha)) u_h$. For $\delta = 4\varepsilon$, $\varepsilon < \varepsilon_0$, there exists $\chi_h \in \mathcal{C}^\infty$ valued in $[0, 1]$, such that

1. $\chi_h = \chi_h(x_1)$ does not depend $x_2$,
2. $b \leq c_0 h$ on $\text{supp}(\chi_h)$,
3. $\| (1 - \chi_h) w_h \|_{L^2(\mathbb{T}^2)} = o(1)$.

• If $b$ is invariant in one direction, $\chi_h = \chi(\frac{b}{c_0 h})$ (Burq-Hitrik '07).
• Assumptions on $b$ used here, together with the $o(h^{2+\delta})$. 
**Smooth damping coefficients: idea of the proof**

**STEP 3:** possible concentration rate for the sequence $w_h$ towards $\Gamma$:

**Lemma**

we have $\langle \mu_\Gamma, a \rangle = \lim_{R \to +\infty} \lim_{h \to 0} \langle \text{Op}_h (a(x, \xi) \chi(\xi_1/Rh^1)) \rangle_{L^2} w_h, w_h$.

Idea of proof: consider 2-microlocal semiclassical measures $\tilde{\nu}_1$ (Fermanian-Kammerer '00, Anantharaman-Macia '11) at scale 1 associated to $w_h$:

- $\langle \tilde{\nu}_1, \langle b \rangle_\Gamma \rangle = 0$, ($\langle b \rangle_\Gamma$ average of $b$ in the direction $\Gamma$).
- Transverse propagation law: $\partial_{x_1} \tilde{\nu}_1 = 0$ (uses $\chi_h$ in an essential way).
- Hence $\tilde{\nu}_1 = 0$.

Consequences (Anantharaman-Macia '11):

- $\mu_\Gamma = 0$,
- $\implies \mu = 0$, (this holds for any $\Gamma$),
- $\implies$ contradiction with $\mu(T^*\mathbb{T}^2) = 1$. 
Rough damping: a particular case

Figure: \( b(x_1, x_2) = \kappa \mathbb{1}_{(0, \sigma)}(x_1) \) characteristic function of a strip

The spectrum has a particular shape (see Asch-Lebeau '03)
Rough damping: localization of the spectrum

Proposition
For $\alpha > 0$, the following assertions are equivalent:

- System (DWE) is stable at rate $\frac{1}{t^\alpha}$,

- $\| (is - A)^{-1} \|_{L(H^1 \times L^2)} \leq C |s|^{1/\alpha}$ for all $s \in \mathbb{R}$, $|s| \geq s_0$ (Batty Duyckaerts '08, Borichev Tomilov '10),

- $\| (z - A)^{-1} \|_{L(H^1 \times L^2)} \leq C |\text{Im}(z)|^{1/\alpha}$ for all $z \in \mathbb{C}$, satisfying $|z| \geq s_0$ and $|\text{Re}(z)| \leq \frac{1}{C|\text{Im}(z)|^{1/\alpha}}$.

Consequence: Decay at rate $\frac{1}{t^\alpha} \implies$ No spectrum in

$$C(\alpha, K) := \left\{ z \in \mathbb{C}, 0 \geq \text{Re}(z) \geq \frac{1}{K|\text{Im}(z)|^{1/\alpha}} \right\},$$

for some $K > 0$. 
Rough damping: the shape of the spectrum

Figure: Full spectrum of the operator $A_h$

Discretization $N = 50$, damping $b(x_1, x_2) = 21_{(0,1/2)}(x_1)$. 
Rough damping: the shape of the spectrum

Figure: Full spectrum of the operator $A_h$

Discretization $N = 40$, damping $b(x_1, x_2) = 21_{(0,0.3)}(x_1)$. 
Rough damping: localization of the spectrum

Theorem (Nonnenmacher ’13)

There exists a sequence \((z_n)_{n \in \mathbb{N}} \in \text{Sp}(A)^{\mathbb{N}}\) such that \(|z_n| \to \infty\) and

\[|\text{Re}(z_n)| \leq \frac{1}{C|\text{Im}(z_n)|^{\frac{3}{2}}}.\]

Decay at rate \(\frac{1}{t^\alpha} \implies\) No spectrum in

\[C(\alpha, K) := \left\{ z \in \mathbb{C}, 0 \geq \text{Re}(z) \geq \frac{1}{K|\text{Im}(z)|^{\frac{1}{\alpha}}} \right\},\]

for some \(K > 0\).

Corollary

Best decay rate possible: \(\frac{1}{t^{\frac{2}{3}}}\)
**Theorem (Nonnenmacher ’13)**

There exists a sequence \((z_n)_{n \in \mathbb{N}} \in \text{Sp}(A)^\mathbb{N}\) such that \(|z_n| \to \infty\) and

\[
|\text{Re}(z_n)| \leq \frac{1}{C|\text{Im}(z_n)|^{\frac{3}{2}}}. 
\]

Decay at rate \(\frac{1}{t^\alpha}\) \(\implies\) No spectrum in

\[
C(\alpha, K) := \left\{ z \in \mathbb{C}, 0 \geq \text{Re}(z) \geq \frac{1}{K|\text{Im}(z)|^{\frac{1}{\alpha}}} \right\},
\]

for some \(K > 0\).

**Corollary**

*Best decay rate possible:* \(\frac{1}{t^{\frac{1}{3}}}\) *and* \(\frac{2}{3} < 1\).

Consequence: our smoothness assumptions on \(b\) to obtain decay at rate \(\frac{1}{t^{1-\varepsilon}}\) cannot completely be disposed of.
Conclusion and open problems

Conclusion:

• Decay rates on the torus seem to depend only on the vanishing rate of \( b \), and not on the number of trapped directions!
• The \textit{a priori} lower bound \( \frac{1}{t} \) is sharp in any geometrical situation (almost reached for smooth \( b \))!
• The \textit{a priori} lower bound \( \frac{1}{t} \) is not reached for \( b = \kappa \mathbb{1}_{(0,\sigma)}(x_1) \)!

Some open problems:

• Is the \textit{a priori} bound \( \frac{1}{\sqrt{t}} \) is sharp on the torus for some rough damping coefficients?
• (Even if \( b \) is invariant in one direction) what is the precise link between the vanishing rate of \( b \) and the decay rate for (DWE)?
• What happens in \( \mathbb{T}^d \), \( d \geq 3 \)?