GWP for the cubic wave equation in 3D
Fabrice Planchon

The purpose of these notes is to give a simpler proof of KPV’s result, which is GWP for $\dot{H}^s$ data with $s > 3/4$. KPV’s result is based on Bourgain’s method: split low and high frequencies, solve the nonlinear problem for the low frequencies, deal with the perturbed equation for the high frequencies, and iterate in a clever way. Here we take a different route, which can somehow be seen as dual. Such a method has previously been successfully used in the context of parabolic equations, in particular construction of infinite energy weak solutions to the Navier-Stokes equations (somehow one could trace these ideas back to nonlinear real interpolation in the 70’s, with work by Peetre, Tartar...). In terms of the method, such a simpler proof (at least for the authors!) was given by I. Gallagher and the author. However, it uses at some stage improvements on the usual Strichartz estimates, and thus is not on par with KPV in term of technology. Not using this improvement leads to $s > 5/6$ with essentially a 1 page proof. The present note disposes with the need of improved Strichartz.

We deal with
\[ \Box \phi + \phi^3 = 0, \]
space dimension $n = 3$. Cut $\phi_0 = u_0 + v_0$, low and high frequencies (at $|\xi| \sim 2^J$ for a large $J$). Remark that this is actually just one way of implementing (real) interpolation, one could just chose another decomposition achieving the same respective size.

Firstly, the high frequency part is small and hence there is a unique small solution to $\Box v + v_3 = 0$, of size $2^{J(\sigma-s)}$ at $\dot{H}^\sigma$ regularity, for any $(0 \leq \sigma \leq s$. Moreover, this solution is small in all the norms one can think of (e.g., Strichartz norms), and one can think of it as really being the (small) linear solution: indeed, the nonlinear term which comes from Duhamel is smaller: at regularity $\sigma$, all its norms are controled by $\|v_0\|_{\dot{H}^\sigma} \|v_0\|^2_{\dot{H}^{1/2}}$.

Next, the pertubed equation
\[ \Box u + 3v^2u + 3vu^2 + u^3 = 0 \]
is well-posed in $\dot{H}^1$, with a local time of existence $T(\|u_0\|_{\dot{H}^1})$ (and the size of $v$, but we just kept the leading term). In fact, this equation is LWP for $\dot{H}^1$ data provided $v$ is anything that satisfies a bunch of estimates similar to those satisfied by our $v$ (e.g., $v$ needs to be small in $L_t^\infty (\dot{H}^{1/2+}_t) \cap L^2_t (W^{-1/2}_\infty)$). In

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short, this is not a limiting step in getting close to $1/2 + \frac{\alpha}{2}$ regularity. Moreover, LWP result is valid starting from any time, not just time zero! (important to finish the argument in the same way one proves GWP for $H^1$ data)

We then aim at controlling the energy, $E = \|\partial u\|_2^2$ (more accurately, the hamiltonian $H = E + \|u\|_4^4$ or the sup$_{0 \leq t \leq T}$ of these quantities, $E_T$ and $H_T$).

We write

$$H_T \leq H_0 + \left| \int_0^T v^2 u \partial_t u + \int_0^T v u^2 \partial_t u \right|.$$  

Up to an irrelevant rescaling, one can ignore the $L^4$ norm and therefore $H_0 \approx 2^{2J(1-s)}$. We want to obtain an inequality of the form

$$H_T \lesssim H_0 + C(T, J, s) H_T^\beta,$$

where $C$ is small, so that by contraction, we control $H_T$ under the condition $H_0 C^{1/(\beta-1)} \lesssim 1$. This then translates into a condition on $s$ in order to be able to choose $T$ arbitrarily large. This a priori control coupled with the local existence result then ensures that the perturbed equation has a solution up to this time $T$.

Let’s go back to the inequality on the hamiltonian : recall that

$$\|v\|_{L^3_t L^6_x} \lesssim \|v_0\|_{\dot{H}^{1/2+1/6}} \sim 2^{J(1/2+1/6-s)}.$$

therefore, we infer that

$$\left| \int_0^T v^2 u \partial_t u \right| \lesssim T^{-1/3} 2^{2J(2/3-s)} E_T$$

which forces a condition on $T$ if one wants to kills this term with the LHS, namely

$$T \leq \frac{1}{2} 2^{2s} T^{s-2/3},$$

which given that we will end up with $s > 3/4$ won’t be a problem. This crude estimate could be improved to yield only an $s > 1/2$ restriction, at the expense of going through bilinear estimates.

The next term is the enoynng part. We dispose of the nonlinear part of $v$ first.

$$\|v_{NL}\|_{L^3_t L^6_x} \lesssim \|v_0\|_{\dot{H}^{1/2}} \|v_0\|_{\dot{H}^{2/3}} \lesssim 2^{J(3/2+1/6-3s)},$$
hence
\[
\left| \int_0^T v_{NL} u^2 \partial_t u \right| \lesssim 2^{3J(5/9 - J)} E_T^{3/2} T^{2/3}.
\]

Next is \( \int v_L u^2 \partial_t u \). This is where the troubles are, and in the first version of this argument, one essentially proves that \( v_L u^2 \in L^2_T L^2_x \) assuming only \( v_0 \in \dot{H}^{1/2} \) (the astute reader will immediately notice that when high frequencies land on \( v_L \), this means we can somehow gain 1/2 derivative over what Strichartz would give). It turns out one can achieve the same purpose by allowing \( u \) to be better behaved (and turn to the equation and Duhamel to quantify this in term of \( H_T \)).

We first dispose of (I am now assuming I have the endpoint. Just shuffle the epsilon around)
\[
\left| \int v_L u^2 \partial_t u \right| \lesssim \|v_0\|_2 \|v_L\|^2_{L^2_T L^\infty_x} E_T^{1/2} \lesssim 2^{2J(1 - s) - Js} E_T^{1/2} \lesssim 2^{J(4 - 6s)} + 1/2 E_T,
\]

which is fine \( (2^{J(4 - 6s)} < 2^{J(2 - 2s)} = H_0) \).

We are left with \( \int v_L u_{NL}^2 \partial_t u \). We resort to the equation to obtain some estimates on \( u_{NL} \). Namely,
\[
u_{NL} = \int_0^T W(t - s)(-3u^2 v^2 - 3u^2 v - u^3) ds = u_{NL}^I + u_{NL}^I + u_{NL}^I.
\]

Let’s start with the first term.
\[
\|u_{NL}^I\|_{L^2 L^\infty_T} \lesssim \|u\|_{L^\infty_T} \|v\|^2_{L^3_x L^6_x} \lesssim T^{1/3} 2^{2J(2/3 - s)} E_T^{1/2},
\]

so that \( \|u_{NL}^I\|_{L^2 L^\infty_T} \) is controled by exactly that (Duhamel!!). Hence,
\[
\left| \int_0^T v_L(u_{NL}^I)^2 \partial_t u \right| \lesssim T^{1/3} 2^{5J(8/15 - s)} E_T.
\]

This gives another constraint on \( T \) which is weaker than the previous one as long as \( s > 4/9 \) (Again, one could improve this to \( s > 1/2 \)). Let us move on to the next term.
\[
\|u^2 v\|_{L^1 L^2} \lesssim T^{2/3} 2^{J(2/3 - s)} E_T.
\]
This yields control of $\|u_{NL}^{II}\|_{L^2L^\infty}$, and therefore

$$|\int_0^T v_L(u_{NL}^{II})^2 \partial_t u| \lesssim \|u_{NL}^{II}\|_{L^3_x L^{\infty}}^2 \|v_L\|_{L^\infty_T L^2_x} \|\partial_t u\|_{L^\infty_T L^2_x} \lesssim T^{4/3} 2^{J(3/2-s)-Js} E^{5/2}.$$ 

This is a potentially bad guy (it can be arranged somewhat differently, but part of it turns out to be as deadly as the next one).

Let’s see what happens with the next one (which is somehow supposed to be the killer). We start with evaluating where $u^3$ is. Seeing as a (symmetrical) product between $u$ and $u^2$, we see that with an $L^2 \times \dot{H}^1$ product we get $\dot{B}_1^1$, so that

$$\|u^3\|_{L^2_T \dot{B}_1^1} \lesssim T^{1/2} H^{1/2}_T E^{1/2}_T,$$

in other words,

$$\|u_{NL}^{III}\|_{L^2L^\infty} \lesssim T^{1/2} H_T.$$

We then obtain

$$|\int_0^T v_L(u_{NL}^{III})^2 \partial_t u| \lesssim T^{-s} J H^{1/2}_T E^{1/2}_T.$$

Now, we notice that this last term is actually like $H^{5/2}$, and the previous term with $E^{5/2}$ is weaker, since $4/3 - 3s < -s$ ($s > 2/3$) and the power of $T$ is smaller (remember, $T$ is large).

There is a middle term with $H^{3/2}$ but by baby Hölder one checks it’s just ok,

$$(\cdot) \lesssim T^{-s} J H^{5/2}_T + T^{1/6} 2^{J(5/6-6s)} E^{1/2}_T,$$

and the second term is fine, it’s controlled by $2^{J(1-s)}$ provided $K + J(13 - 20s) < 0$ (here think of $T \approx 2^K$) which is weaker than the final requirement.

The final estimate is then (modulo the restriction on $T, J$ mentioned earlier, which turns out to be weaker),

$$H < 2^{J(1-s)} + T^{-s} J H^{5/2}_T.$$ 

Setting $T = 2^K$, this reduce to $3J(1-s) + K - Js < 0$ which, since we want $K$ to be large, impose $s > 3/4$.

Of course, the same method can be applied to other equations, as mentioned before. In the parabolic case, this turns out to be very efficient, since parabolic smoothing essentially kills the problem of low vs high regularity.
In the present context, the absence of derivatives in the nonlinearity plays a similar role. If one considers the NLS equation, local smoothing allows to recover some smoothness and carry out the method (alternatively one could use $X^{s,b}$ spaces). It doesn’t not, however give very sharp results, but certainly is a cheap way to get an $\varepsilon$ below $H^1$. 