\[ L^p \text{ ESTIMATES FOR} \]
\[ \text{THE WAVE EQUATION WITH THE} \]
\[ \text{INVERSE-SQUARE POTENTIAL} \]

FABRICE PLANCHON  
Laboratoire d'Analyse Numérique, URA CNRS 189  
Université Pierre et Marie Curie, 175 rue Chevaleret, 75252 Paris, France

JOHN G. STALKER  
Department of Mathematics  
Princeton University, Princeton NJ 08544

A. SHADI TAHVILDAR-ZADEH  
Department of Mathematics, Rutgers, The State University of New Jersey  
110 Frelinghuysen Road, Piscataway NJ 08854

Abstract. We prove that Strichartz-type \( L^p \) estimates hold for solutions of the linear wave equation with the inverse square potential, under the additional assumption that the Cauchy data are spherically symmetric. The estimates are then applied to prove global well-posedness in the critical norm for a nonlinear wave equation.

1. Introduction. Consider the following linear wave equation

\[
\begin{align*}
\Box_n u + \frac{a}{r} u &= h(x,t) \\
u(0, x) &= f(x) \\
\partial_t u(0, x) &= g(x)
\end{align*}
\]

where \( \Box_n = \partial^2_t - \Delta_n \) is the D'Alembertian in \( \mathbb{R}^{n+1} \) and \( a \) is a real number. The interest in this equation comes from the potential term being homogeneous of degree -2 and therefore scaling the same way as the D'Alembertian term. Such a potential arises in the problem of the stability of certain singular stationary solutions of nonlinear wave equations, as well as in the study of the wave equation on conic manifolds [4]. We also note that the heat and Schrödinger flows for the elliptic operator \( -\Delta + a|x|^{-2} \) have been studied in the theory of combustion (see [20] and references therein), and in quantum mechanics (see [10] and references therein) respectively. In particular, the unusual spectral properties of this operator are well-known. These are described in more detail at the end of this section.

In the case of the linear wave equation with no potential, there are three types of basic estimates. First is the energy estimate which gives a bound for the \( L^2 \) norm of the first derivatives of the solution at time \( t \) in terms of the same quantity at time zero. Secondly, the pointwise, or dispersive estimate, which gives a bound for the \( L^\infty \) norm of the solution at time \( t \) in terms of the \( L^1 \) norm of an appropriate number of derivatives of the data, with a constant that decays in \( t \). These two classical estimates are for the homogeneous case \( \Box u = 0 \), and by interpolating between them one can obtain a decay-in-time estimate for the \( L^p \) norm of the solution at time \( t \) for \( 1 < p < \infty \) in terms of the dual \( L^{p'} \) norm of a certain number

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of derivatives of the data (since this could be fractional the appropriate norms are Besov-space norms, see [15, pp. 50-60]).

The third kind of estimate (or family of estimates) for the free wave equation is a bound for the $L^p$ norm in spacetime of the solution $u$. In the case of zero Cauchy data for example, this estimate has the form

$$
\|u\|_{L^p([0,T]; L^q) \cap L^2_x L^\infty_t)} \leq C \|\Box u\|_{L^p([0,T]; L^q) \cap L^2_x L^\infty_t)} , \quad p = \frac{2(n+1)}{n-1} , \quad p' = \frac{2(n+1)}{n+3} . \quad (1.2)
$$

(Hence for this particular $p$ there is no gain in regularity for $\Box^{-1}$ but the gain in integrability is the maximum possible, i.e. the same as $\Delta^{-1}$). This remarkable estimate was first obtained by Strichartz [18] and subsequently reproved, refined and generalized by many others (see [7], [12], and references therein).

Now for the wave equation with a potential, the energy estimate is still valid and easy to obtain if the potential is nonincreasing in $t$. In particular, for the case of the inverse-square potential, one can obtain an energy estimate by multiplying the equation in (1.1) by $\partial_t u$ and integrating on a hyperplane $t = \text{const}$, to get

$$
\frac{d}{dt} \sqrt{E(u, t)} \leq \frac{1}{\sqrt{2}} \|h(\cdot, t)\|_{L^2(\mathbb{R}^n)} ,
$$

where the quantity $E(u, t)$ is the energy of the solution $u$ at time $t$,

$$
E(u, t) := \frac{1}{2} \int_{\mathbb{R}^n} |\partial_t u|^2 + |\nabla_x u|^2 + \frac{a}{|x|} |u|^2 \, dx.
$$

An application of Hardy’s inequality shows that the above energy bounds the square of the $\dot{H}^1$ Sobolev norm of $u$ provided that $n \geq 2$ and

$$
a > -\frac{(n-2)^2}{4} .
$$

In this case it follows from the above calculation that

$$
\|u(\cdot, t)\|_{\dot{H}^1(\mathbb{R}^n)} \leq C \left( \|\nabla f\|_{L^2} + \|g\|_{L^2} + \int_0^t \|h(\cdot, s)\|_{L^2} \, ds \right) ,
$$

which is the energy estimate for (1.1).

For the wave equation with a smooth potential, various $L^p-L^q$ estimates are also known to hold if the potential is either nonnegative ([1], [22], [6]), or rapidly decaying ([9], [2], [21], [5]). One method of proving estimates in the presence of a potential is to deduce them from the case with no potential by studying the intertwining or conjugation operator, which takes the solution of one equation to the other, proving that it maps $W^{k,p}$ into $W^{k,p}$ for an appropriate range of $k$ and $p$. This is the approach taken in [21] for example. Since the techniques for doing this typically fail to cover the extreme cases $p = 1, \infty$, the pointwise estimate needs a different approach.

In this paper, we are only concerned with the spherically symmetric wave equation, i.e. considering solutions of (1.1) that depend only on $|x|$ and $t$. In this case, because of the special form of the potential, the conjugation operator to the free case can be computed explicitly, allowing us to obtain the $L^p$ estimate (1.2) and its generalizations, with $\Box + \frac{a}{|x|^2}$ in place of $\Box$ (see Corollaries 3.3 and 3.9 for precise statements). In subsequent papers, we will consider the dispersive inequality in the radial case, as well as generalizations of these estimates to non-radial cases.

Before proceeding further we must clarify what we mean by the operator $A = -\Delta_n + \frac{a}{|x|^2}$. We adopt the standard notation $D(\Omega)$ for the space of smooth functions
on a domain $\Omega \subset \mathbb{R}^n$. Let
\[ \lambda := \frac{n - 2}{2}. \]  
(1.3)
For $\nu \in \mathbb{R}$, we denote by $\mathcal{D}_\nu(\mathbb{R}^n)$ the space of smooth functions $\phi$ on $\mathbb{R}^n \setminus \{0\}$ such that $r^{\lambda - \nu} \phi$ extends to a smooth function on $\mathbb{R}^n$ vanishing to infinite order at infinity. We have $\mathcal{D}_\nu(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ if and only if $\nu > -1$. Let $D$ be the dilatation operator $(D\phi)(x) = \phi(c^{-1}x)$, $c > 0$ and $R$ the rotation operator $(R\phi)(x) = \phi(Q^{-1}x)$, $Q \in O_n(\mathbb{R})$. $A$ is well-defined as an operator from $\mathcal{D}(\mathbb{R}^n \setminus \{0\})$ satisfying the equations
\[ RA\phi = AR\phi, \]  
(1.4)
\[ \langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle, \]  
(1.5)
and
\[ AD\phi = c^{-2}DA\phi, \]  
(1.6)
for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$, $c > 0$ and $Q \in O_n(\mathbb{R})$. Moreover,
\[ \langle \phi, A\phi \rangle > 0 \]  
(1.7)
for all $\phi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ if and only if $a > -\lambda^2$. Let $\nu$ be a real solution of
\[ \nu^2 = \lambda^2 + a, \quad \nu > -1. \]  
(1.8)
Then $A$ is well defined as an operator on $\mathcal{D}_\nu(\mathbb{R}^n)$ satisfying (1.4-1.7) for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$, $c > 0$ and $Q \in O_n(\mathbb{R})$. This operator has a unique extension to a closed, densely defined operator on $L^2(\mathbb{R}^n)$ whose domain is invariant under dilations and rotations. This extension satisfies (1.4-1.7) for all $\phi, \psi \in \text{Dom}(A)$, $c > 0$ and $Q \in O_n(\mathbb{R})$. In this paper, we will consider only these extensions. If $a = -\lambda^2$ or $a \geq 1 - \lambda^2$ there is only one such extension. If $-\lambda^2 < a < 1 - \lambda^2$ there are two extensions corresponding to $A$, one with $-1 < \nu < 0$ and one with $0 < \nu < 1$. We denote these extensions by $A_\nu$. The Cauchy problem we are studying in the radial case, is therefore
\[ \begin{cases} \\
\left( \partial_t^2 + A_\nu \right)u = h(t,r) \\
u(0,r) = f(r) \\
\partial_t u(0,r) = g(r) 
\end{cases} \]  
(1.9)

Remark 1.1. If we are prepared to drop some of conditions (1.4-1.7) then there are more closed, densely defined extensions. The rotationally invariant ones have been completely classified. For $a \geq 1 - \lambda^2$ there is only one described above. If $-\lambda^2 < a < 1 - \lambda^2$ there is a one-parameter family of extensions. This includes the two described above and others which are not dilation invariant. Half of them satisfy (1.7) while half have a single negative eigenvalue of multiplicity one. If $a < -\lambda^2$ there is a finite-dimensional family of extensions. All of these have negative point spectrum extending to $-\infty$. The eigenfunctions are of the form
\[ K \frac{(\lambda + 1)^{-\frac{n}{2} + s}}{(\lambda + 1)^{-\frac{n}{2} + s} + c} |x|^{n/2} Y_{\frac{n}{2}}(x/|x|) \]
where $K$ is a modified Bessel function of the second kind and $Y$ is a spherical harmonic of order $l$. The preceding results are contained, with much more detailed information on the spectral decomposition, in [19], [3], [14], and [10]. Nothing much seems to be known about extensions which are not rotationally invariant.

2. Construction of the Conjugation Operator. In this section we show how to construct an operator that conjugates the elliptic operator $A_\nu$ in the above section to $-\Delta$. The construction is most conveniently done via the Mellin transform, which at the same time will provide us with a good definition for arbitrary powers of $A_\nu$. We thus begin by collecting what we need on the Mellin transform.
2.1. Test functions. Let \( \mu_1, \ldots, \mu_M \) and \( \nu_1, \ldots, \nu_N \) be complex numbers such that \( 2\lambda + \mu_j - \nu_k \) is not a negative even integer for any pair of indices \( j \) and \( k \). We define \( P(\sigma) \) and \( Q(\sigma) \) to be, respectively, the number of indices \( j \) such that \( \sigma - \lambda - \mu_j \) is a positive even integer and the number of indices \( k \) such that \( \sigma - \lambda + \nu_k \) is a non-positive even integer. We let \( R \) and \( L \) be, respectively, the sets of complex numbers \( \sigma \) such that \( P(\sigma) \) and \( Q(\sigma) \) are positive. We define \( \mathcal{D}_{\nu_1, \ldots, \nu_N}^{\mu_1, \ldots, \mu_M} \) to be the set of smooth functions \( \phi \) on \( (0, \infty) \) such that

\[
\phi(r) \approx \sum_{\sigma \in R} \sum_{i=0}^{P(\sigma)-1} a_{\sigma,i} r^{-\sigma} \log^i r
\]

for \( r \) large and

\[
\phi(r) \approx \sum_{\sigma \in L} \sum_{i=0}^{Q(\sigma)-1} b_{\sigma,i} r^{-\sigma} \log^i r
\]

for \( r \) small. The most important special case is \( \mathcal{D}_{\nu} \), the space of smooth functions vanishing to infinite order at infinity and satisfying

\[
\phi(r) \approx c_m r^{-\nu+2m}
\]

for \( r \) small. This is the natural domain of definition of the operator

\[
A_{\nu} := -\partial_r^2 - (n - 1) r^{-1} \partial_r + (\nu^2 - \lambda^2) r^{-2}.
\]

When \( \nu \) is real and greater than \(-1\), \( A_{\nu} \) is self-adjoint on \( \mathcal{D}_{\nu} \). The spaces with multiple superscripts and subscripts will arise when we consider fractional powers of \( A_{\nu} \).

2.2. The Mellin and inverse Mellin transforms. If \( L \) and \( R \) are such that the strip \( \max \text{Re} L < \text{Re} z < \min \text{Re} R \) is nonempty then for \( \phi \in \mathcal{D}_{\nu_1, \ldots, \nu_N}^{\mu_1, \ldots, \mu_M} \) the integral

\[
(M\phi)(z) := \int_0^\infty r^{-n} \phi(r) r^{n-1} dr
\]

eexists and is an analytic function of \( z \). If, in addition,

\[
\max \text{Re} L < \alpha \leq \text{Re} z \leq \beta < \min \text{Re} R
\]

then Hölder’s inequality bounds \( |(M\phi)(z)| \) by

\[
\left( \int_0^\infty r^{\alpha-n} |\phi(r)| r^{n-1} dr \right)^{\frac{\beta-n}{\beta-\alpha}} \left( \int_0^\infty r^{\beta-n} |\phi(r)| r^{n-1} dr \right)^{\frac{\alpha-n}{\beta-\alpha}},
\]

so \( M\phi \) is uniformly bounded in the strip.

We define the Euler operators

\[
E^\mu := \frac{1}{2} (r \partial_r + \lambda + \mu + 2), \quad E_{\nu} := -\frac{1}{2} (r \partial_r + \lambda - \nu)
\]

and note that if \( \phi \in \mathcal{D}_{\nu_1, \ldots, \nu_N}^{\mu_1, \ldots, \mu_M} \) then, for positive integers \( S_1, \ldots, S_M, T_1, \ldots, T_N \), the function

\[
\phi^\ast := \prod_{j=1}^{M} \prod_{s_j=0}^{S_j-1} E^{\mu_j+2s_j} \prod_{k=1}^{N} \prod_{t_k=0}^{T_k-1} E_{\nu_k+2t_k} \phi
\]

\(^{1}\)Here and throughout whenever we write an asymptotic expansion it is understood that the derivatives of all orders satisfy similar expansions, the coefficients being obtained by formal term by term differentiation.
belongs to $\mathcal{D}_{\nu_1, \ldots, \nu_M}$, where $\mu_j := \mu_j + 2S_j$ and $\nu_k := \nu_k + 2T_k$. Let $R'$ and $L'$ bear the same relation to $\mathcal{D}_{\nu_1, \ldots, \nu_M}$ that $R$ and $L$ bear to $\mathcal{D}_{\nu_1, \ldots, \nu_N}$. Repeated integration by parts shows that

$$(\mathcal{M}\phi)(z) = \frac{M}{\pi} \prod_{j=1}^{S_j-1} \prod_{s_j=0}^{1} \left(1 - \frac{z - \lambda - \mu_j}{2}\right)^{-1} \prod_{k=1}^{N} \prod_{t_k=0}^{T_k-1} \left(1 - \frac{z - \lambda + \nu_k}{2}\right)^{-1} (\mathcal{M}\phi')(z).$$

The right hand side of the above is meromorphic for

$$\max \text{Re } L' < \text{Re } z < \min \text{Re } R',$$

the poles being located at the points of $R \cup L$ in the strip and the order of the pole at $\sigma$ being at most $P(\sigma) + Q(\sigma)$. If, in addition,

$$\max \text{Re } L' < \alpha \leq \text{Re } z \leq \beta < \min \text{Re } R'$$

then

$$(\mathcal{M}\phi)(z) \leq C \prod_{j=1}^{S_j-1} \prod_{s_j=0}^{1} \left|1 - \frac{z - \lambda - \mu_j}{2}\right|^{-1} \prod_{k=1}^{N} \prod_{t_k=0}^{T_k-1} \left|1 - \frac{z - \lambda + \nu_k}{2}\right|^{-1}$$

where $C$ is

$$\left(\int_0^{\infty} r^{n-1} |\phi'(r)| r^{n-1} dr \right)^{\frac{\alpha - \beta - n}{\beta - n}} \left(\int_0^{\infty} r^{\beta-n} |\phi'(r)| r^{n-1} dr \right)^{\frac{\alpha + n}{\beta - n}}.$$ 

It follows that $(\mathcal{M}\phi)(z) = O(|z|^{-K})$ for large $z$ in the strip, where $K = S_1 + \cdots S_M + T_1 + \cdots + T_N$. Given any real $\alpha$, $\beta$, we can choose $S_1, \ldots, S_M$, and $T_1, \ldots, T_N$ such that $\max L' < \alpha$, $\beta < \min R'$, and $K$ is as large as we please. In this way we obtain a meromorphic extension to any strip which vanishes to arbitrary order at infinity. In the remainder of this paper by $\mathcal{M}\phi$ we will always mean this extension.

For later convenience we gather together the properties of $\mathcal{M}\phi$ as follows:

**Definition 2.1.** Let $\mathcal{D}_{\nu_1, \ldots, \nu_M}$ be the set of meromorphic functions $f$ whose only poles are the points of $L \cup R$, the order of the pole at $\sigma$ being at most $P(\sigma) + Q(\sigma)$, which are $O(|z|^{-K})$ for large $z$ in any vertical strip of finite width for every integer $K$.

We can summarize what we have proven about the Mellin transform by saying that $\mathcal{M}\phi \in \mathcal{D}_{\nu_1, \ldots, \nu_M}$ whenever $\phi \in \mathcal{D}_{\nu_1, \ldots, \nu_M}$.

To define the inverse Mellin transform, let $C$ be a path from $-i\infty$ to $+i\infty$ which is contained in a vertical strip of finite width and such that the points of $L$ lie to the left while those of $R$ lie to the right of the path. We define

$$(\mathcal{M}^{-1}f)(r) := -\frac{1}{2\pi i} \int_C r^{-z} f(z) \, dz.$$ 

The integral is easily seen to be independent of the path chosen, provided that it is satisfies the conditions above. Moreover, $\mathcal{M}^{-1}f \in \mathcal{D}_{\nu_1, \ldots, \nu_M}$ whenever $\int \in \mathcal{D}_{\nu_1, \ldots, \nu_M}$. The notation $\mathcal{M}^{-1}$ is meant to suggest that it is the inverse of $\mathcal{M}$, a fact which we must now establish.

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2This, of course, requires that $L$ and $R$ be disjoint sets. That was the reason for the requirement that $2\lambda + \mu_j - \nu_k$ is not a negative even integer for any pair of indices $j$ and $k$. 

We begin by showing that $M^{-1}f \in \mathcal{D}_{\nu_1, \ldots, \nu_N}$ whenever $f \in \mathcal{D}_{\mu_1, \ldots, \mu_N}$. Let $X$ be a real number greater than $\max \Re L$ and unequal to $\Re \sigma$ for any $\sigma \in R$. An easy residue calculation shows that

$$
(M^{-1}f)(r) = \sum_{\sigma \in R, \Re \sigma < X} \sum_{r=0}^{P(\sigma)-1} a_{\sigma,r} r^{-\sigma} \log^r r - \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} r^{-z} f(z) \, dz,
$$

where the coefficients $a_{\sigma,r}$ are determined by the principal parts of $f$ at the poles,

$$
f(z) = -\sum_{m=1}^{P(\sigma)} (-1)^m (m-1)! a_{\sigma,m-1} (z-\sigma)^{-m} + O(1),
$$

for $z$ near $\sigma$, the path of integration being the line $\Re z = X$. The integral on the right is $O(r^{-\alpha})$ for $r$ large, so, since $X$ can be taken arbitrarily large, we have established the required asymptotic expansion for large $r$. The argument for $r$ small is entirely analogous. We shift the contour of integration to the right and we see that the coefficients in the asymptotic expansion about zero are related to the principal parts at the poles in $L$ by

$$
f(z) = \sum_{m=1}^{Q(\sigma)} (-1)^m (m-1)! b_{\sigma,m-1} (z-\sigma)^{-m} + O(1).
$$

We now prove that $M M^{-1} f = f$. By definition

$$
(M M^{-1} f)(z) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \int_C r^{-z-w-n} f(w) \, dw \, r^{n-1} \, dr
$$

We split the integral,

$$
(M M^{-1} f)(z) = \frac{1}{2\pi i} \left( \int_{0}^{1} \int_C r^{-z-w-n} f(w) \, dw \, r^{n-1} \, dr + \int_{1}^{\infty} \int_C r^{-z-w-n} f(w) \, dw \, r^{n-1} \, dr \right)
$$

and shift the contours of integration,

$$
(M M^{-1} f)(z) = \frac{1}{2\pi i} \left( \int_{0}^{1} \int_{\alpha-i\infty}^{\alpha+i\infty} r^{-z-w-n} f(w) \, dw \, r^{n-1} \, dr + \int_{1}^{\infty} \int_{\beta-i\infty}^{\beta+i\infty} r^{-z-w-n} f(w) \, dw \, r^{n-1} \, dr \right),
$$

where $\alpha < \Re z < \beta$. Next we switch the order of integration, which is now justified by Fubini’s theorem, and evaluate the inner integral,

$$
(M M^{-1} f)(z) = \frac{1}{2\pi i} \left( \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{f(w)}{w-z} \, dw + \int_{\beta-i\infty}^{\beta+i\infty} \frac{f(w)}{z-w} \, dw \right).
$$

The rapid decay of $f$ allows us to deform this to a finite contour and apply Cauchy’s theorem,

$$
(M M^{-1} f)(z) = f(z).
$$

Next we prove that $M^{-1} M \phi = \phi$. We define an operator $\Omega^\sigma$ from $\mathcal{D}_{\nu_1, \ldots, \nu_N}$ to $\mathcal{D}_{\mu_1, \ldots, \mu_N, \nu_1, \ldots, \nu_N + \sigma}$ by

$$
(\Omega^\sigma \phi)(\omega) = \omega^\sigma \phi(\omega).
$$

It is clear that

$$
(M \Omega^\sigma f)(z) = (M \phi)(z + \sigma).
$$
From this we see that $\mathcal{M}^{-1} \mathcal{M}$ commutes\(^3\) with $\Omega^{\sigma}$. The only such operators are multiplication operators. On the other hand it is easy to see that $\mathcal{M}^{-1} \mathcal{M}$ commutes with dilations, so it must be multiplication by a constant. Finally, from the previously established relation $\mathcal{M} \mathcal{M}^{-1} = 1$, we see that $\mathcal{M}^{-1} \mathcal{M}$ is equal to its own square, and so the constant is one.

2.3. **Fractional powers of $A_\nu$.** Now that $\mathcal{D}^{\mu_1, \ldots, \mu_M}_{v_1, \ldots, v_M}$ is identified with $\mathcal{D}^{\mu_1, \ldots, \mu_M}_{v_1, \ldots, v_M}$ via the Mellin transform, we can define operators on $\mathcal{D}^{\mu_1, \ldots, \mu_M}_{v_1, \ldots, v_M}$ by describing their action on the Mellin transforms of functions. We define $\mathcal{H}_\nu$ by

$$ (\mathcal{M} \mathcal{H}_\nu \phi)(z) = 2^{z-\lambda-1} \frac{\Gamma(\frac{z+\lambda+\nu}{2})}{\Gamma(1-\frac{z-\lambda-\nu}{2})} (\mathcal{M} \phi)(n-z) $$

$\mathcal{H}_\nu$ is the Hankel transform, but we do not need this for any of the arguments below. For completeness, however, we include a proof. Taking the inverse Mellin transform and interchanging the order of integration in the (absolutely convergent) double integral,

$$ (\mathcal{H}_\nu \phi)(r) = \int_0^\infty \left( \frac{-1}{2\pi i} \int_C 2^{z-\lambda-1} \frac{\Gamma(\frac{z+\lambda+\nu}{2})}{\Gamma(1-\frac{z-\lambda-\nu}{2})} r^{-z} s^{z-n} dz \right) \phi(s)s^{n-1} ds, $$

The inner integral may be evaluated by shifting the contour of integration to the right, picking up the residues at each of the poles. A straightforward residue calculation then shows that

$$ \frac{-1}{2\pi i} \int_C 2^{z-\lambda-1} \frac{\Gamma(\frac{z+\lambda+\nu}{2})}{\Gamma(1-\frac{z-\lambda-\nu}{2})} r^{-z} s^{z-n} dz = (rs)^{-\lambda} J_\nu(rs), $$

where $J_\nu$ is the usual Bessel function of the first kind.

It follows immediately from the definitions that $\mathcal{H}_\nu^2 = 1$ and that $\Omega^\sigma \Omega^\tau = \Omega^{\sigma+\tau}$. Next we define $\mathcal{K}_{\mu, \nu}^{\sigma}$ by

$$ \mathcal{K}_{\mu, \nu}^{\sigma} := \mathcal{H}_\mu \mathcal{H}_\nu. $$

It follows that

$$ \mathcal{K}_{\mu, \nu}^{\sigma} \mathcal{K}_{\nu, \rho}^{\tau} = \mathcal{K}_{\mu, \rho}^{\sigma+\tau}, $$

and a simple calculation shows that

$$ (\mathcal{M} \mathcal{K}_{\mu, \nu}^{\sigma} \phi)(z) = 2^\sigma \frac{\Gamma(\frac{z+\lambda+\nu}{2})}{\Gamma(1-\frac{z-\lambda-\nu}{2})} \frac{\Gamma(\frac{\sigma+\lambda+\nu}{2})}{\Gamma(1-\frac{\sigma-\lambda-\nu}{2})} (\mathcal{M} \phi)(z - \sigma). $$

On the other hand, differentiating under the integral sign in the definition of the inverse Mellin transform,

$$ (\mathcal{M} A_\nu \phi)(z) = (z-\lambda-\nu-2)(z-\lambda+\nu-2)(\mathcal{M} \phi)(z-2). $$

Comparing these two formulae we see that

$$ \mathcal{K}_{\nu, \nu}^{2} = A_\nu. $$

We then define complex powers of $A_\nu$ by

$$ (A_\nu)^{\sigma/2} := \mathcal{K}_{\nu, \nu}^{\sigma}. $$

In particular

$$ (-\Delta)^{\sigma/2} = \mathcal{K}_{\lambda, \lambda}^{\sigma}. $$

\(^3\)More precisely, $\Omega^\sigma$ intertwines the actions of $\mathcal{M}^{-1} \mathcal{M}$ on $\mathcal{D}^{\mu_1, \ldots, \mu_M}_{v_1, \ldots, v_M}$ and $\mathcal{D}^{\mu_1+\sigma, \ldots, \mu_M+\sigma}_{v_1, \ldots, v_M+\sigma}$ in the sense that $\mathcal{M}^{-1} \mathcal{M} \Omega^\sigma = \Omega^\sigma \mathcal{M}^{-1} \mathcal{M}$. This is not strictly a commutation relation since the operators $\mathcal{M}^{-1} \mathcal{M}$ on the right and left hand sides are acting on different spaces.
2.4. Integral kernels. It is clear from its definition that $K^\sigma_{\mu,\nu}$ maps $D^{\mu+\sigma}_{\nu}$ to $D_{\mu}$.
In order to obtain Sobolev mapping properties we need an explicit integral kernel. We have
\[
(K^\sigma_{\mu,\nu}\phi)(r) = (\mathcal{M}^{-1}M K^\sigma_{\mu,\nu}\phi)(r) = \frac{-1}{2\pi i} \int_{C} r^{-z} (\mathcal{M} K^\sigma_{\mu,\nu}\phi)(z) \, dz,
\]
or,
\[
(K^\sigma_{\mu,\nu}\phi)(r) = \frac{-2\sigma}{2\pi i} \int_{C} r^{-z} \int_{0}^{\infty} \frac{\Gamma\left(\frac{-\lambda+\mu}{2}\right)}{\Gamma\left(\frac{-\lambda+\sigma+\mu}{2}\right)} \frac{\Gamma\left(1 - \frac{z - \sigma - \lambda - \mu}{2}\right)}{\Gamma\left(1 - \frac{z - \sigma - \lambda - \mu}{2}\right)} s^{-\sigma-n} \phi(s) s^{n-1} \, ds \, dz.
\]
Assume now that
\[
\text{Re} \, \sigma < -1.
\]
The double integral is absolutely convergent, so, interchanging the order of integration,
\[
(K^\sigma_{\mu,\nu}\phi)(r) = \int_{0}^{\infty} k^\sigma_{\mu,\nu}(r, s) \phi(s) s^{n-1} \, ds,
\]
where
\[
k^\sigma_{\mu,\nu}(r, s) = \frac{-2\sigma}{2\pi i} \int_{C} \frac{\Gamma\left(\frac{-\lambda+\mu}{2}\right)}{\Gamma\left(\frac{-\lambda+\sigma+\mu}{2}\right)} \frac{\Gamma\left(1 - \frac{z - \sigma - \lambda - \mu}{2}\right)}{\Gamma\left(1 - \frac{z - \sigma - \lambda - \mu}{2}\right)} r^{-z} \phi(z) s^{-\sigma-n} \, dz.
\]
Convergent expansions for $s < r$ and $s > r$ can be derived by the usual procedure of shifting the contour of integration to the right and left, respectively. These expansions make sense even without the restriction to $\text{Re} \, \sigma < -1$ and define $k^\sigma_{\mu,\nu}(r, s)$ as a function off the diagonal. For later purposes we note that, for any small $\varepsilon > 0$,
\[
|k^\sigma_{\mu,\nu}(r, s)| = O(r^{-\lambda-\varepsilon-2+\varepsilon} s^{\lambda+\varepsilon-\varepsilon})
\]
for $s < r$ and
\[
|k^\sigma_{\mu,\nu}(r, s)| = O(r^{\mu-\lambda-\varepsilon} s^{-\lambda-\mu-\sigma-2+\varepsilon})
\]
for $s > r$.

For $\text{Re} \, \sigma \geq -1$ we proceed as follows. For fixed $r$, $(K^\sigma_{\mu,\nu}\phi)(r)$ is a linear function of $\phi$, i.e. a distribution. When $\text{Re} \, \sigma < -1$ this distribution coincides with the $L_1$ function $k^\sigma_{\mu,\nu}(r, s)$. Differentiation under the integral sign shows that it satisfies the differential equation
\[
r^{-2} \left[(\lambda + \mu)^2 - \nu^2\right] k^\sigma_{\mu,\nu}(r, s) = s^{-2} \left[(\lambda + \mu)^2 - \nu^2\right] k^\sigma_{\mu,\nu}(r, s).
\]
When $\text{Re} \, \sigma \geq -1$ it continues to satisfy this equation in the sense of distributions. This can be checked directly by integration by parts, but it follows immediately from the uniqueness of the analytic continuation. The restriction of this distribution to $(0, r) \cup (r, \infty)$ is just $k^\sigma_{\mu,\nu}(r, s)$. We are primarily interested in the case $\sigma = 0$. In this case the differential equation above has a regular singular point at $s = r$, with indices $-1$ and $0$. It follows that $k^0_{\mu,\nu}(r, s)$ looks like
\[
c(r - s)^{-1} + O(-\log |r - s|)
\]
for $s$ near $r$. The constant $c$ is the same for $r < s$ and $r > s$, and $K^0_{\mu,\nu}$ is given by integration against this kernel in the principal value sense plus a multiple of the identity.

\[\text{Here we need the fact that we have a distribution solution on } (0, \infty) \text{ and not simply a pair of classical solutions on } (0, r) \text{ and } (r, \infty).\]
The facts above suffice for our purposes in this paper. We have, however, calculated explicit expansions in the case $\sigma = 0$ and include these without proof.

$$k_{\mu, \nu}^0 (r, s) = \begin{cases} C_{\mu, \nu} \frac{\Gamma (\mu + \nu)}{\Gamma (\frac{\mu - \nu}{2})} F \left( \frac{\mu + \nu}{2} + 1, \frac{\mu - \nu}{2} + 1; \frac{1}{2}; \frac{s^2}{r^2} \right) & \text{for } s < r \\ C_{\nu, \mu} \frac{\Gamma (\nu + \mu)}{\Gamma (\frac{\nu - \mu}{2})} F \left( \frac{\nu + \mu}{2} + 1, \frac{\nu - \mu}{2} + 1; \frac{1}{2}; \frac{s^2}{r^2} \right) & \text{for } s > r, \end{cases}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function, and

$$C_{\mu, \nu} = \frac{2 \Gamma (\frac{\mu + \nu}{2} + 1)}{\Gamma (\frac{\mu - \nu}{2} + 1)}$$

for $s$ close to but less than $r$,

$$k_{\mu, \nu}^0 (r, s) = \frac{2}{\pi} \sin \frac{\pi}{2} (\mu - \nu) s^{\nu - \lambda} \left\{ \frac{1}{r^2} + \frac{\mu^2 - \nu^2}{4r^2} \left[ \log \left( 1 - \frac{s^2}{r^2} \right) + 2 \gamma - 1 + \frac{2}{\mu - \nu} - \pi \cot \frac{\pi (\mu - \nu)}{2} - \frac{4(\mu - \nu)}{4 - (\mu - \nu)^2} + \ldots \right] \right\}.$$
It then follows that the kernel $k_{\mu,\nu}^0$ is homogeneous of degree zero and that

$$
k_{\mu,\nu}^0(r,s) = \begin{cases} 
O((s/r)^{\lambda+\nu+2-\frac{2d}{p}}) & s < r \\
O((s/r)^{\lambda-\mu-\frac{2d}{p}}) & s > r.
\end{cases}
$$

After going into logarithmic coordinates, $k_{\mu,\nu}^0$ is a convolution kernel, and by Young's inequality we need to show that $k_{\mu,\nu}^0$ is bounded in $L^1$ on the diagonal in order to establish that the corresponding operator maps $L^p$ into $L^p$. From the above it is clear that we will have the required bounds for $k_{\mu,\nu}^0$ away from $r \sim s$ provided that

$$
\lambda - \mu < \frac{n}{p} < \lambda + \nu + 2.
$$

On the other hand, for $r \sim s$, (2.11) shows that $k_{\mu,\nu}^0(r,s)$ is a Calderon-Zygmund kernel and the $L^p$ estimates would then follow.

We have thus proved the following

**Theorem 3.1.** The operator $K_{\mu,\nu}^0$ defined above is continuous on $L^p_{\text{rad}}(\mathbb{R}^n)$ (radial $L^p$ functions on $\mathbb{R}^n$) provided

$$
\max\left\{\frac{\lambda - \mu}{n}, 0\right\} < \frac{1}{p} < \min\left\{\frac{\lambda + \nu + 2}{n}, 1\right\}.
$$

The operator that conjugates away the potential in $-\Delta + ar^{-2}$ is a special case of the above, namely $K_{\lambda,\nu}^0$. We therefore have

**Corollary 3.2.** The conjugation operator $K_{\lambda,\nu}^0$ is continuous on $L^p_{\text{rad}}(\mathbb{R}^n)$ provided

$$
0 < \frac{1}{p} < \min\left\{\frac{\lambda + \nu + 2}{n}, 1\right\}
$$

while the $L^p_{\text{rad}}$-continuity of its inverse, $K_{\nu,\lambda}^0$, holds if

$$
\max\{0, \frac{\lambda - \nu}{n}\} < \frac{1}{q} < 1.
$$

As mentioned in the Introduction, the above continuity result allows us to prove the $L^p$ estimate analogous to (1.2) for solutions of (1.9):

**Corollary 3.3.** Consider the Cauchy problem (1.9) with $\nu > \frac{1}{n+1}$ and $f = g = 0$. Then

$$
\|u\|_{L^q(\mathbb{R}^{n+1})} \leq C\|h\|_{L^p(\mathbb{R}^{n+1})}, \quad q = \frac{2(n+1)}{n-1}, \quad p = \frac{2(n+1)}{n+3}.
$$

**Proof.** Since $K_{\lambda,\nu}^0$ commutes with $\partial_t$, from (3.12) we have that $\Box K_{\lambda,\nu}^0 u = K_{\lambda,\nu}^0 (\Box + ar^{-2}) u = K_{\lambda,\nu}^0 h$, and thus using (1.2) we have $\|K_{\lambda,\nu}^0 u\|_{L^q} \leq C\|K_{\lambda,\nu}^0 h\|_{L^p}$. It is now enough to check that $p$ and $q$ satisfy (3.13) and (3.14) respectively. Since $p$ and $q$ are dual, these conditions are the same: $\nu > \lambda - n/q = -1/(n+1)$. Thus by the above Theorem,

$$
\|u\|_{L^q} \leq \|K_{\nu,\lambda}^0\|_{L^q} \|K_{\lambda,\nu}^0 u\|_{L^q} \leq C\|K_{\nu,\lambda}^0\|_{L^q} \|K_{\lambda,\nu}^0\|_{L^p} \|h\|_{L^p}.
$$

Notice that all the norms here are space-time norms. However, since the conjugation operators $K_{\mu,\nu}^0$ are time-independent, their continuity on $L^p(\mathbb{R}^{n+1})$ follows from the continuity on $L^p(\mathbb{R}^n)$.

**Remark 3.4.** We note that the same continuity result allows us to take any of the known $L^p$-$L^q$ estimates for the free heat, Schrödinger, or Klein-Gordon equation and conjugate it in this way to obtain a corresponding estimate for the heat, Schrödinger, or Klein-Gordon flow of the operator $A_\nu$, in the radial case, as long as (3.14) and (3.13) are satisfied.
3.2. \( \dot{H}^s_p \)-continuity of the conjugation operator. In order to obtain the corresponding Strichartz estimates for (1.1) with nonzero Cauchy data, and also for the sake of the nonlinear applications, we need to allow for derivatives of the data, the solution and the right hand side to appear in our estimates. For the free wave equation this poses no problems since differentiation in any direction commutes with \( \Box \), but that is not the case in presence of a potential term. Therefore in order to have estimates involving derivatives of the solution we need to exchange powers of \( -\Delta = A_\lambda \) with powers of \( -\Delta + a|x|^{-2} = A_\nu \), which amounts to studying the \( \dot{H}^s_p \) continuity of the operators \( \mathcal{K}^0_{\lambda, \nu} \) and \( \mathcal{K}^0_{\nu, \lambda} \). Note that the Sobolev spaces \( \dot{H}^s_p \) are defined, for real \( \sigma \), via fractional powers of the Laplacian, namely
\[
f \in \dot{H}^s_p \iff \| (A_\lambda)^{\frac{s}{2}} f \|_{L^p} < \infty.
\] (3.16)

let \( B^s_{p, \nu} \) be the following exchange operator:
\[
B^s_{p, \nu} := (A_\mu)^{\frac{s}{2}} (A_\nu)^{-\frac{s}{2}}.
\]

Our strategy is to obtain the \( \dot{H}^s_p \) continuity of \( \mathcal{K}^0_{\lambda, \nu} \) from its \( L^p \) continuity by showing the \( L^p \) continuity of \( B^s_{p, \nu} \) and its inverse \( B^s_{p, \lambda} \). We do this by using the Mellin transformed representation
\[
(\mathcal{M} B^s_{p, \nu} \phi)(z) = \frac{\Gamma(1 - \frac{z - \lambda - \mu}{2}) \Gamma(1 - \frac{z - \lambda - \nu}{2}) \Gamma(1 - \frac{z - \nu - \lambda + \mu}{2}) \Gamma(1 - \frac{z - \nu - \lambda - \mu}{2})}{\Gamma(1 - \frac{z + \nu + \lambda - \mu}{2}) \Gamma(1 - \frac{z + \nu + \lambda + \mu}{2})} (\mathcal{M} \phi)(z)
\] (3.17)

and proceeding as in the case of \( \mathcal{K}^0_{\mu, \nu} \). We note that \( B^s_{p, \nu} \) maps \( D^{s, \nu + \sigma}_{\mu, \nu} \) into \( D^{s, \nu + \sigma}_{\mu, \nu} \cdot \nu \). A procedure similar to that carried out in sections 2.4 and 3.1 then yields

**Proposition 3.5.** The operator \( B^s_{p, \nu} \) defined above is continuous on \( L^p_{rad}(\mathbb{R}^n) \) provided
\[
\max\{\lambda + \nu + \sigma, \lambda - \mu\} < \frac{n}{p} < \min\{\lambda + \mu + 2 + \sigma, \lambda + \nu + 2\}.
\]

**Remark 3.6.** When \( \sigma \) is a positive even integer, \( \sigma = 2m \), there are obvious cancellations in (3.17), and it is easy to see that as a result, the \( L^p \)-continuity of \( B^s_{\mu, \nu} \) holds with fewer restrictions, namely,
\[
\lambda - \nu + 2m < \frac{n}{p} < \lambda + \nu + 2,
\]

independent of \( \mu \).

**Remark 3.7.** The continuity of \( B^s_{\mu, \nu} \) can be viewed as a regularity result for the elliptic operator \( A_\nu \): For \( f \in L^p_{rad}(\mathbb{R}^n) \), there is a unique \( u \in \dot{H}^s_p(\mathbb{R}^n) \) with \( A_\nu u = f \), and \( \|u\|_{\dot{H}^s_p} \leq C\|f\|_{L^p} \), provided \( \nu > 0 \) and \( \lambda - \nu + 2 < \frac{n}{p} < \lambda + \nu + 2 \).

Combining the above Proposition with Corollary 3.2 we obtain

**Theorem 3.8.** Let \( \nu, \alpha, \beta \in \mathbb{R} \), \( \nu > -1 \), \( -n < \alpha < 2(\nu + 1) \), \( -2(\nu + 1) < \beta < n \).
The conjugation operator \( \mathcal{K}^\alpha_{\lambda, \nu} \) is continuous on \( \dot{H}^s_{p, rad}(\mathbb{R}^n) \) provided
\[
\max\{0, \frac{\lambda - \nu}{n} \} < \frac{1}{p} < \min\left\{ \frac{\lambda + \nu + 2}{n}, \frac{\lambda + \nu + 2 + \beta}{n}, 1 \right\},
\] (3.18)
while the inverse operator \( \mathcal{K}^\alpha_{\nu, \lambda} \) is continuous on \( \dot{H}^s_{p, rad}(\mathbb{R}^n) \) provided
\[
\max\{0, \frac{\lambda - \nu}{n}, \frac{\lambda - \nu + \alpha}{n} \} < \frac{1}{q} < \min\left\{ \frac{\lambda + \nu + 2}{n}, 1, 1 + \frac{\alpha}{n} \right\}.
\] (3.19)
Proof. We know that
\[(A_{\lambda})^{\beta/2}K_{\lambda,\nu}^{0} = K_{\lambda,\nu}^{0}(A_{\nu})^{\beta/2},\]
for all real \( \beta \), and thus by the \( L^p \) continuity of \( K_{\lambda,\nu}^{0} \) and \( B_{\nu,\lambda}^{\beta} \), which hold by Proposition 3.5 and Corollary 3.2 under the condition (3.18), we have
\[\|K_{\lambda,\nu}^{0}u\|_{H^\beta_p} = \|A_{\lambda}^{\beta/2}K_{\lambda,\nu}^{0}u\|_{L^p} = \|K_{\lambda,\nu}^{0}A_{\nu}^{\beta/2}u\|_{L^p} \leq C\|A_{\nu}^{\beta/2}u\|_{L^p} \leq nC\|B_{\nu,\lambda}^{\beta}A_{\lambda}^{\beta/2}u\|_{L^p} \leq C\|u\|_{H^\beta_p},\]
and similarly for \( K_{\nu,\lambda}^{0} \).
\[\Box\]

We can now state the generalized Strichartz estimates that hold for solutions of (1.9):

**Corollary 3.9.** Let \( n \geq 2 \) and let \( p, q, \rho, \sigma, \alpha, \beta, \mu \) be nonnegative real numbers, with \( 1 < p \leq 2 \leq q < \infty \) and \( \mu < n/2 \), satisfying
\[\frac{1}{\sigma} \leq \min\left\{ \frac{1}{2}, \frac{n-\frac{1}{2}}{2}, \frac{\sigma - \frac{1}{q}}{2} \right\}, \quad \alpha - \mu = \frac{1}{\rho} - n\left( \frac{1}{2} - \frac{1}{q} \right), \quad (3.20)\]
and
\[\frac{1}{\rho} \geq \max\left\{ \frac{1}{2}, 1 - \frac{n - \frac{1}{2}}{2}, \frac{1}{2}, 1 + \frac{n - \frac{1}{2}}{2} \right\}, \quad \beta - \mu = \frac{1}{\rho} + n\left( \frac{1}{2} - \frac{1}{q} \right) - 2. \quad (3.21)\]

For \( \nu > -1 \), let \( u \) be a solution of (1.9). Then there exists a constant \( C > 0 \) such that for all \( T > 0 \),
\[\|u\|_{L^r_{t\nu}([0,T],[H^\mu_p(\mathbb{R}^n)])} + \|\partial_t u\|_{L^r_{t\nu}([0,T],[H^\mu_p(\mathbb{R}^n)])} \leq C\left(\|u\|_{L^r_{t\nu}([0,T],[H^\mu_p(\mathbb{R}^n)])} + \|f\|_{L^r_{t\nu}([0,T],[H^\mu_p(\mathbb{R}^n)])} + \|g\|_{L^{q-1}_{t\nu}([0,T],[H^\mu_p(\mathbb{R}^n)])}\right), \quad (3.22)\]
provided
\[\nu > \max\{\lambda + \alpha - \frac{n}{q}, \frac{n}{p}, \lambda - 2\}. \quad (3.23)\]

**Remark 3.10.** Estimate (3.22) is essentially quoted from [7]. It is slightly less general than the one there in that the regularity parameters \( \alpha, \beta, \mu \) are here taken to be nonnegative (which simplifies greatly the condition on \( \nu \)) and that it is stated in terms of Sobolev instead of Besov norms.

**Proof.** Since (3.22) holds for solutions of the free wave equation, the proof of the above Corollary consists of checking the relevant continuity properties of the conjugation operators involved: It is clear that we need \( K_{\nu,\lambda} \) to be continuous on \( H^\beta_p \) and \( K_{\lambda,\nu} \) to be continuous on \( H^\mu_p \). Since \( p \leq 2 \leq q \), the third requirement does not impose any further restrictions than the first two. Also note that by (3.21), \( \mu < n/2 \) implies that \( \beta < n/p \). By the nonnegativity of \( \alpha \) and \( \beta \), all that remains from (3.18) and (3.19) is that \( n/p < \lambda + \nu + 2 \) and \( n/q > \lambda - \nu + \alpha \).
\[\Box\]

4. **Application to a non-linear problem.** In [17] the authors study the following equation,
\[\Box u + V(x)u = g(x, u), \quad (4.24)\]
where \( V(x) = C|x|^{2-\delta} \) and \( g \) behaves like \( |u|^\gamma \) for some \( \gamma > 1 \). When \( C = 0 \) (no potential term) the behavior of global solutions of small amplitude depend on whether \( \gamma \) is larger than a critical value \( \gamma_c \) (8). In the range \( \gamma > \gamma_c \) no blow-up occurs. However, adding a potential term can affect the solution, leading to blow-up in finite time. Essentially in [17] blow-up is proved when \( \delta > 0 \) for \( \gamma > 1 \) while global existence of smooth solutions is proved when \( \delta < 0 \) for \( \gamma > \gamma_c \) and \( |C| \) small.
The inverse square potential corresponds to the borderline case $\delta = 0$. In that case, the potential scales like the Laplacian, and therefore it cannot be treated as a small perturbation, unless we require $C$ to be very small. However, relying on the estimates from the previous sections, we will be able to show well-posedness for the non-linear equation in the radially symmetric case, for data at the critical regularity level, and for a large range of powers $p$ and constants $C$.

We would like to study the following Cauchy problem
\[
\begin{cases}
\partial_t^2 u - \Delta u + \frac{\alpha}{r^2} u = \pm |\nu|^p, \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = u_1(x)
\end{cases}
\quad (4.25)
\]
for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. As explained in the Introduction however, the operator $-\Delta + \alpha r^{-2}$ is ambiguous when $-\lambda^2 < \alpha < 1 - \lambda^2$, and the problem should thus be phrased in terms of the selfadjoint extensions $A_{s},$
\[
\begin{cases}
\hat{\Box} u := \partial_t^2 u + A_s u = \pm |u|^p, \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = u_1(x)
\end{cases}
\quad (4.26)
\]
Set $c := \frac{2}{\gamma - 1}$. We intend to show well-posedness for (4.26) for radial data $u_0 \in H^s$ and $u_1 \in H^{s-1}$. A simple computation shows the equation to be invariant by the scaling $u_\theta = \theta^c u(\theta x, \theta^2).$ This suggests the equation should be well-posed at the critical level $H^s$, where $s_c = \frac{\gamma}{\gamma - 1}$, as the $H^s$ norm of $u$ is invariant by the rescaling. Indeed, for the usual wave equation with no potential, well-posedness holds for all $s \geq s_c$ when $s_c \geq \frac{1}{2}$ (or equivalently $\gamma \geq \frac{2}{\gamma - 1}$). Moreover, if the data are radial then well-posedness holds in an extended range [13]. We intend to prove the same kind of results depending on the value of $\nu$. Moreover, one may add terms like $\frac{u^{p-2}u}{\|u\|_{E}}$ in the nonlinearity (under appropriate restrictions on $\nu, \eta$). This illustrates how one can deal with nonlinearities which are in some sense rougher than the simple power $|u|^p$. Indeed $r^{-\alpha}$ has the same scaling as $u$ but unlike $u$ it does not disperse.

Let us outline the strategy of proof. Recall $\mathcal{K}_{\lambda, \nu}^0$ is the conjugation operator such that $\mathcal{K}_{\lambda, \nu}^0 \hat{\Box} u = \mathcal{K}_{\lambda, \nu}^0 u$. Suppose one is willing to solve (4.26) by an iterative procedure. Then we need estimates for
\[
\mathcal{K}_{\lambda, \nu}^0 \hat{\Box} u = \Box \mathcal{K}_{\lambda, \nu}^0 u
\]
(4.27)
where $N$ is our non-linearity. Applying $\mathcal{K}_{\lambda, \nu}^0$, we get
\[
\Box \mathcal{K}_{\lambda, \nu}^0 (w_{l+1} - w_l) = \mathcal{K}_{\lambda, \nu}^0 (N(w_l) - N(w_{l-1})).
\quad (4.28)
\]
Now, suppose we have at our disposal two functional spaces $E$ and $F$ such that
\[
\|\Box^{-1} N \|_E \lesssim \| N \|_F
\]
\quad (4.29)
\[
\| N(u) - N(v) \|_F \lesssim \| u - v \|_E (\| u \|_E + \| v \|_E)^\eta,
\quad (4.30)
\]
for some $\eta > 0$. All we need then to proceed is good continuity properties of $\mathcal{K}_{\lambda, \nu}^0$ and $\mathcal{K}_{\lambda, \nu}^0$ to write
\[
\| w_{l+1} - w_l \|_E \lesssim \| \mathcal{K}_{\lambda, \nu}^0 (w_{l+1} - w_l) \|_E
\]
\quad (4.31)
\[
\lesssim \| \mathcal{K}_{\lambda, \nu}^0 (N(w_l) - N(w_{l-1})) \|_E
\]
and thus using (4.30) we see that we have a contraction if the $E$ norm is small. Assuming therefore that we have the continuity of $\mathcal{K}_{\lambda, \nu}^0$ and its inverse, we will
derive our existence theorem from the existence theorem for the equation
\[ \Box w = \pm |w|^{\alpha} \] (4.32)
which we expect to be well-posed in \( \dot{H}^{s_c}(\mathbb{R}^n) \). Recall that well-posedness for a power-like nonlinearity is well-known (For well-posedness in Sobolev spaces see [16, 11] and [13]).

We are now in a position to state our theorem for the Cauchy problem (4.26). We will use the following notation: If \( X \) is a Banach space of functions on \( \mathbb{R}^n \), we denote by \( C(X) \) the set of bounded continuous functions on \( \mathbb{R}^+ \) with values in \( X \). Similarly, \( L^q(X) \) denotes the set of \( X \)-valued functions that are in \( L^q(\mathbb{R}^+) \).

**Theorem 4.1.** Let \( n \geq 2 \), \( \lambda = \frac{n-2}{2} \), \( \gamma \geq \frac{n+\beta}{n-1} \), \( s_c := \frac{n}{2} - \frac{2}{\gamma-1} \). Suppose that
\[ \nu > \lambda - \frac{2}{\gamma-1} + \max \left\{ \frac{1}{2}, \frac{2}{2^{\gamma-1}(n+1)(\gamma-1)} \right\}. \]

Let \( (u_0, u_1) \in (\dot{H}^{s_c}, \dot{H}^{s_c-1}) \) be radial functions with small norms. Then there exists a unique global radial solution to (4.26) such that
\[ u(x, t) \in C_1(\dot{H}^{s_c}) \cap L^q(\dot{H}^{s_c}) \cap \partial_t u(x, t) \in C_1(\dot{H}^{s_c-1}) \cap L^q(\dot{H}^{s_c-1}), \] (4.33)
where \( q, \sigma, \alpha \) are as follows:

1. For \( n \leq 3 \) or \( \frac{n+3}{n-1} \leq \gamma \leq \frac{n+1}{n-3} \),
\[ \frac{1}{q} = \frac{1}{2} - \frac{4}{(n^2 - 1)(\gamma - 1)}, \quad \frac{1}{\sigma} = \frac{2}{(n + 1)(\gamma - 1)}, \quad \alpha = s_c - \frac{2}{(n - 1)(\gamma - 1)}. \]

2. For \( n \geq 4 \) and \( \gamma > \frac{n+1}{n-3} \),
\[ \frac{1}{q} = \frac{1}{2} - \frac{1}{(n-1)\gamma}, \quad \frac{1}{\sigma} = \frac{1}{2\gamma}, \quad \alpha = s_c - \frac{n+1}{2(n-1)\gamma}. \]

We believe the above result holds true for non-radial data as well. The restriction to the radial case in the above is solely due to the present lack of linear estimates in the non-radial case.

**Proof.** The equation will be solved by an iteration procedure outlined in the above, in the space
\[ \mathcal{E} = C_1(\dot{H}^{s_c}) \cap L^q(\dot{H}^{s_c}). \]
We need to find a space \( \mathcal{F} \) such that (4.29) and (4.30) hold, with \( N(u) = \pm |u|^{\alpha} \). We are going to use the linear estimate (3.22). The critical regularity assumption means that \( \mu = s_c \). The following fractional derivative and Sobolev estimate shows that we can take \( \beta = \alpha \) (and this turns out to be the best possible choice in terms of later restrictions):
\[ \|u\|_{\dot{H}^p} \leq C\|u\|_{L_{q_1}}^{q_1-1}\|u\|_{\dot{H}^{s_c}_q}, \quad \frac{1}{q_1} = \frac{1}{\gamma-1}\left(\frac{1}{p} - \frac{1}{q}\right) \]
\[ \leq C\|u\|_{\dot{H}^{s_c}_q}^{q_1} \quad \text{if} \quad \frac{1}{q_1} = \frac{1}{q} - \frac{\alpha}{n}, \]
and therefore
\[ \|u\|_{L^p(\dot{H}^{s_c}_q)} \leq C\|u\|_{L^p(\dot{H}^{s_c}_q)}^{q_1} \quad \text{if} \quad \sigma = \gamma p. \]
The above imposes two relations between the four numbers \( q, \sigma, p, \rho \). We need to pick \( q \) and \( \sigma \) such that the two inequalities in (3.20) and (3.21) are satisfied, while at
the same time the restriction (3.23) imposed on the potential is as weak as possible, which means that \( q \) needs to be as small as possible. Therefore we set

\[
\frac{1}{\sigma} = \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{q} \right)
\]

which in turn implies that

\[
\frac{1}{p} = \frac{8 - (\gamma - 1)(n-1)}{4n} + \frac{2n + (\gamma - 1)(n-1)}{q} \frac{1}{2n}
\]

Since \( \frac{1}{p} = \frac{1}{\sigma} \), the inequality in (3.21) now restricts \( q \) from below, and saturating it leads to the choices for \( q \) in the statement of the Theorem. \( \square \)

We end this section with a few additional remarks. Since we are dealing exclusively with radial data, one may extend Theorem 4.1 in the range \( \gamma < \frac{n+2}{n-1} \) using the results specific to the radial case from [13]. This amounts to another bookkeeping of continuity properties on different Sobolev spaces therefore we have omitted it. A more interesting issue regards the regularity properties of our solutions. We obtained global solutions with the minimal amount of regularity, and it is worth mentioning that, unlike the case without the potential, one cannot expect to have smooth solutions for (4.26), even for smooth initial data. Indeed this fails for the linear equation (1.1), where one can use (2.10) to obtain an expansion near \( r = 0 \) of the form \( u(r,t) \approx \sum_{\ell=0}^{\infty} a_{\ell}(t) r^{\ell} \) for the solution, with an explicit formula for the coefficients \( a_{\ell}(t) \) in terms of the initial data. For example,

\[
a_{\ell}(t) = \begin{cases} 
0 & \text{if } t < r, \\
\frac{\alpha_{\ell}}{\Gamma(\ell+1)\Gamma(-s+1/2)} \int_0^\infty (t^2 - r^2)^{\ell-\lambda} g(r) r^{s-1} dr & \text{if } t > r.
\end{cases}
\]

This shows that when \( \nu - \lambda \) is not a nonnegative integer, the solution will generally have a singularity at \( r = 0 \).

REFERENCES


E-mail address: fabrice.math@jussieu.fr
E-mail address: stalker@math.princeton.edu
E-mail address: shadi@math.rutgers.edu