Lecture 3: ARIMA(p,d,q) models

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Motivation

- Characterize the main properties of ARIMA(p) models.
- Discuss asymptotic equivalence with ARMA(p,q) models

Road map

1. Definition

- To some extent, ARIMA(p,d,q) models are a generalization of ARMA(p,q) models : the d-differenced process $\Delta^d X_t$ is (asymptotically) an ARMA(p,q) process :
- On the other hand, the statistical properties of the two models are different, especially in terms of forecasting.

Definition

A stochastic process $(X_t)_{t \geq -p-d}$ is said to be an $ARIMA(p,d,q)$—an integrated mixture autoregressive moving average model—if it satisfies the following equation :

$$\Phi(L)(1-L)^d X_t = \mu + \Theta(L)\epsilon_t \ \forall t \geq 0$$

where ϵ_t is a (weak) white noise process with variance σ_ϵ^2, the lag polynomials are given by :

$$\begin{aligned} \Phi(L) &= 1 - \phi_1 L - \cdots - \phi_p L^p \text{ with } \phi_p \neq 0 \\ \Theta(L) &= 1 + \theta_1 L + \cdots + \theta_q L^q \text{ with } \theta_q \neq 0, \end{aligned}$$

and the initial conditions :

$$Z_{-1} = \{X_{-1}, \cdots, X_{-p-d}, \epsilon_{-1}, \cdots, \epsilon_{-q}\}$$

are such that :

Remarks :

1. The stochastic process X_t also writes :

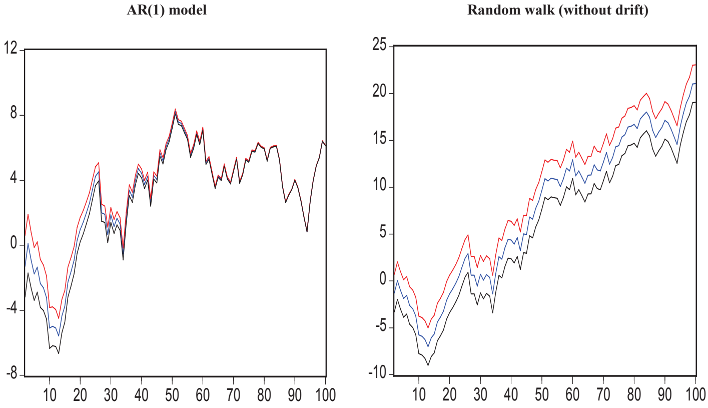
$$\Phi(L)\Delta^d X_t = \mu + \Theta(L)\epsilon_t \text{ or } \Phi(L)Y_t = \mu + \Theta(L)\epsilon_t$$

where $Y_t = \Delta^d X_t = (1-L)^d X_t$.

2. Initial conditions are fundamental.
Example : Consider the two stochastic processes :

$$\begin{aligned} X_t &= \rho X_{t-1} + \epsilon_t \\ Y_t &\equiv X_t - X_{t-1} = \eta_t \end{aligned}$$

where $\rho = 0.9$, ϵ_t and η_t are (weak) white noises.



Note: The black, blue, and red solid lines correspond respectively to X_0 = 0, X_0 = 2, and X_0 = -2.

FIGURE: Initial conditions and stationarity

1.2. Fundamental representations

Definition (Fundamental representation)

Let $(X_t)_{t \geq -p-d}$ denote the following ARIMA(p,d,q) stochastic process :

$$\Phi(L)\Delta^d X_t \quad = \quad \mu + \Theta(L)\epsilon_t \ \forall t \geq 0$$

where $\Phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ and $\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$, $\theta_q \neq 0$,$\phi_p \neq 0$, μ is a constant term, (ϵ_t) is a weak white noise, and the initial conditions are uncorrelated with ϵ_t $(t \geq 0)$. This representation is said to be causal or fundamental if and only if :

(i) All of the roots of the (inverse) characteristic equation associated to Φ are of modulus less (larger) than one.

(ii) All of the roots of the (inverse) characteristic equation associated to Θ are of modulus less (larger) than one.

Definition (Fundamental minimal representation)

Let $(X_t)_{t \geq -p-d}$ denote the following ARIMA(p,d,q) stochastic process :

$$\Phi(L)\Delta^d X_t = \mu + \Theta(L)\epsilon_t \ \forall t \geq 0$$

where $\Phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ and $\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$, $\theta_q \neq 0$,$\phi_p \neq 0$, μ is a constant term, (ϵ_t) is a weak white noise, and the initial conditions are uncorrelated with ϵ_t $(t \geq 0)$. This representation is said to be causal or fundamental if and only if :

(i) All of the roots of the (inverse) characteristic equation associated to Φ are of modulus less (larger) than one.

(ii) All of the roots of the (inverse) characteristic equation associated to Θ are of modulus less (larger) than one.

(iii) The (inverse) characteristic polynomials have no common roots.

1.3. Equivalence with ARMA(p,q) models

Proposition

Let $(X_t)_{t \geq -p-d}$ *denote a minimal and causal* $ARIMA(p,d,q)$ *stochastic process :*

$$\Phi(L)(1-L)^d X_t = \mu + \Theta(L)\epsilon_t.$$

The stochastic process defined by :

$$Y_t = \Delta^d X_t = (1-L)^d X_t$$

is asymptotically equivalent to an $ARMA(p,q)$ *process.*

1.4. Autoregressive approximation

Definition

The autoregressive approximation (and not the AR(∞) representation) of a causal and minimal ARIMA(p,d,q) stochastic process is given by :

$$A_t(L)X_t = \mu_0 + \epsilon_t + h(t)' Z_{-1}.$$

where $A_t(L) = \sum_{j=0}^{t} a_j L^j$, $a_0 = 1$, and the a_j terms are the coefficients of the division (by increasing powers) of $\Phi(u)$ by $\Theta(u)$, μ_0 is a constant term, and $h(t) \in \mathbb{R}^{p+d+q}$ (with $\lim_{t \to +\infty} h(t) = 0$).

1.5. Moving average approximation

Definition

The moving average approximation (and not the MA(∞) representation) of a causal and minimal ARIMA(p,d,q) stochastic process is given by :

$$X_t = \mu_1 + B_t(L)\epsilon_t + \tilde{h}(t)' Z_{-1}.$$

where $B_t(L) = \sum_{j=0}^{t} b_j L^j$, $b_0 = 1$, and the b_j terms are the coefficients of the division (by increasing powers) of $\Theta(u)$ by $\Phi(u)$, μ_1 is a constant term, and $\tilde{h}(t) \in \mathbb{R}^{p+d+q}$ (with $\lim_{t \to +\infty} \tilde{h}(t) = 0$).

- Example
  - Starting from :

$$(1-\phi L)(1-L)X_t = \epsilon_t - \theta \epsilon_{t-1} \ \forall t \geq 0$$

    with the initial conditions ϵ_{-1}, X_{-1} and X_{-2} ;
  - One gets :

$$\begin{aligned} \Delta X_t &= \frac{1-\theta L}{1-\phi L}\tilde{\epsilon}_t - \phi^t \theta \epsilon_{-1} + \phi^{t+1}(X_{-1} - X_{-2}) \\ &= \frac{1-\theta L}{1-\phi L}\tilde{\epsilon}_t + h_t' Z_{-1}. \end{aligned}$$

2. Application : modelling of the US stock market dividends

- Visual inspection of the autocorrelation function may indicate that the (log) US dividend is not stationary. The autocorrelogram of the first-differenced variable decreases quickly to zero—the first-difference variable may be stationary.
- Unit root tests tends to favor the assumption of non-stationarity (see later on).
- The chosen specification is then an ARIMA(p,1,q) model :

$$\Phi(L)(1-L)d_t = \Theta(L)\epsilon_t$$

where ϵ_t is a white noise process, and :

$$\begin{aligned}\Phi(L) &= 1-\phi_1 L - \cdots - \phi_p L^p \\ \Theta(L) &= 1+\theta_1 L + \cdots + \theta_q L^q.\end{aligned}$$

Figure 34: (Partial) autocorrelogram of (log) US stock market dividents

Level

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.993	0.993	440.93	0.000
		2	0.986	-0.009	876.75	0.000
		3	0.979	-0.011	1307.4	0.000
		4	0.972	-0.014	1732.7	0.000
		5	0.965	-0.009	2152.7	0.000
		6	0.958	-0.011	2567.3	0.000
		7	0.950	-0.025	2976.3	0.000
		8	0.942	-0.001	3379.6	0.000
		9	0.935	-0.027	3777.2	0.000
		10	0.927	-0.010	4168.9	0.000
		11	0.918	-0.008	4554.7	0.000
		12	0.910	-0.031	4934.4	0.000
		13	0.902	-0.008	5307.9	0.000
		14	0.893	-0.005	5675.2	0.000
		15	0.884	-0.019	6036.2	0.000
		16	0.876	-0.008	6390.9	0.000
		17	0.867	-0.001	6739.4	0.000
		18	0.858	-0.004	7081.7	0.000
		19	0.849	-0.005	7417.8	0.000
		20	0.841	0.017	7748.0	0.000
		21	0.832	-0.007	8072.3	0.000
		22	0.824	-0.011	8390.6	0.000
		23	0.815	0.007	8703.2	0.000
		24	0.807	-0.013	9009.9	0.000
		25	0.798	-0.001	9310.9	0.000
		26	0.789	-0.012	9606.1	0.000
		27	0.781	-0.004	9895.6	0.000
		28	0.772	-0.010	10179.	0.000
		29	0.763	-0.009	10457.	0.000
		30	0.755	-0.004	10730.	0.000
		31	0.746	-0.003	10996.	0.000
		32	0.737	-0.004	11258.	0.000
		33	0.728	-0.006	11513.	0.000
		34	0.720	-0.006	11763.	0.000
		35	0.711	-0.002	12008.	0.000
		36	0.702	-0.001	12248.	0.000

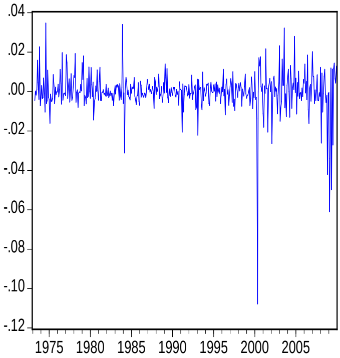
First-difference

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.190	0.190	16.157	0.000
		2	0.266	0.239	47.827	0.000
		3	0.232	0.163	72.018	0.000
		4	0.134	0.024	80.084	0.000
		5	0.140	0.034	88.909	0.000
		6	0.184	0.108	104.22	0.000
		7	0.094	0.003	108.18	0.000
		8	0.113	0.016	114.01	0.000
		9	0.138	0.060	122.63	0.000
		10	0.068	-0.008	124.72	0.000
		11	0.062	-0.023	126.49	0.000
		12	0.097	0.035	130.83	0.000
		13	0.074	0.032	133.37	0.000
		14	0.028	-0.040	133.73	0.000
		15	0.102	0.048	138.56	0.000
		16	0.013	-0.030	138.64	0.000
		17	0.040	-0.003	139.37	0.000
		18	-0.032	-0.087	139.84	0.000
		19	0.015	0.010	139.95	0.000
		20	0.015	0.027	140.06	0.000
		21	0.081	0.078	143.16	0.000
		22	0.009	-0.019	143.19	0.000
		23	-0.007	-0.047	143.21	0.000
		24	-0.015	-0.035	143.32	0.000
		25	-0.043	-0.040	144.17	0.000
		26	-0.011	0.010	144.24	0.000
		27	-0.013	0.003	144.32	0.000
		28	-0.007	0.010	144.34	0.000
		29	-0.008	0.001	144.37	0.000
		30	0.012	0.017	144.44	0.000
		31	-0.012	0.007	144.50	0.000
		32	-0.044	-0.053	145.43	0.000
		33	0.030	0.055	145.87	0.000
		34	0.019	0.036	146.05	0.000
		35	-0.037	-0.044	146.70	0.000
		36	-0.038	-0.077	147.41	0.000

Estimation of an ARIMA(2,1,1) model

Parameter	Estimate	Std. Error	t-stat.	p-value
c	0.0007	0.0003	1.8777	0.0611
ϕ_1	0.7342	0.1096	6.6982	0.0000
ϕ_2	0.1206	0.0619	1.9477	0.0521
θ_1	-0.6442	0.1038	-6.2008	0.0000

Figure 35: Adjusted error terms



	AC	PAC	Q-Stat	Prob
1	0.001	0.001	0.0010	
2	0.014	0.014	0.0864	0.769
3	0.023	0.023	0.3279	0.849
4	-0.075	-0.075	2.8206	0.420
5	-0.033	-0.034	3.3126	0.507
6	0.056	0.058	4.7386	0.449
7	-0.039	-0.035	5.4104	0.492
8	0.008	0.002	5.4404	0.606
9	0.057	0.051	6.8873	0.549
10	-0.021	-0.013	7.0953	0.627
11	-0.013	-0.017	7.1738	0.709
12	0.045	0.040	8.1109	0.703
13	0.022	0.035	8.3220	0.759
14	-0.019	-0.022	8.4876	0.810
15	0.079	0.068	11.316	0.661
16	-0.019	-0.009	11.486	0.717
17	0.004	0.008	11.492	0.778
18	-0.073	-0.087	13.963	0.670
19	-0.017	-0.004	14.094	0.723
20	0.007	0.017	14.114	0.777
21	0.097	0.088	18.536	0.552
22	0.007	-0.000	18.559	0.613
23	-0.015	-0.024	18.660	0.666
24	-0.026	-0.031	18.983	0.702
25	-0.053	-0.044	20.304	0.679
26	-0.011	-0.001	20.358	0.728
27	-0.006	-0.011	20.377	0.773
28	0.001	0.001	20.378	0.815
29	0.004	-0.003	20.387	0.850
30	0.029	0.019	20.783	0.867
31	-0.006	0.005	20.801	0.894
32	-0.041	-0.048	21.616	0.895
33	0.063	0.072	23.533	0.861
34	0.057	0.063	25.109	0.836
35	-0.014	-0.011	25.209	0.863