OVERLAPPING SCHWARZ METHODS FOR FEKETE AND GAUSS–LOBATTO SPECTRAL ELEMENTS*

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Abstract. The classical overlapping Schwarz algorithm is here extended to the triangular/tetrahedral spectral element (TSEM) discretization of elliptic problems. This discretization, based on Fekete nodes, is a generalization to nontensorial elements of the tensorial Gauss–Lobatto–Legendre quadrilateral spectral elements (QSEM). The overlapping Schwarz preconditioners are based on partitioning the domain of the problem into overlapping subdomains, solving local problems on these subdomains, and solving an additional coarse problem associated with either the subdomain mesh or the spectral element mesh. The overlap size is generous, i.e., one element wide, in the TSEM case, while it is minimal or variable in the QSEM case. The results of several numerical experiments show that the convergence rate of the proposed preconditioning algorithm is independent of the number of subdomains $N$ and the spectral degree $p$ in case of generous overlap; otherwise it depends inversely on the overlap size. The proposed preconditioners are also robust with respect to arbitrary jumps of the coefficients of the elliptic operator across subdomains.

Key words. spectral elements, Fekete nodes, Gauss–Lobatto–Legendre nodes, overlapping Schwarz preconditioners

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1. Introduction. Spectral and $hp$-finite elements are among the most successful high-order methods for the numerical approximation of partial differential equations. These methods can achieve spectral convergence of the discrete solution by allowing both $h$-refinement of the elements mesh and $p$-refinement of the polynomial degree in each element. While hierarchical $hp$-finite elements employ modal basis functions (see [29, 27]), spectral element methods on quadrilateral elements (QSEM) employ a nodal approach based on the tensorial product of one-dimensional Gauss–Lobatto–Legendre (GLL) points (see, e.g., [1, 17, 7]). The need to extend spectral element formulations to complex geometries and unstructured meshes has recently led to the construction and study of triangular/tetrahedral spectral elements (TSEM); see, e.g., [15, 16, 31, 33, 12, 13] and the previous studies of interpolation nodes on triangles [5, 6, 14] and on $p$-version substructuring methods [2]. In this paper, we consider TSEM based on the Fekete nodes [3, 30], as in our previous works [19, 20, 21].

The discrete systems produced by these high-order methods are much more ill-conditioned than in the standard finite element or finite difference cases, and therefore the design and analysis of efficient iterative solvers is still a challenging research topic. In this paper, we construct and study overlapping Schwarz methods for Fekete–Gauss TSEM, obtaining solvers that are scalable (independent of the number of subdomains $N$) and optimal (independent of the spectral degree $p$). This last property holds because we employ generous overlap (at least one element wide) between subdomains. In the QSEM case it is possible to construct Schwarz preconditioners with minimal

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overlap by extending each subdomain with a few rows of GLL nodes of the neighboring elements [4, 10, 11, 18], but the design of a similar technique in the TSEM case is still an open problem, since Fekete points are not distributed as a tensor product. The good convergence properties of our solver are retained also for heterogeneous problems with discontinuous coefficients across subdomain boundaries. Preliminary results with only one element per subdomain and constant coefficients have been reported in the conference paper [21]. A different approach involving preconditioners based on nonoverlapping methods of Neumann–Neumann type for TSEM has been recently studied in [22].

In the rest of the paper, we introduce the model elliptic problem and its QSEM and TSEM discretizations (section 2), the overlapping Schwarz preconditioners in additive and multiplicative form (section 3), and the results of several numerical experiments with the additive Schwarz preconditioners for TSEM and QSEM for homogeneous and heterogeneous problems (section 4).

2. The model problem and SEM formulation. Let \( \Omega \in \mathbb{R}^d \), \( d = 2, 3 \), be a bounded Lipschitz domain with piecewise smooth boundary \( \partial \Omega \). For simplicity, we consider a model elliptic problem in the plane \( (d = 2) \), with homogeneous Dirichlet boundary data, but the numerical methods and results presented in this paper can be applied equally well in three dimensions and to more general elliptic problems:

\[
- \text{div}(\alpha \nabla u) + \beta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( \alpha, \beta > 0 \) are piecewise constant in \( \Omega \) and \( f \) is a given function in \( L^2(\Omega) \). We denote by \( L^2(\Omega) \) the space of square integrable measurable functions in \( \Omega \), and by \( H^1(\Omega) \) the space of functions in \( L^2(\Omega) \) whose gradient is in \( [L^2(\Omega)]^2 \). Then let \( V \) be the Sobolev space

\[
V \equiv H^1_0(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega \}.
\]

The weak formulation of (2.1) reads (see, e.g., [26]): Find \( u \in V \) such that

\[
a(u, v) := \int_{\Omega} (\alpha \nabla u \cdot \nabla v + \beta uv) \, dx = (f, v) := \int_{\Omega} fv \, dx \quad \forall v \in V.
\]

The variational problem (2.2) is discretized by the standard conforming spectral element method, either quadrilateral-based (QSEM) or triangle-based (TSEM). The method can be viewed as a nodal version of the \( hp \)-finite element method, which uses GLL points of the quadrilateral for the QSEM and the Fekete points of the triangle for the TSEM, and employs a discrete space consisting of continuous piecewise polynomials of degree \( p \) within each quadrilateral or triangular spectral element; see [1, 7, 17] for a general introduction and an analysis of the method. For the QSEM, let \( Q_{\text{ref}} \) be the reference square \( (-1, 1)^2 \) and let \( \mathbb{Q}_p(Q_{\text{ref}}) \) be the set of polynomials on \( Q_{\text{ref}} \) of degree \( \leq p \) in each variable. Then, for the TSEM, let \( T_{\text{ref}} = \{(r, s) : -1 \leq r, s, r + s \leq 0\} \) be the reference triangle and let \( \mathbb{P}_p(T_{\text{ref}}) \) be the set of polynomials on \( T_{\text{ref}} \) of total degree \( \leq p \). We assume that the original domain \( \Omega \) is decomposed into \( K \) quadrilateral or triangular spectral elements \( T_k \),

\[
\overline{\Omega} = \bigcup_{k=1}^{K} T_k.
\]

This is a conforming finite element partition; in particular, the intersection between two distinct elements \( T_k \) is either the empty set or a common vertex or a common side.
We denote by $h$ the maximum diameter of the elements $T_k$ and by $\tau_h$ the associated finite element mesh. Each element $T_k$ is the image of the reference square $Q_{ref}$ or triangle $T_{ref}$ by means of a suitable mapping $g_k$, $k = 1, \ldots, K$, i.e., $T_k = g_k(Q_{ref})$ or $T_k = g_k(T_{ref})$.

Finally, the space $V$ is discretized by continuous piecewise polynomials of degree $\leq p$ in each variable for QSEM,

$$V^Q_{K,p} = \{ v \in V : v|_{\Omega_k} \circ g_k \in Q_p(Q_{ref}), \ k = 1, \ldots, K \},$$

or of total degree $\leq p$ for TSEM,

$$V^T_{K,p} = \{ v \in V : v|_{\Omega_k} \circ g_k \in P_p(T_{ref}), \ k = 1, \ldots, K \}.$$

The spectral element approximation of the variational elliptic problem (2.2) is obtained by replacing the $L^2$-inner product and the bilinear form defined in (2.2) with their approximations based either on GLL points for QSEM or on Gauss points for the Fekete–Gauss TSEM.

2.1. The GLL QSEM. We consider conforming QSEM based on GLL quadrature points, which also allows the construction of a tensor-product basis for $V^Q_{K,p}$. We denote by $\{\xi_j\}_{j=0}^p$ the set of GLL points of $[-1, 1]$ that are the $(p+1)$ zeros of the polynomial

$$(1 - \zeta^2) \frac{\partial L_p(\zeta)}{\partial \zeta},$$

where $L_p$ is the $p$th Legendre polynomial in $[-1, 1]$. Then we denote by $\sigma_j$ the quadrature weight associated with $\xi_j$. Let $l_j(x)$ be the Lagrange interpolating polynomial of degree $\leq p$ which vanishes at all the GLL nodes except $\xi_j$, where it equals 1. The Lagrangian nodal basis functions on the reference square $Q_{ref}$ are defined by building tensor products

$$l_j(x)l_\ell(y), \quad 0 \leq j, \ell \leq p.$$ 

Each function $u \in Q_p(Q_{ref})$ is expanded in this nodal GLL basis through its values at GLL nodes $u(\xi_j, \xi_\ell)$, $0 \leq j, \ell \leq p$, as

$$u(x, y) = \sum_{j=0}^p \sum_{\ell=0}^p u(\xi_j, \xi_\ell)l_j(x)l_\ell(y).$$ 

Then, on $Q_{ref}$, the discrete $L^2$-inner product is

$$(u, v)_{Q_{ref},p} = \sum_{j=0}^p \sum_{\ell=0}^p u(\xi_j, \xi_\ell)v(\xi_j, \xi_\ell)\sigma_j\sigma_\ell,$$

and in general on $\Omega$,

$$(u, v)^Q_{K,p} = \sum_{k=1}^K \sum_{j=0}^p \sum_{\ell=0}^p (u \circ g_k)(\xi_j, \xi_\ell)(v \circ g_k)(\xi_j, \xi_\ell)|J^Q_k|\sigma_j\sigma_\ell,$$

where $|J^Q_k|$ is the Jacobian of the mapping $g_k$. We then obtain the discrete variational problem: Find $u \in V^Q_{K,p}$ such that

$$(a^Q_{K,p}(u, v) = (f, v)^Q_{K,p} \quad \forall v \in V^Q_{K,p},$$
where \( a^{Q}_{K,p} (\cdot, \cdot) \) is obtained by substituting each integral in (2.2) with the GLL quadrature rule (2.3).

The discrete problem (2.4) can be written equivalently in matrix form as a linear system \( A_{Q} \mathbf{u} = \mathbf{b} \), where \( A_{Q} \) is here the assembled QSEM matrix, \( \mathbf{b} \) is the load vector accounting for the contribution of \( f \), and \( \mathbf{u} \) is the vector of the unknown nodal values of the function \( u \) at the GLL nodes.

### 2.2. The Fekete–Gauss TSEM

For TSEM it is no longer possible to define spectral elements by tensor product as for QSEM. With \( n = (p + 1)(p + 2)/2 \), let \( \{ \psi_j \}_{j=1}^{n} \) be a polynomial basis of \( \mathbb{P}_{p}(T_{\text{ref}}) \) for the usual \( L^2(T_{\text{ref}}) \)-inner product, e.g., the Koornwinder–Dubiner orthogonal basis [9]. The Fekete points on \( T_{\text{ref}} \) are defined as the points \( \{ \hat{x}_i \}_{i=1}^{m} \) that maximize the determinant of the Vandermonde matrix \( V \) with elements

\[
V_{ij} = \psi_j(\hat{x}_i), \quad 1 \leq i, j \leq n,
\]

and an algorithm to accurately solve the corresponding optimization problem is described in [30]. For TSEM introduced in [31], the Fekete points are used as approximation points; i.e., the Lagrange polynomials \( \{ \phi_i \}_{i=1}^{n} \) built on these points are used as basis functions. Among the main properties of Fekete points proved in [3, 30], we recall that for the cube, Fekete points and Gauss–Lobatto points coincide, thus providing a strong link with the usual QSEM.

Unlike GLL points, a quadrature formula based on Fekete points is exact only for integrands in \( \mathbb{P}_{p}(T_{\text{ref}}) \). This fact has suggested for the TSEM that we might separate the sets of approximation and quadrature points, using the Fekete points \( \{ \hat{x}_i \}_{i=1}^{n} \) for the first set and Gaussian points \( \{ \hat{y}_i \}_{i=1}^{m} \) for the second set, obtained by imposing an exact integration of polynomials, e.g., in \( \mathbb{P}_{2p}(T_{\text{ref}}) \); see [20]. Given the values of a polynomial \( u_{p} \in \mathbb{P}_{p}(T_{\text{ref}}) \) at the Fekete points, one can set up interpolation and differentiation matrices to compute, at the Gauss points, the values of \( u_{p} \) and of its derivatives, respectively. For instance, denoting by \( \mathbf{y} \) the vector of the \( u_{p}(\hat{x}_i), 1 \leq i \leq n \), and by \( \mathbf{y}' \) that of the \( u_{p}(\hat{y}_i), 1 \leq i \leq m \), we have \( \mathbf{y}' = V'V^{-1}\mathbf{y} \), where \( V'_{ij} = \psi_j(\hat{y}_i) \). On a generic triangle \( T_{k} = g_k(T_{\text{ref}}) \), the same relation between \( \mathbf{y}' \) and \( \mathbf{y} \) holds true, provided that \( u_{i} = (u_{p} \circ g_{k})(\hat{x}_i) \) and \( u_{j}' = (u_{p} \circ g_{k})(\hat{y}_j) \). Similarly, in \( T_{\text{ref}} \) one has \( (\partial \mathbf{y})' = W'W^{-1}\mathbf{y} \), with \( \partial \) for differentiation with respect to any coordinates and where \( W_{ij}' = \partial \psi_j(\hat{y}_i) \). Then, using the chain rule, one can compute derivatives in the generic triangle.

The TSEM requires, of course, the use of highly accurate integration rules based on Gauss points, which are still an open subject of research, but in any case, we can use integration rules based on Gauss points for the quadrilateral and then map them to \( T_{\text{ref}} \); see [17]. On a generic triangle \( T_{k} = g_k(T_{\text{ref}}) \),

\[
(u, v)_{T_{k},p} = \sum_{j=1}^{m} u_{j}' \psi_{j}' |J_{k}(\hat{y}_j)| \omega_j,
\]

where \( \omega_j > 0, 1 \leq j \leq m \), are the quadrature weights and \( |J_{k}^T| \) is the Jacobian of the mapping \( g_k \) between \( T_{\text{ref}} \) and \( T_{k} \), and in general on \( \Omega \),

\[
(u, v)^T_{K,p} = \sum_{k=1}^{K} (u, v)_{T_{k},p}.
\]
As for (2.4), we obtain a discrete problem
\begin{equation}
 a_T^{K,p}(u,v) = (f,v)^{K,p}_T \quad \forall v \in V^{T}_K,p,
\end{equation}
which can be written in matrix form as a linear system $A_T u = b$. The TSEM matrix $A_T$ is less sparse than the QSEM matrix $A_Q$ and more ill-conditioned, since its condition number grows as $O(p^4 h^{-2})$ rather than $O(p^3 h^{-2})$ for $d = 2$ (see section 4).

3. Decomposition into subdomains and overlapping Schwarz preconditioners. We now apply some domain decomposition techniques and build some preconditioning techniques for the iterative solution of the QSEM or TSEM discrete systems $A u = b$ by the preconditioned CG (PCG) method, with $A = A_Q$ for QSEM or $A = A_T$ for TSEM. The preconditioner belongs to the family of overlapping Schwarz methods and is built from the solution of parallel independent local elliptic problems on overlapping subdomains, in addition to the solution of coarse problems on the coarse mesh, needed to ensure scalability. For a general introduction to domain decomposition methods and overlapping Schwarz preconditioners we refer to [8, 32, 28, 26] and the references therein.

Preliminary results on overlapping Schwarz preconditioners for Fekete TSEM have been reported in the conference paper [21], where we considered only the case of one spectral element per subdomain and constant coefficients in the elliptic operator. For the classical GLL QSEM case with only one element per subdomain, several works are available in the literature, both for the scalar and vector cases; see, e.g., [4, 10, 24, 25]. In this paper, we extend our investigation to the more general case where $\Omega$ is decomposed into subdomains $\Omega_i$, which are in turn partitioned into spectral elements, for both the triangular and quadrilateral case. To this aim, we assemble the elements $T_k$, $k = 1, \ldots, K$, of the QSEM or TSEM partitions into $N$ quadrilateral or triangular subdomains $\Omega_i$, each consisting of $K_i$ spectral elements $T_k$. With a local renumbering of the triangles we then have
\begin{equation}
 \Omega_i = \bigcup_{k=1}^{K_i} T_k,
\end{equation}
where $K = \sum_i K_i$ (see Figures 1–3 below for some examples). We assume also that the subdomain partition is conforming, i.e., that the intersection between two distinct subdomains is either the empty set or a common vertex or a common side, and we denote by $H$ the maximum diameter of the subdomains $\Omega_i$.

The coarse mesh $\tau_0$ is defined to be either the subdomain mesh $\tau_H$ or the element mesh $\tau_h$. Let $\tau_p$ be the fine discretization mesh determined by either the GLL or the Fekete nodes introduced in each element $T_k$ in sections 2.1 and 2.2. We can define different discretizations and overlapping partitions, according to the spectral element domain decomposition of $\Omega$ and the set of nodes within each subdomain. In the special case where $H = h$, each element $T_k$ coincides with a whole subdomain. When $H > h$, we associate groups of triangular or quadrilateral elements to each subdomain, and two different choices for the coarse problem are possible, one associated with the subdomain mesh $\tau_H$, the other with the element mesh $\tau_h$, and both using coarse basis functions with $p = 1$ polynomial degree. We will denote them as $\tau_H$-coarse space and $\tau_h$-coarse space.

QSEM. The coarse triangulation $\tau_0$ is given by quadrilateral subdomains $\Omega_i$ (resp., elements $T_k$) providing a coarse problem on $\tau_0 \equiv \tau_H$ (resp., $\tau_0 \equiv \tau_h$) with bilinear finite
Fig. 1. QSEM examples with $N = 3 \times 3$ subdomains $\Omega_i'$ corresponding to an overlap of $\delta = 2$ GLL points. The internal overlapping subdomain is depicted with thick dashed lines. Left: spectral degree $p = 9$ and $K_i = 1$; i.e., each subdomain consists of only one element. Right: spectral degree $p = 6$ and $K_i = 2 \times 2$; i.e., each subdomain consists of four elements.
Fig. 2. TSEM example with spectral degree $p = 6$, $K = 6 \times 6 \times 2$ triangular elements, and $K_i = 1$; i.e., each subdomain consists of only one element. In this case, an internal overlapping subdomain (depicted with thick dashed lines) consists of 13 elements.

Fig. 3. TSEM example with spectral degree $p = 6$. Left: $N = 3 \times 3$ square subdomains, $K_i = 2 \times 2 \times 2$ elements per subdomain; in this case, an internal overlapping subdomain (depicted with thick dashed lines) consists of 30 elements. Right: $N = 3 \times 3 \times 2$ triangular subdomains, $K_i = 4$ elements per subdomain; in this case, an internal overlapping subdomain (depicted with thick dashed lines) consists of 22 elements.

$N = 6 \times 6 \times 2$ triangular subdomains ($H = 1/3$), polynomial degree $p = 6$, and one element per subdomain ($H = h$). Moreover, see Figure 3 (left) for a two-dimensional example corresponding to $N = 3 \times 3$ quadrilateral subdomains ($H = 2/3$), polynomial degree $p = 6$, and $K_i = 2 \times 2 \times 2$ triangular elements per subdomain, and Figure 3 (right) for a two-dimensional example corresponding to $N = 3 \times 3 \times 2$ triangular subdomains, polynomial degree $p = 6$, and $K_i = 2 \times 2$ triangular elements per subdomain. Overlapping techniques involving a smaller number of elements (e.g., sharing edges of $T_k$ only) proved unsuccessful, whereas less generous overlapping partitions considering few nodes around $\Omega$, similarly to the QSEM case cannot be designed straightforwardly, since the internal Fekete nodes are not distributed regularly as in a tensor product.
Overlapping Schwarz preconditioners. Overlapping Schwarz methods can now be described either in matrix form or in terms of a space decomposition of the discrete spaces. Let us consider the matrix form first. The overlapping Schwarz preconditioner $B^{-1}$ for $A$ is based on the following:

(a) the solution of a coarse problem on the coarse mesh $\tau_0$, with bilinear elements for QSEM and bilinear or linear ones for TSEM ($p = 1$);

(b) the solutions of local problems on the overlapping subdomains $\Omega_i'$.

For the coarse solve, we need to define:

(a) a restriction matrix $R_0$; its transpose $R_0^T$ interpolates coarse bilinear (resp., bilinear or linear) functions on $\tau_0$ to spectral elements functions on the fine GLL (resp., Fekete) mesh $\tau_p$;

(b) a coarse stiffness matrix $A_0 = R_0 A R_0^T$ needed for the solution of the coarse problem with $p = 1$ on $\tau_0$.

For the local solves, we need to define the following:

(b) local stiffness matrices $A_i = R_i A R_i^T$ needed for the solution of the $i$th local problem on $\Omega_i'$ with zero Dirichlet boundary conditions on $\partial \Omega_i'$.

Finally, we define

$$P_i = R_i^T A_i^{-1} R_i A, \quad i = 0, 1, \ldots, N.$$

These are the building blocks of the proposed preconditioners. The additive form of the overlapping Schwarz preconditioner is

$$B_{add}^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^N R_i^T A_i^{-1} R_i, \quad \text{i.e.,} \quad P_{add} \equiv B_{add}^{-1} A = P_0 + \sum_{i=1}^N P_i,$$

while the multiplicative form is

$$B_{mul}^{-1} A = I - (I - P_N) \cdots (I - P_1)(I - P_0).$$

More general hybrid variants can be considered as well; see [28].

These preconditioners are associated with the space decomposition

$$V_{K,p} = V_0 + \sum_{i=1}^N V_i,$$

where either $V_{K,p} \equiv V_{K,p}^Q$ or $V_{K,p} \equiv V_{K,p}^T$. Here, $V_0$ is the subspace of $V_{K,p}$ consisting of piecewise bilinear or linear functions on the coarse mesh $\tau_0$ and for QSEM

$$V_i = \{ v \in V_{K,p}^Q : v = 0 \text{ at all the GLL nodes outside } \Omega_i' \text{ and on } \partial \Omega_i' \},$$

while for TSEM

$$V_i = \{ v \in V_{K,p}^T : v = 0 \text{ at all the Fekete nodes outside } \Omega_i' \text{ and on } \partial \Omega_i' \}.$$
where \( a_{K,p} \equiv a_{K,p}^Q \) for QSEM and \( a_{K,p} \equiv a_{K,p}^T \) for TSEM, it is then easy to check that \( B^{-1}_{\text{add}}A \) given in (3.1) is exactly the matrix form of the additive Schwarz operator

\[
T_{\text{add}} = T_0 + T_1 + \cdots + T_N.
\]  

Analogously, \( B^{-1}_{\text{mul}}A \) given in (3.2) is the matrix form of the multiplicative Schwarz operator

\[
T_{\text{mul}} = I - (I - T_N) \cdots (I - T_1)(I - T_0).
\]

**Condition number estimates.** The spectral equivalence between the QSEM matrices (mass and stiffness) and the usual \( P_1 \) finite element matrices obtained when using the GLL nodal mesh (see Casarin [4]) allows us to transfer the main domain decomposition results from the finite element case to QSEM (see, e.g., Toselli and Widlund [32]).

**Theorem 3.1.** The condition number \( \kappa_2(T_{\text{add}}) = \lambda_{\text{max}}(T_{\text{add}})/\lambda_{\text{min}}(T_{\text{add}}) \) of the overlapping Schwarz QSEM operator \( T_{\text{add}} \) in additive form is bounded by

\[
\kappa_2(T_{\text{add}}) \leq C \left( 1 + \frac{H}{\delta^*} \right),
\]

with the constant \( C \) independent of \( p, H, h, \delta^* \).

As in the standard finite element case, we cannot guarantee that in the case of QSEM this bound is independent of the coefficient jumps in the elliptic operator, in our case \( \alpha \), but the numerical experiments reported in the next section show that such independence holds.

The bound (3.5) implies that in case of minimal overlap \( \delta = 1 \) GLL points, i.e., \( \delta^* \approx h/p^2 \), we have

\[
\kappa_2(T_{\text{add}}) \leq C \left( 1 + \frac{H}{h} p^2 \right);
\]

i.e., the number of iterations is expected to scale like \( p \) for fixed \( H/h \), while it scales like \( \sqrt{H/h} \) for fixed \( p \). In case of generous overlap \( \delta^* = h \), we have

\[
\kappa_2(T_{\text{add}}) \leq C \left( 1 + \frac{H}{h} \right);
\]

i.e., the number of iterations is expected to be independent of \( p \) and on the order of \( \sqrt{H/h} \). This was already proved in Pavarino [23] for \( hp \)-finite elements and generous overlap. For unstructured \( hp \) elements on nontensorial elements, such as TSEM, the spectral equivalence used for QSEM in [4, 23] does not hold, and preconditioners with small overlap are not known. Nevertheless, we can build preconditioners with generous overlap as shown before, and the numerical results of the next section show that they are optimal and scalable. Hence we conjecture that a bound as in Theorem 3.1 also holds for TSEM.

**Complexity estimates for TSEM.** We briefly examine here the computational costs of solving both the original TSEM discrete problem and the TSEM preconditioned problem in two dimensions on a serial computer. Analogous estimates can be derived in three dimensions and for QSEM.

We consider for simplicity the square domain \( \Omega = (0,1)^2 \) with a uniform subdivision in triangular elements as in Figure 3, where the total number of degrees of freedom
is $d_T = O(p^2 NK)$, since we have $O(p^2)$ degrees of freedom per element, $K_i$ elements per subdomain, and $N$ subdomains (in this geometry, we have $h^{-1} = \sqrt{NK_i}$ and $H^{-1} = \sqrt{N}$). The band of the stiffness matrix is $b_T = O(p^2 \sqrt{NK_i})$, i.e., the number of degrees of freedom on one spectral element row (or column) of the square domain.

(a) The computational cost of solving the original TSEM system with a banded direct solver is $O(d_T b_T^2) = O(p^6 (NK_i)^2)$). The cost of solving the same system with a CG method would be proportional to the number of required CG iterations (i.e., the square root of the condition number), i.e., $O((p^2 h^{-1}) = O(p^2 \sqrt{NK_i})$) times the cost of each CG iteration. The latter is dominated by the matrix-vector products, which for our banded matrix involve $O(d_T b_T) = O(p^4 (NK_i)^{3/2})$ operations. Hence the CG cost would be $O(p^6 (NK_i)^2)$, i.e., similar to the direct solver’s one.

(b) The computational cost of solving the TSEM system with the Schwarz PCG method is again given by the product of the required PCG iterations and the cost of each PCG iteration; see [28, Chapter 3.6]. From (3.7), the condition number of our two-level Schwarz preconditioner with generous overlap is $O(H/h) = O(\sqrt{K_i})$. Since $H/h$ can be kept constant by decreasing $h$ and $H$ proportionally, we have that the PCG iterations are $O(1)$ (otherwise the extra cost would only be a factor $\sqrt{K_i}$), the cost per iteration is dominated by the matrix-vector products and the preconditioner solves. We assume that we have factored the local and coarse matrices before starting the iterations, so that the local and coarse solves are just banded back-substitutions, with band $b_i = O(p^2 \sqrt{K_i})$ for the $i$th local matrix and band $b_0 = \sqrt{N}$ for the coarse matrix (linear elements on the coarse mesh $\tau_0 = \tau_H$). Denoting by $d_i = O(p^2 K_i)$ and $d_0 = O(N)$ the number of local and coarse degrees of freedom, we have that the cost of the local solves is $O(N d_i b_i) = O(p^4 NK_i^{3/2})$, while the cost of the coarse solve is $O(N^{3/2})$. The overlap costs amount to lower-order terms, since the factor $\sqrt{K_i}$ in the local band $b_i$ is only increased by 1 (if the subdomain touches the domain boundary) or 2 (if the subdomain is internal), and similarly the local number of degrees of freedom $d_i$ is only increased by a factor $O(p^2 \sqrt{K_i})$ corresponding to the elements in the overlapping subregions $\Omega_i'$ outside subdomain $\Omega_i$. Adding up the matrix-vector products and all the local and coarse solves of the preconditioner, we then estimate the total PCG cost as $O(p^4 (NK_i)^{3/2} + p^4 NK_i^{3/2} + N^{3/2})$. This cost is lower than the cost of solving the original TSEM system as estimated in (a). Better estimates could be obtained by considering better local and coarse solvers, matrix-vector products, etc. We remark that these serial estimates have a relative importance, since domain decomposition preconditioners should be considered on parallel architectures, where more involved complexity estimates need to include architecture-dependent quantities such as communication and memory costs (see [28, Chapter 3.6]), and are beyond the scope of this paper.

4. Numerical results. We tested the properties of the overlapping Schwarz preconditioned CG method for TSEM and QSEM partitions, by performing a $p$-, $H$-, and $h$-convergence study. We consider either a homogeneous or a heterogeneous material, given by the model problem (2.1) over the domain $\Omega = [-1,1]^2$. The body force $f$ is consistent with $u(x) = \sin(\pi x) \sin(\pi y)$ as an exact solution of (2.1). The discrete problems are solved by either the CG method or the PCG method, with zero initial guess and stopping criterion $\|r^{(v)}\|_2/\|b^{(v)}\|_2 \leq 10^{-7}$, where $r^{(v)}$ is the residual of the linear systems $A_T u = b$ or $A_Q u = b$ at the $v$th iterate. The Schwarz preconditioner is here applied in the additive version (3.1) without or with the coarse correction given by the term $R_H^T A_0^{-1} R_0$ in (3.1). We recall that the overlapping Schwarz preconditioners are built using generous overlap in the TSEM case and using minimal overlap ($\delta = 1$) in the QSEM case.
The homogeneous case: The Fekete–Gauss TSEM. In this section, we report the results of numerical experiments for the model problem (2.1) discretized with triangular TSEM using the Fekete and Gauss nodes. We consider a homogeneous material with \( \alpha = \beta = 1 \).

4.1.1. Case \( K_i = 1 \). We start with the simple choice of associating each triangular element with a subdomain (i.e., \( H = h \), or equivalently, \( N = K \)). The mesh is obtained by first dividing \( \Omega \) into \( M^2 \) identical squares and then by dividing each of them into two triangles, thus providing a partition of \( N = 2M^2 \) triangular subdomains. The overlapping partition is described in Figure 2. The grid-size parameter \( H \) is equal to \( 2/M \).

In Table 1, we report the iteration counts (It.), spectral condition number (\( \kappa_2 \)), and extreme eigenvalues (\( \lambda_{\text{max}}, \lambda_{\text{min}} \)), fixing \( p = 6 \) and varying the parameter \( 1/h = 1/H \) from 2 to 7, i.e., varying the number of subdomains from \( N = 2 \times 4 \times 4 \) to \( N = 2 \times 14 \times 14 \).

Columns 2–3 refer to unpreconditioned CG, columns 4–7 to PCG without coarse problem, and columns 8–11 to PCG with coarse problem (\( \tau_h = \tau_H \) in this case).

The same quantities are reported in Table 2, now fixing \( h = H = 1/2 \) (\( N = 2 \times 4 \times 4 \) subdomains) and varying the degree \( p \) from 3 to 18.

The condition numbers and extreme eigenvalues (not reported in the tables) of the unpreconditioned Fekete matrix are plotted in Figure 4 as a function of \( H \) (left) and \( p \) (right). These results show that the very ill-conditioned TSEM original matrix \( A_T \) has a condition number that grows as \( O(p^4h^{-2}) \). The maximum eigenvalue \( \lambda_{\text{max}} \) grows as \( O(p^2) \), and \( 1/\lambda_{\text{min}} \) grows as \( O(p^2h^{-2}) \). The results also show that the overlapping Schwarz preconditioned operator with or without coarse solver is optimal in \( p \), since all the quantities are bounded by constants independent of \( p \) (Table 2).

Without the coarse problem the preconditioner loses its scalability (Table 1), since
\(\lambda_{\text{min}}\) tends to 0 with increasing \(N\), while the addition of a coarse problem seems to guarantee scalability (higher values of \(1/H\) would be necessary in order to clearly assess scalability).

4.1.2. Case \(K_i > 1\). We now consider the more general case where each subdomain \(\Omega_i\) contains several triangular elements. The TSEM partition is built by first dividing the square computational domain \(\Omega\) into \(N = M^2\) identical square subdomains \(\Omega_i\), with \(M = 2, \ldots, 6\). Then a fine local partition is considered within each subdomain, consisting of \(2 \times 2, \ldots, 6 \times 6\) smaller squares, each of them being divided into two identical triangular elements, so that \(K_i = 8, 18, \ldots, 72\). Each subdomain \(\Omega_i\) is then extended into an overlapping subdomain \(\Omega'_i\) by adding all the surrounding elements that intersect its boundary; see Figure 3(left). The numerical results with the overlapping Schwarz preconditioner are obtained with both the \(\tau_h\)-coarse space and \(\tau_H\)-coarse space.

First, we check the convergence properties of our TSEM preconditioners with respect to the number of subdomains, fixing \(p = 6\) as total degree of the polynomials within each triangle and with \(h = H/3\). The results are reported in Table 3 and plotted in Figure 5(left). As expected, one observes an increase of the condition number when no coarse solver is used. However, nice results are obtained when the preconditioner employs either one of the \(\tau_h\)- or \(\tau_H\)-coarse spaces, and the condition number appears to be bounded above by a constant of about 20.

Second, we vary the number \(K_i\) of triangular elements in each subdomain; i.e., we vary the ratio \(H/h\). The results are reported in Table 4 and plotted in Figure 6(left). Again, one observes the efficiency of the overlapping Schwarz preconditioner, especially when a coarse solver is included. Here it should be remarked that the result is better than expected, since the condition number appears to be independent of \(H/h\), i.e., \(K_i\).

Third, we vary the TSEM polynomial degree \(p\). The results are reported in Table 5 and plotted in Figure 7(left). Of course, the coarse solver has no influence here, and all the three preconditioners have slowly increasing condition numbers that seem to be bounded above by a constant independent of \(p\). We note that such slow increase toward an upper bound was not observed in the case \(K_i = 1\) (Table 2), where the independence of \(p\) was very clear.
Table 3

<table>
<thead>
<tr>
<th>N</th>
<th>CG</th>
<th>PCG without coarse pb.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>It. κ₂</td>
<td>It. λₘₐₓ λₘᵢₙ κ₂</td>
</tr>
<tr>
<td>2 x 2</td>
<td>129 1641.54</td>
<td>37 7.81 0.43 18.07</td>
</tr>
<tr>
<td>3 x 3</td>
<td>176 3687.55</td>
<td>39 8.52 0.38 21.99</td>
</tr>
<tr>
<td>4 x 4</td>
<td>221 6471.91</td>
<td>47 8.71 0.22 39.36</td>
</tr>
<tr>
<td>5 x 5</td>
<td>261 9348.38</td>
<td>46 7.87 0.15 49.95</td>
</tr>
<tr>
<td>6 x 6</td>
<td>329 12152.89</td>
<td>54 7.52 0.11 64.86</td>
</tr>
</tbody>
</table>

These results show that the overlapping Schwarz preconditioner with a coarse problem is optimal and scalable also in the nontrivial case \( H \neq h \), i.e., \( K_i > 1 \). The results are not as clear as in the trivial case with only one element per subdomain \( (H = h, \text{i.e., } K_i = 1) \), but we conjecture that numerical experiments with higher values of \( p \) and \( N \) would confirm these properties.

4.2. The homogeneous case: The GLL QSEM. Overlapping Schwarz methods for QSEM based on GLL points have been successfully applied to many elliptic and parabolic problems (see, e.g., [32, 7] and the references therein). Numerical results are usually carried out by considering the simpler choice \( H = h \) with only one element per subdomain.

In this section, we report the results of numerical experiments with GLL QSEM for the same tests and model problem (2.1) of the previous section. We focus on the nontrivial case \( H \neq h \) and perform analogous \( p \)-, \( H \)-, and \( h \)-convergence tests. The QSEM partition is built by dividing the square computational domain \( \Omega \) into \( N = M^2 \) identical square subdomains \( \Omega_i \) of characteristic diameter \( H \). Then a local partition is considered within each subdomain, consisting of \( K_i \) square elements of
Table 4
TSEM: iteration counts, condition number, and extreme eigenvalues of the original and pre-conditioned matrix, fixing $p = 6$, $N = 3 \times 3$, and varying $K_i$.

<table>
<thead>
<tr>
<th>$K_i$</th>
<th>CG</th>
<th>PCG without coarse pb.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>It.</td>
<td>$\kappa_2$</td>
</tr>
<tr>
<td>$2 \times 2 \times 2$</td>
<td>129</td>
<td>1641.54</td>
</tr>
<tr>
<td>$2 \times 3 \times 3$</td>
<td>176</td>
<td>3687.55</td>
</tr>
<tr>
<td>$2 \times 4 \times 4$</td>
<td>221</td>
<td>6471.91</td>
</tr>
<tr>
<td>$2 \times 5 \times 5$</td>
<td>261</td>
<td>9348.38</td>
</tr>
<tr>
<td>$2 \times 6 \times 6$</td>
<td>329</td>
<td>12152.89</td>
</tr>
</tbody>
</table>

Table 5
TSEM: iteration counts, condition number, and extreme eigenvalues of the original and pre-conditioned matrix, fixing $N = 3 \times 3$, $K_i = 2 \times 3 \times 3$, and varying $p$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>CG</th>
<th>PCG without coarse pb.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>It.</td>
<td>$\kappa_2$</td>
</tr>
<tr>
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<td>69</td>
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<tr>
<td>6</td>
<td>176</td>
<td>3687.55</td>
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<tr>
<td>9</td>
<td>437</td>
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</tr>
<tr>
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<td>940</td>
<td>109050.07</td>
</tr>
<tr>
<td>18</td>
<td>1460</td>
<td>236603.08</td>
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</tbody>
</table>

Fig. 6. Condition number of PCG with and without coarse problem as a function of the number of elements per subdomain $H/h$; TSEM data from Table 4 (left), QSEM data from Table 7 (right).
characteristic diameter $h$, and a tensorial GLL mesh is introduced in each element. We then extend each subdomain $\Omega_i$ to an overlapping subdomain $\Omega_i'$ with minimal overlap, consisting of just $\delta = 1$ layer of GLL nodes outside the boundary; see Figure 1(right) and section 3.

We first study the convergence properties of overlapping Schwarz preconditioners for QSEM with respect to the number of subdomains. We compare the overlapping Schwarz preconditioner operator based either on the $\tau_h$- or the $\tau_H$-coarse space. In Table 6 we report the same quantities of the previous section, and we plot them in Figure 5(right). We fix the local polynomial degree $p = 6$ and the number of elements per subdomain $K_i = 3 \times 3$, which corresponds to $H = 3h$, varying the number of subdomains from $N = 2 \times 2$ to $N = 6 \times 6$. While both unpreconditioned CG and PCG without coarse problem are not scalable, PCG with either the $\tau_h$- or $\tau_H$-coarse space is.

We then consider the convergence properties with respect to the number $K_i$ of elements per subdomain in Table 7 and plot the results in Figure 6(right). We fix the local polynomial degree $p = 6$ and the number of subdomains $N = 3 \times 3$, varying the number of elements per subdomain from $K_i = 2 \times 2$ to $K_i = 6 \times 6$. In agreement
with the theoretical estimate of Theorem 3.1, the preconditioner with the $\tau_H$-coarse space yields condition numbers that grow linearly with $K_i$. On the other hand, the $\tau_h$-coarse space yields condition numbers that are independent of $K_i$.

We then study the convergence properties with respect to the polynomial degree $p$ in Table 8 and plot the results in Figure 7(right). We fix the number of subdomains $N = 3 \times 3$, and the number of elements per subdomain $K_i = 3 \times 3$, varying the local polynomial degree from $p = 3$ to $p = 18$. In agreement with the bound of Theorem 3.1, all PCG iteration counts display a linear growth with $p$, corresponding to a quadratic growth of the condition number associated with a quadratic decay of $\lambda_{\min}$ (while $\lambda_{\max}$ has a constant value of about 4).

Finally, we consider the convergence properties of our preconditioner with respect to the overlap $\delta$ (number of GLL points extending $\Omega_i$ in each direction) in Table 9 and plot the same results with respect to the corresponding values of $\delta^*$ (size of the overlapping region) in Figure 8. We fix the number of subdomains $N = 3 \times 3$, the number of elements per subdomain $K_i = 2 \times 2$, the local polynomial degree $p = 9$, varying the overlap from $\delta = 1$ GLL points (minimal overlap) to $\delta = 9$ GLL

**Table 7**

QSEM: iteration counts, condition number, and extreme eigenvalues of the original and preconditioned matrix, fixing $p = 6$, $N = 3 \times 3$, minimal overlap $\delta = 1$, and varying $K_i$.

<table>
<thead>
<tr>
<th>$K_i$</th>
<th>It.</th>
<th>$\kappa_2$</th>
<th>CG</th>
<th>It.</th>
<th>$\lambda_{\max}$</th>
<th>$\lambda_{\min}$</th>
<th>$\kappa_2$</th>
<th>PCG without coarse pb.</th>
<th>It.</th>
<th>$\lambda_{\max}$</th>
<th>$\lambda_{\min}$</th>
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<td>0.10</td>
<td>38.03</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>$3 \times 3$</td>
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<td>603.09</td>
<td>25</td>
<td>4.00</td>
<td>0.07</td>
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**Table 8**

QSEM: iteration counts, condition number, and extreme eigenvalues of the original and preconditioned matrix, fixing $N = 3 \times 3$, $K_i = 3 \times 3$, minimal overlap $\delta = 1$, and varying $p$.

<table>
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<tr>
<th>$p$</th>
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<th>CG</th>
<th>It.</th>
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<th>$\lambda_{\min}$</th>
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<th>$\lambda_{\min}$</th>
<th>$\kappa_2$</th>
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<th>It.</th>
<th>$\lambda_{\max}$</th>
<th>$\lambda_{\min}$</th>
<th>$\kappa_2$</th>
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<th>$\lambda_{\min}$</th>
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Table 9

QSEM: iteration counts, condition number, and extreme eigenvalues of $P_{add}$, fixing $N = 3 \times 3$, $K_i = 2 \times 2$, $p = 9$, and varying the overlap $\delta$. Columns 2–5 refer to the coarse mesh $\tau_0 = \tau_H$, and columns 6–9 refer to the richer coarse mesh $\tau_0 = \tau_h$.

<table>
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<th>$\delta$</th>
<th>It. $\lambda_{max}$</th>
<th>$\lambda_{min}$</th>
<th>$\kappa_2$</th>
<th>It. $\lambda_{max}$</th>
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<td>4.50</td>
<td>0.97</td>
<td>4.63</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>14</td>
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</tr>
<tr>
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<td>4.74</td>
<td>1.00</td>
<td>4.74</td>
<td>12</td>
<td>5.00</td>
</tr>
</tbody>
</table>

Fig. 8. QSEM: condition number and iteration counts for the overlapping Schwarz preconditioned operator, as a function of the overlap size $\delta^*$ normalized with respect to $h$, from the corresponding data expressed in terms of $\delta$ in Table 9.

Again in agreement with the theory, the condition numbers improve with increasing $\delta$ because $\lambda_{min}$ improves considerably. The improvement is large in going from $\delta = 1$ to $\delta = 2$ and 3, while it becomes rapidly marginal (or negative) for larger values of $\delta$, due to the slight growth of $\lambda_{max}$.

4.3. The heterogeneous case: The Fekete–Gauss TSEM. To check the efficiency of the additive Schwarz preconditioner for TSEM, we vary the diffusion coefficient $\alpha$ in the elliptic operator of our model problem (2.1). The computations are done for the reference case with $p = 6$, $N = 3 \times 3$, and $K_i = 2 \times 3 \times 3$.

First, we use two different values of $\alpha$, $\alpha_1 = 1$ in some subdomains and $\alpha_2 = 10^4$, with $t = -3, -2, \ldots, 3$ in the others. These two values are distributed in a checkerboard layout on the subdomains, starting with $\alpha_1$ on the top left subdomain; see Figure 9(left). The results are reported in Table 10. We remark that the efficiency of the overlapping Schwarz preconditioner is not affected by the jump of $\alpha$ at the interfaces of the subdomains, while unpreconditioned CG greatly suffers the severe
Fig. 9. Values of the discontinuous coefficient $\alpha$ of the elliptic operator in the heterogeneous test of Tables 10 and 11: checkerboard (left) and random (right) layout.

Table 10

| TSEM for heterogeneous case: iteration counts, condition number, and extreme eigenvalues of the original and preconditioned matrix, fixing $p = 6$, $K_i = 3 \times 3 \times 2$, $N = 3 \times 3$, and considering the coefficient $\alpha$ discontinuous across subdomains. |
|---|---|---|---|---|
| $t$ | Iter. $\kappa_2$ | Iter. $\lambda_{\text{max}}$ | Iter. $\lambda_{\text{min}}$ | Iter. $\kappa_2$ |
| $10^0$ | $1$ | $10^0$ | $1$ | $10^{-3}$ | $10^2$ | $10^{-1}$ |
| $10^1$ | $10^1$ | $10^1$ | $10^{-5}$ | $10^0$ | $10^1$ | $10^{-5}$ |
| $10^2$ | $10^2$ | $10^2$ | $10^{-10}$ | $10^0$ | $10^1$ | $10^{-10}$ |

We then consider a harder case with “random” distribution of $\alpha$ shown in Figure 9(right), and report the results in the last line of Table 10. Even in this very difficult case, where unpreconditioned CG does not seem to converge, the overlapping Schwarz preconditioner remains very efficient.

4.4. The heterogeneous case: The GLL QSEM. Analogous conclusions are obtained with GLL QSEM; see Table 11. Here the condition numbers of the preconditioned operator seem to even improve with increasing coefficient jumps, while their maximum is attained in absence of jumps for $t = 0$. For the difficult test case with the same random distribution of $\alpha$, unpreconditioned CG again fails to converge, while the overlapping Schwarz preconditioner requires about the same iterations as in absence of jumps.

5. Conclusions. We have constructed and studied overlapping Schwarz preconditioners for the triangular/tetrahedral spectral element (TSEM) discretization of scalar elliptic problems based on Fekete nodes. The proposed preconditioners employ generous overlap among subdomains, since it is not currently know how to construct TSEM preconditioners with small overlap. We also compared the TSEM nontensorial ill-conditioning of the problems due to the increasing coefficient jumps.
discretization with the classical Gauss–Lobatto–Legendre QSEM discretization, where it is possible to construct Schwarz preconditioners with variable overlap (from generous to minimal). In spite of the more severe ill-conditioning of the resulting TSEM discrete problem (where $\kappa_2 \approx O(p^4 h^{-2})$), the convergence rate of the proposed preconditioning algorithm is independent of the number of subdomains $N$ when a coarse problem is employed and of the spectral degree $p$ in case of generous overlap; otherwise it depends inversely on the overlap size. Moreover, the proposed preconditioners are robust with respect to arbitrary jumps across subdomains of the coefficients of the elliptic operator. Our results are limited to regular structured meshes in the plane. Future work will extend this study to three dimensions and to unstructured meshes for more general domains.

### REFERENCES


