Review

Non-conforming high order approximations of the elastodynamics equation

P.F. Antonietti, I. Mazzieri, A. Quarteroni, F. Rapetti

MOX-Modelling and Scientific Computing, Department of Mathematics, Politecnico di Milano, P.za Leonardo da Vinci 32, 20133, Milano, Italy
Université de Nice Sophia Antipolis, Laboratoire de Mathématiques J.A. Dieudonné, Parc Valrose, 06108 Nice, Cedex 02, France
CMCS-MATHICSE, École Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 16 March 2011
Received in revised form 24 September 2011
Accepted 4 November 2011
Available online 15 November 2011

Keywords:
Spectral methods
Non-conforming domain decomposition techniques
Computational seismology
Numerical approximations and analysis

ABSTRACT

In this paper we formulate and analyze two non-conforming high order strategies for the approximation of elastic wave problems in heterogeneous media, namely the Mortar Spectral Element Method and the Discontinuous Galerkin Spectral Element Method. Starting from a common variational formulation we make a full comparison of the two techniques from the points of view of accuracy, convergence, grid dispersion and stability.

© 2011 Elsevier B.V. All rights reserved.

Contents

1. Introduction and motivations ........................................................................................................... 213
2. Problem formulation ......................................................................................................................... 213
3. Non-conforming Galerkin spectral formulations ............................................................................... 214
   3.1. Discontinuous Galerkin spectral formulation .............................................................................. 216
   3.2. Mortar spectral formulation ....................................................................................................... 217
4. Algebraic formulations and time integration .................................................................................... 218
   4.1. Algebraic formulation of the problem ......................................................................................... 218
   4.2. Structural damping .................................................................................................................... 220
   4.3. Time integration scheme ............................................................................................................ 220
5. Error estimates .................................................................................................................................. 220
   5.1. Semi-discrete error estimates-DGSEM ....................................................................................... 220
   5.2. Semi-discrete error estimates-MSEM ....................................................................................... 221
   5.3. Fully-discrete error estimates – DGSEM/MSEM ...................................................................... 221
6. Analysis of grid dispersion and stability ......................................................................................... 222
   6.1. Grid dispersion – DGSEM ......................................................................................................... 222
   6.2. Grid dispersion – MSEM .......................................................................................................... 223
   6.3. Grid dispersion – numerical results .......................................................................................... 223
   6.4. Stability – DGSEM and MSEM ............................................................................................... 226
   6.5. Stability – numerical results ..................................................................................................... 228
7. Accuracy and order of convergence ................................................................................................. 228
8. An application of geophysical interest ............................................................................................. 231
9. Error analysis .................................................................................................................................... 234
   9.1. Semi-discrete error analysis – DGSEM ..................................................................................... 234
   9.2. Semi-discrete error analysis – MSEM ...................................................................................... 235
10. Conclusions .................................................................................................................................... 237
1. Introduction and motivations

The possibility of inferring the physical parameter distribution of the Earth’s substratum, from information provided by elastic wave propagations, has increased the interest towards computational seismology. Recent developments in this scientific discipline concern with different numerical strategies as finite differences, finite elements, but the major efforts apply to spectral element methods (see [1–7]).

A motivation is that, in geophysical or industrial applications, finite difference discretizations require very large systems of equations to model realistic rock properties and uniform meshes are needed. On the other hand, when classical finite element methods are employed for treating complex geometries, it is necessary to invert the mass matrix.

The reasons for using spectral element-based approximations can be summarized in the following lines. Firstly, the flexibility in handling complex geometries, retaining the spatial exponential convergence for locally smooth solutions. Secondly, since spectral element methods are based on the weak formulation of the elastodynamics equations, they handle naturally both interface continuity and free boundary conditions, allowing very accurate resolutions of evanescent interface and surface waves (of major interest in seismology). Finally, spectral element methods retain a high level parallel structure, thus well suited for parallel computers.

However, when dealing with complex wave phenomena, such as soil–structure interaction problems or seismic response of sedimentary basins, the geometrical and polynomial flexibility is an important task for simulating correctly the wave-front field.

For this reason we consider two different non-conforming high-order techniques, namely the Mortar Spectral Element Method (MSEM) [8,9] and the Discontinuous Galerkin Spectral Element Method (DGSEM) [10–12] to simulate seismic wave propagation in heterogeneous media. In contrast to standard conforming discretizations, as Spectral Element Method (SEM) [13,14], these techniques have the further advantages that they can accommodate discontinuities, not only in the parameters, but also in the wavefield, while preserving the energy.

Depending on the involved materials it is possible to make a partition of the computational domain. Then, in each non-overlapping subregion a spectral finite element discretization is employed. The quadrilaterals/hexahedras do not have to match between neighbouring subdomains, and different spectral approximation degrees are allowed. Therefore, the continuity of the solution at the skeleton of the decomposition is imposed weakly, either by means of a Lagrange multiplier for the MSEM, or by penalizing the jumps of the displacement on the skeleton in the DGSEM.

In the present work, starting from a displacement-based weak formulation of the elastodynamics equation, we analyze stability, convergence, accuracy, dissipation and dispersion for the MSEM and DGSEM for the space discretization combined with second order time integration scheme. In particular we prove a priori error bounds for both the semi-discrete and fully-discrete non-conforming methods.

A similar analysis is provided in the existing literature for a slightly different Discontinuous Galerkin formulation, for dynamic linear elasticity and viscoelasticity [12,15]. In fact the above formulation involves an additional penalty term whose physical meaning is unclear. Yet, other authors refer to that analysis when discussing their Discontinuous Galerkin schemes [16,17]. Here we modify and update the results of [12] to analyze the presented DGSEM.

In the MSEM case, at the best of our knowledge, such analysis has never been carried out before in elastodynamics, but only for elliptic and parabolic equations [8,18–20].

Since we are dealing with time-dependent problems, we also take into account of the stability and dispersion property of our numerical scheme.

For wave propagation problems, the grid dispersion criterion determines the lowest number of nodes per wavelength such that the numerical solution has an acceptable level of accuracy, while the stability criterion determines the largest time step allowed for explicit time integration schemes.

A general framework to study the numerical dispersion for the SEM was developed in [21] and analyzed for the acoustic case up to polynomial approximation degree equal to three. In [22] a complete description for the elastic case is given, based on a Rayleigh quotient approximation of the eigenvalue problem characterizing the dispersion relation.

For the DGSEM, grid dispersion has been analyzed in [23,16]. In particular in [23] the dispersion and dissipation errors of the acoustic wave equation in one space dimension are derived using the flux formulation. The results include polynomial approximation degree equal to three and conjectures on the extension to higher degrees are given. Making use of the plane wave analysis, in [16] a complete description of the grid dispersion properties is carried out for both the acoustic and the elastic case.

At the best of our knowledge, for the MSEM no results are available for the grid dispersion properties regarding the elastic wave equation.

For what concerns the stability, a classical numerical approach to solve a second order initial value problem is provided by the family of the Newmark methods [24]. The Leap-Frog Method, stability bounds linking the time step with the size of the elements and the maximum wave velocity. All results obtained are compared to those obtained with the conforming SEM case.

After introducing the elastodynamics problem and its variational formulation in Section 2, we describe in Section 3 the geometrical and functional discretization of the problem within the context of non-conforming approximations. In particular we derive the Mortar and the Discontinuous Galerkin Spectral Formulations. The algebraic aspects of the two methods are then described in Section 4. Section 5 is focused on the convergence estimates while Section 6 is devoted to the grid dispersion and stability analysis, which are carried out for 2-d case. In Sections 7 and 8 we discuss the property of accuracy and convergence of the MSEM and the DGSEM, and present a geophysical application, namely the seismic response of an alluvial basin, respectively. Finally in Section 9 we report the proofs of the convergence estimates given in Section 5.

2. Problem formulation

Let us consider an elastic medium occupying a finite region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma = \partial \Omega$ and unit outward normal...
The boundary is assumed to be composed of portions $\Gamma_D$, where the displacement vector $u$ is prescribed, $\Gamma_N$ where external loads apply, and $\Gamma_{NR}$ where suitable non-reflecting conditions are imposed. The portion $\Gamma_{NR}$ is in fact a fictitious boundary of the computational domain which is introduced to bound the physical domain for the numerical approximation of wave propagation problems in unbounded media. We make the assumptions that either $\Gamma_D$ or $\Gamma_N$ can be empty, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_N \cap \Gamma_{NR} = \emptyset$.

Here and in the sequel, an underlying bar denotes matrix or tensor quantities, while vectors are typed in bold. Having fixed the temporal interval $[0,T]$, with $T$ real and positive, the equilibrium equations for an elastic medium, subjected to an external force $f$ read:

$$\begin{align*}
\rho \partial_\tau u - \nabla \cdot (\sigma(u)) &= f, & \text{in } \Omega \times [0,T], \\
u = 0, & \text{on } \Gamma_D \times [0,T], \\
\sigma(u) \cdot n &= t, & \text{on } \Gamma_N \times [0,T], \\
\text{non reflecting boundary conditions} & \text{on } \Gamma_{NR} \times [0,T], \\
\partial_\tau u &= u_\tau, & \text{in } \Omega \times \{0\}, \\
\rho u &= u_0, & \text{in } \Omega \times \{0\},
\end{align*}$$

(1)

where $u$ is the medium displacement vector, $\sigma$ the stress tensor, $\tau$ the time variable and $\rho$ the material density. Without loss of generality (see, for instance, [28]) we make the following further assumptions on $\Gamma$: on $\Gamma_D$ the medium is rigidly fixed in the space and on $\Gamma_N$ we prescribe surface tractions $t$. Finally, on $\Gamma_{NR}$ non-reflecting boundary conditions are imposed: from the mathematical point of view, the latter have the effect of introducing a fictitious traction $t'$ which is a linear combination of space and time derivatives of the displacement $u$ (cf. [29,27], for example). In particular, for $d = 2$, if $\Gamma_{NR}$ has outward unit normal $n = (n_x, n_y)$ and tangential unit vector $\tau = (\tau_x, \tau_y)$, the non-reflecting conditions in coordinate frame $(\tau, n)$ take the form

$$\begin{align*}
\frac{\partial}{\partial \tau} u \cdot n &= \frac{1}{c_p^2} \left( \frac{\partial}{\partial \tau} u \cdot n + \frac{\partial}{\partial \tau} u \cdot \tau \right), \\
\frac{\partial}{\partial \tau} u \cdot \tau &= \frac{1}{c_s^2} \left( \frac{\partial}{\partial \tau} u \cdot \tau + \frac{\partial}{\partial \tau} u \cdot n \right).
\end{align*}$$

(2)

For $d = 3$ non-reflecting boundary conditions are given by

$$\begin{align*}
\frac{\partial}{\partial \tau} u \cdot n &= \frac{1}{c_p^2} \left( \frac{\partial}{\partial \tau} u \cdot n + \frac{\partial}{\partial \tau} u \cdot \tau_1 + \frac{\partial}{\partial \tau} u \cdot \tau_2 \right), \\
\frac{\partial}{\partial \tau} u \cdot \tau_1 &= \frac{1}{c_s^2} \left( \frac{\partial}{\partial \tau} u \cdot \tau_1 + \frac{\partial}{\partial \tau} u \cdot n \right), \\
\frac{\partial}{\partial \tau} u \cdot \tau_2 &= \frac{1}{c_s^2} \left( \frac{\partial}{\partial \tau} u \cdot \tau_2 + \frac{\partial}{\partial \tau} u \cdot n \right),
\end{align*}$$

(3)

where $\tau_1$ and $\tau_2$ are two arbitrary mutually orthogonal unit vectors on the plane orthogonal to $n$, the normal to $\Gamma_{NR}$, such that $(\tau_1, \tau_2, n)$ defines a right handed Cartesian frame.

The quantities $c_p$ and $c_s$ appearing in (2) and (3) are respectively the compressional and the shear wave velocities, defined as

$$c_p = \sqrt{\frac{\lambda + 2 \mu}{\rho}} \quad \text{and} \quad c_s = \sqrt{\frac{\mu}{\rho}}.$$

(4)

where $\lambda$ and $\mu$ are the Lamé elastic coefficients. We remark that for heterogeneous media $\rho$, $\lambda$ and $\mu$ are bounded functions of the spatial variable, not necessarily continuous, i.e., $\rho$, $\lambda$, and $\mu$ in $L^\infty(\Omega)$. We observe that Neumann type boundary conditions can be simply given by (2) or (3), where the right-hand side is substituted with the known value of the external load $t$.

To complete the system in (1), we prescribe initial conditions $u_0$ and $u_1$ for the displacement and the velocity, respectively. When we consider viscoelastic materials, see Section 7, we introduce in the system (1) an additional term in the form of volume forces $f^{vis} = -2\rho\ddot{u} - \rho\dot{\zeta}^2 u$, where $\zeta$ is a suitable decay factor with dimension inverse of time. Correspondingly, the equation of motion becomes

$$\rho \partial_\tau u - \nabla \cdot (\sigma(u)) = f + f^{vis}.$$  

(5)

The parameter $\zeta$ is spatially variable (i.e. piecewise constant), as in [27], in order to model absorbing regions, thus providing an alternative or a complement to the absorbing boundary conditions. In other cases, like seismic wave propagation through heterogeneous media with strong elastic impedance, this model is used to prevent the onset of non-physical oscillations of the numerical solution.

We consider the strain tensor $\varepsilon$ defined as the symmetric gradient of $u$, i.e.,

$$\varepsilon(u) = \frac{1}{2} \left( \nabla u + \nabla u^\top \right),$$

so that the stress tensor $\sigma$ satisfies the constitutive relation (Hooke’s law)

$$\sigma(u) = \lambda \nabla \cdot u + 2\mu \varepsilon(u) = D \varepsilon(u),$$

where $I$ is the $d$-dimensional identity tensor and $D$ is the fourth order positive definite Hooke’s tensor, satisfying the symmetries $D_{ijk} = D_{jik} = D_{kji}$.

Here and in the sequel we use the standard notation [30] to define the $L^2$-inner product $(\cdot, \cdot)$ for scalar, vector and tensor quantities.

By multiplying the first equation in (1) for a regular enough function $v$ (candidate to represent an admissible displacement), integrating by parts over the domain $\Omega$, using the Green’s formula:

$$-\langle \nabla \cdot (\sigma(u)) , v \rangle = \langle (\sigma(u), \varepsilon(v)) , u \rangle - \langle (\varepsilon(u) , \varepsilon(v)) , f \rangle,$$

and imposing the boundary conditions, the variational formulation of (1) reads: $\forall t \in (0,T)$ find $u(t) \in V$ such that

$$d_{ict}(\rho(u(t)) + A(u(t), v) = \mathcal{L}(v) \quad \forall v \in V,$$

(6)

where the bilinear form $A : V \times V \rightarrow \mathbb{R}$ is defined as

$$A(u, v) = (\sigma(u), \varepsilon(v)),$$

and the linear functional $\mathcal{L} : V \rightarrow \mathbb{R}$ as

$$\mathcal{L}(v) = (\tau, v)_{\Gamma_N} + (t, v)_{\Gamma_{NR}} + (f, v)_{\Gamma_D}.$$

Here $V$ is the Sobolev space $V = \{ v \in [H^1(\Omega)]^d : v = 0 \text{ on } \Gamma_D \}$, where $L^2(\Omega)$ is the space of square integrable functions over $\Omega$ and $H^1(\Omega)$ is the space of functions in $L^2(\Omega)$ with gradient in $[L^2(\Omega)]^d$. We recall that the bilinear form $A(\cdot, \cdot)$ is symmetric, V-elliptic and continuous [31]. These conditions imply that problem (6) admits a unique solution $u \in C^0([0,T]; V) \cap C^1((0,T); [L^2(\Omega)]^d)$ satisfying stability estimates [32,31], provided that $\rho \in L^\infty(\Omega)$ is a strictly positive function, and that $u_0 \in V$, $u_1 \in [L^2(\Omega)]^d$ and $f \in [L^2(\Omega \times (0,T))]^d$.

By introducing a finite dimensional space $V_h$ which is a suitable approximation of $V$, the semi-discrete approximation of (6) reads: $\forall t \in (0,T)$ find $u_h(t) \in V_h$ such that

$$d_{ict}(\rho(u_h(t)) + A(u_h, v)_{V_h} = \mathcal{L}(v) \quad \forall v \in V_h.$$

(7)

In the next section we will explain how to construct $V_h$ for two different families of non-conforming domain decomposition methods, namely, the Mortar Spectral Element Method (MSEM) and the Discontinuous Galerkin Spectral Element Method (DGSEM). Both methods are well suited to allow: (1) variable approximation orders, that is an elementwise polynomial degree, (2) unstructured and non-matching meshes, and (3) exponential rates of convergence in case of smooth solutions, [21].

3. Non-conforming Galerkin spectral formulations

In a domain decomposition approach we start by a discretization of the spatial differential operators in $\Omega$, that relies on a time-independent three-level spatial decomposition of the domain $\Omega$, as follows. At the first level, we subdivide $\Omega$ into $K$ non-
overlapping regions \( \Omega_k, k = 1, \ldots, K \), such that \( \widehat{\Omega} = \bigcup_{k=1}^{K} \Omega_k \) with \( \Omega_k \cap \Omega_\ell = \emptyset \) if \( k \neq \ell \) and we define the skeleton of this (macro) decomposition as \( S = \bigcup_{k=1}^{K} \partial \Omega_k \setminus \partial \Omega \). Note that this (macro) decomposition can be geometrically non-conforming, i.e., for two neighbouring subdomains \( \Omega_k, \Omega_\ell \), the interface \( \gamma = \partial \Omega_k \cap \partial \Omega_\ell \) may not be a complete side (for \( d = 2 \)) or face (for \( d = 3 \)) of \( \Omega_k \) or \( \Omega_\ell \). Then problem (1) is solved in each \( \Omega_k \) together with transmission conditions to ensure that the local solution is the restriction to \( \Omega_k \times [0,T] \) of the global solution. For the elastic problem (1) the transmission conditions reads: (TC1) \( [u] = 0 \) and (TC2) \( [\sigma] = 0 \), where \( [\ ] \) denotes the jump of a quantity across a given interface.

To get the second level, in each \( \Omega_k \) we introduce a partitioning \( \mathcal{T}_{h_k} \), made by elements \( \Omega_k^j \) (quadrilaterals if \( d = 2 \)) or hexahedras if \( d = 3 \), with typical linear size \( h_k \). Let \( \Omega_{\mathcal{T}} = \bigcup_{k=1}^{K} \bigcup_{j=1}^{N_k} \Omega_k^j \) (see Fig. 1). Let us set \( \Omega = (-1,1)^d \) and suppose that there exists a suitable invertible mapping \( \mathcal{F} : \Omega \to \Omega_{\mathcal{T}} \) with (positive) Jacobian \( J_{\mathcal{F}} \). This (meso) partition is instead geometrically conforming in each \( \Omega_k \), thus the intersection of two elements \( \Omega_k^j, \Omega_\ell^j, \ell \neq j \), is either empty, or a vertex, or an edge, or a face of both \( \Omega_k^j \) and \( \Omega_\ell^j \). We thus have that

\[
\int_{\Omega_k} f = \sum_{j=1}^{N_k} \int_{\Omega_j^k} f = \sum_{j=1}^{N_k} \int_{\partial \Omega_j^k} (f \circ \mathcal{F}_j^k) J_{\mathcal{F}_j^k}.
\]

The third (micro) level will be represented by the so-called Gass–Batto–Legendre (GLG) points in each mesh element \( \Omega_k^j \). Let \( \mathbf{Q}_{N_k} (\Omega_k) \) be the space of functions defined on \( \Omega \) that are algebraic polynomials of degree less than or equal to \( N_k \geq 2 \) in each variable \( x_1, \ldots, x_d \) and

\[
\mathbf{Q}_{N_k} (\Omega_k^j) = \left\{ v \in L^2 (\Omega_k^j) : v_{|\partial \Omega_k^j} \in \mathbf{Q}_{N_k} (\partial \Omega_k^j), \forall \partial \Omega_k^j \in \mathcal{T}_{h_k} \right\}.
\]

and finally

\[
\mathbf{V}_h = \{ \mathbf{v} \in [L^2 (\Omega)]^d : \mathbf{v}|_{\partial \Omega} \in [\mathbf{X}_h (\Omega)^d], \forall k = 1, \ldots, K : \partial \Omega_k = \emptyset \},
\]

where \( \delta = (h, N) \) with \( h = (h_1, \ldots, h_d) \) and \( N = (N_1, \ldots, N_d) \) \( K \)-uplets of discretization parameters. Each component \( h_k \) and \( N_k \) represents the mesh size and the degree of the polynomial interpolation in the region \( \Omega_k \), respectively. In order to construct a nodal basis for \( \mathbf{V}_h \) we introduce on each element \( \Omega_k^j \) a set of interpolation points \( \{ p \} \) and corresponding degrees of freedom which allow to identify uniquely a generic function in \( \mathbf{V}_h \). We remark that the definition of the space \( \mathbf{V}_h \), the basis functions will not be globally continuous on the whole domain \( \Omega \). In the spectral element approximation, the interpolation points are the GLL points. On the reference element \( \Omega \), these points are tensor product of points defined in the interval \([-1,1]\) as the zeros of \((1-x^2)^{C_2} L_n \) where \( L_n \) is the derivative of the Legendre polynomial \( L_n \). This means that there exist \( N_k + 1 \) points \( \{ p \} \) for the interpolation of a polynomial of degree \( N_k \) in \([-1,1]\) [14]. As previously observed, in higher dimensions, the spectral nodes \( \{ p \} \) are defined on the reference element \( \Omega \) via tensor product of the one dimensional distribution, and are then mapped onto the generic element \( \Omega_k^j \) in the physical space by \( \mathcal{F}_j^k \).

In the SEM, the interpolation points are used also as quadrature points. Thus, we have

\[
\int_{\Omega_k} (f \circ \mathcal{F}_j^k) J_{\mathcal{F}_j^k} \approx \sum_{t=1}^{N_k+1} (f \circ \mathcal{F}_j^k)(p_t) w_t,
\]

where \( w_t \) are the weights of the GLL quadrature formula which is exact for all \( (f \circ \mathcal{F}_j^k) J_{\mathcal{F}_j^k} \in \mathbf{Q}_{N_k+1} (\Omega) \). The spectral shape functions \( \mathbf{V} \) in \( \mathbf{V}_h \) are defined as \( \mathbf{V} (\{ p \}) = \delta_{p, p_t} \). The strong form of the problem (7) is then equivalent to: for all \( \mathbf{v} = (\mathbf{v}, \mathbf{n}) \in \mathbf{V} \),

\[
\sum_{k=1}^{K} \mathbf{G}_k (\mathbf{u}; \mathbf{v}) + \mathbf{A}_k (\mathbf{u}; \mathbf{v}) + \frac{1}{2} \mathbf{B}_k (\mathbf{u}; \mathbf{v}) = \mathbf{f}_k (\mathbf{v}),
\]

where \( \mathbf{f}_k = \mathbf{f} \big|_{\Omega_k} \big|_{\partial \Omega_k} \) and \( \mathbf{G}_k = \mathbf{G} \big|_{\Omega_k} \big|_{\partial \Omega_k} \). The bilinear form \( \mathbf{B}(\cdot, \cdot) \) may either be zero or gather all the contributions \( \mathbf{B} (\mathbf{u}, \mathbf{v}) = \mathbf{B} (\mathbf{u}; \mathbf{v}) \big|_{\partial \Omega_k} \). Depending on the chosen non-conforming approach, the functional space \( \mathbf{V}_h \) is completed by additional conditions on \( u_{\partial \Omega_k} = 1, \ldots, K \), on the skeleton of the macro decomposition which ensure that \( u_{\partial \Omega_k} \) is the restriction to \( \Omega_k \) of \( u \in [H^1 (\Omega)]^d \). The bilinear form \( \mathbf{B}(\cdot, \cdot) \) may either be zero or gather all the contributions \( \mathbf{B} (\mathbf{u}, \mathbf{v}) = \mathbf{B} (\mathbf{u}; \mathbf{v}) \big|_{\partial \Omega_k} \). Depending on the chosen approach. In fact, (TC1) is imposed by introducing a weak continuity condition on \( \mathbf{V} \) compatible with the considered formulations while (TC2) is enforced strongly. In both situations this lead to a strongly consistent numerical method. This means that the exact solution satisfies the numerical scheme for each choice of \( \mathbf{h} \) and \( N \) [33].

Eq. (8) represents the starting point to introduce the Discontinuous Galerkin variational formulation and the Mortar variational formulation. With both formulations we will be able to treat more general situations like (i) geometric non-conformity and (ii) polynomial degree non-conformity.

In (i) the partitions \( T_k \) and \( T_\ell \), of different regions \( \Omega_k \) and \( \Omega_\ell \), can have mesh sizes \( h_k \) and \( h_\ell \), significantly different: in fact, the practical importance of the proposed methods for elastodynamics problems lies on the possibility of using computational grids with different local mesh sizes to take into account sharp variations in the physical parameters of the media.

Fig. 1. Example of a two dimensional subdomain partitioning. In this case \( K = 3 \) and \( \widehat{\Omega} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \), with \( \Omega_1 = \bigcup_{j=1}^{3} \Omega_1^j \), \( \Omega_2 = \bigcup_{j=1}^{3} \Omega_2^j \) and \( \Omega_3 = \bigcup_{j=1}^{3} \Omega_3^j \).
Furthermore, the vertices of elements $\Omega_i^h$ and $\Omega_j^h$ lying on the skeleton $S$ do not necessarily have to match, not even in the case $h_i = h_i$ (Fig. 2).

In (ii) we use different polynomial approximation degrees in each region to get higher precision without refining too much the grid. Moreover, as we show in Section 6, it is evident that high order methods do not significantly suffer from numerical dispersion. The combination of (i) and (ii) yields approximated solutions that are both numerically accurate and computationally cheap.

Obviously, interface conditions other than those we consider are possible as well: an intuitive alternative is offered by pointwise matching conditions which require different spectral solutions to match on a particular set of points lying on $S$. The Discontinuous Galerkin or Mortar approach is preferred to the pointwise matching conditions since it brings optimal convergence rate, which is not the case for methods based on pointwise conditions (see [34] for the elliptic case), without affecting significantly the computational cost.

In the sequel, we describe in detail the non-conforming methods. To ease the presentation, we suppose that each partition $\mathcal{P}_h$ of $\Omega_i$ consists in only one element, this means that each region is a spectral element. The more general case follows from similar arguments.

### 3.1. Discontinuous Galerkin spectral formulation

Before going into the detail of the Discontinuous Galerkin spectral formulation let us introduce some notation that will be useful in the sequel. Let us subdivide the skeleton $S$ in elementary components as follows:

$$S = \bigcup_{j=1}^M \gamma_j, \quad \text{with } \gamma_i \cap \gamma_j = \emptyset, \quad \text{if } i \neq j.$$  

![Fig. 2. Example of non-conforming decomposition.](image)

where each element $\gamma_j = (\partial \Omega_{ij} \cap \partial \Omega_{ij}) \setminus \partial \Omega$, for some different positive integers $k$ and $\ell$: this decomposition is unique (see Fig. 3). Next we collect all the edges (faces if $d = 3$) in the set $\Gamma_f$.

For any pair of neighbouring regions $\Omega_i$ and $\Omega_j$ that share a non trivial edge (face) $\gamma_j \in \Gamma_f$, we denote by $\mathbf{v}_i$, $\sigma_i$ (resp. $\mathbf{v}_j$, $\sigma_j$) the restriction to $\Omega_i$ (resp. $\Omega_j$) of regular enough functions $\mathbf{v}$, $\sigma$. We also denote by $\mathbf{n}_i$ (resp. $\mathbf{n}_j$) the exterior unit normal to $\Omega_i$ (resp. $\Omega_j$).

On each $\gamma_j \in \Gamma_f$ we define the average and jump operators for $\mathbf{v}$ and $\sigma$ as follows:

$$\left\{ \mathbf{v} \right\} = \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_j), \quad \left[ \mathbf{v} \right] = \mathbf{v}_i \otimes \mathbf{n}_i + \mathbf{v}_j \otimes \mathbf{n}_j, \quad (10)$$  

and

$$\left\{ \sigma \right\} = \frac{1}{2}(\sigma_i + \sigma_j), \quad \left[ \sigma \right] = \sigma_i \cdot \mathbf{n}_i + \sigma_j \cdot \mathbf{n}_j, \quad (11)$$

where $\mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{d \times d}$ is the tensor with entries $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$, $1 \leq i, j \leq d$, for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$.

After integration by parts over each region, the application of jump and average operators defined in (10) and (11) and the imposition of condition $\text{TC2}$, i.e., continuity of tractions across $S$, we deduce that:

$$\sum_{k=1}^M \left( \left\{ \mathbf{u} \right\} \cdot \left[ \sigma \right] \right)_{\partial \Omega_{ij} \setminus \partial \Omega} = \sum_{k=1}^M \left( \left\{ \mathbf{v} \right\} \cdot \left[ \mathbf{v} \right] \right)_{\partial \Omega_{ij} \setminus \partial \Omega}, \quad (12)$$

Since also $\text{TC1}$ holds, i.e., $\left[ \mathbf{u} \right] = \mathbf{0}$ is zero across $S$, we can further add other terms in (12) that penalize and control the jumps of the numerical solution, such as

$$\theta \sum_{j=1}^M \left( \left\{ \mathbf{u} \right\}, \left\{ \mathbf{v} \right\} \right)_{\gamma_j} + \sum_{j=1}^M \eta_{\gamma_j} \left( \left[ \mathbf{u} \right], \left[ \mathbf{v} \right] \right)_{\gamma_j},$$

with $\theta \in (-1,0,1)$ and $\eta_j$ positive constants depending on the discretization parameters $h$ and $N$ and on the Lamé coefficients. The terms do not affect consistency of the method and are added with the purpose of providing more generality and better stability properties to the scheme (see [11,35]).

In this context we choose $\eta_j = \tilde{\alpha}(\lambda + 2\mu)_{\partial \Omega_{ij}} \frac{h_j^2}{h_{ij}}$, where $q_{ij}$ represents the harmonic average of the quantity $q$, defined by $q_{ij} = \frac{2q_{ij}q_{ji}}{q_{ij} + q_{ji}}$, $N_{ij} = \max(N_{ij}, N_{ji})$, $h_j = \min(h_{ij}, h_{ji})$ and $\tilde{\alpha}$ is a positive constant at disposal. The semi-discrete DG formulation reads:

$$\forall t \in [0, T] \text{ find } \mathbf{u}_h = (\mathbf{u}_{1,h}(t), \ldots, \mathbf{u}_{K,h}(t)) \in V_{\sigma}^{DG} \equiv V_s \text{ such that}$$

$$\sum_{k=1}^K \left( d_h(\mathbf{p} \cdot \mathbf{u}_h)_{\Omega_k} + A(\mathbf{u}_h, \mathbf{v}_h)_{\Omega_k} \right) + \sum_{j=1}^M \left( B(\mathbf{u}_h, \mathbf{v}_h)_{\gamma_j} \right) = \sum_{k=1}^K \mathcal{L}(\mathbf{v}_k), \quad (13)$$

for all $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_K) \in V_{\sigma}^{DG}$, with
\begin{align}
B(\mathbf{u}, \mathbf{v})_2 = -((\mathbf{g}(\mathbf{u}) \cdot \mathbf{v})_2) + \theta((\mathbf{u} \cdot \mathbf{g}(\mathbf{v}))_2) + \eta_1((\mathbf{u} \cdot \mathbf{v})_2).
\end{align}

Corresponding to different values of $\theta$ we obtain different DG schemes, namely: $\theta = -1$ (resp. $\theta = 1$) leads to the symmetric interior penalty method S1IPG (resp. non-symmetric NIPG), while $\theta = 0$ corresponds to the so-called incomplete interior penalty method IIIPG (see [10,11,35,12] for more details).

3.2. Mortar spectral formulation

In this section we introduce the Mortar Spectral Element Method for the solution of (8). The emphasis is on the numerical formulation, implementation and on the illustration of its flexibility and accuracy. To illustrate the key points, we consider the free-vertex variant of the MSEM [36,37]. The constrained-vertex strategy can be implemented in a similar framework. For this latter technique the theoretical analysis is given in [8,34,38].

The MSEM relaxes the $H^1$-continuity requirements of the conforming spectral-element method by considering each element (or in the general case each macro region) individually and achieving matching or patching conditions through a variational process. The mortars play the role of gluing the bricks of the spectral construction. Through the use of mortars, one can also couple domains where spectral elements are employed with others treated by finite elements [34]. However, in this context we focus on non-conforming spectral methods.

To begin, we denote by $I_k^v$, $\ell = 1, \ldots, 2d$, the edges (faces) of each subdomain $\Omega_k$, $k = 1, \ldots, K$, so that

$$\partial \Omega_k = \bigcup_{\ell=1}^{2d} I_k^\ell.$$ 

We then identify the skeleton $S$ as the union of elementary non-empty components called mortars (or masters), more precisely

$$S = \bigcup_{k=1}^K (\partial \Omega_k \setminus \partial \Omega) = \bigcup_{m=1}^M \tilde{\gamma}_m, \quad \text{with} \quad \gamma_m \cap \gamma_n = \emptyset, \quad \text{if} \quad m \neq n,$$

where each mortar is a whole edge (or face) $I_{k(m)}^v$ of a specific element $\Omega_{k(m)}$ and $m$ is an arbitrary numbering $m = 1, \ldots, M$, with $M$ a positive integer. Those edges or faces $I_k^\ell$ that do not coincide with a mortar are called non-mortars (or slaves) and provide a dual description of the skeleton, as

$$S = \bigcup_{m \text{ mortar}} \gamma_m^m = \bigcup_{n \text{ non-mortar}} \gamma_n^n.$$

The intersection of the closures of the mortars defines a set of vertices or cross-points

$$(\mathbf{x}_q = (\tilde{\gamma}_i^+ \cap \tilde{\gamma}_j^-), \mathbf{x}_q \notin \tilde{\gamma}_m, \ m = 1, \ldots, M),$$

where $q$ is an arbitrary numbering $q = 1, \ldots, \nu$. We define as well the set $\mathcal{V}$ of virtual vertices (that are not cross-points) as

$$\mathcal{V} = \{\mathbf{x}_q = (\tilde{\gamma}_i^+ \cap \tilde{\gamma}_j^-)\},$$

where $q$ is an arbitrary numbering $q = 1, \ldots, \nu$ (see Fig. 4).

We define $A_i(I_k^V) = Q_{\Omega_k}(I_k^V)$, the space of the traces of functions of $X_k(\Omega_k)$ over $I_k^V$ and we also introduce $A_i(I_k^V) = Q_{\Omega_k-2}(I_k^V)$.

We can now define the non-conforming spectral element discretization space $V_m$ as the space of functions $\mathbf{v}_k \in V_m$ that satisfy the following additional mortar matching condition:

\[ (MC1) \text{ let } \Phi \text{ be the mortar function associated with } \mathbf{v}_k, \text{ i.e., a function that is continuous on } S, \text{ zero on } \partial \Omega \text{ and such that on each mortar } \gamma_m \cap I_{k(m)}^v \text{ it coincides with the restriction of } \mathbf{v}_{\ell(k)} \text{ to } \gamma_m \text{ then, for all indices } (k,\ell) \text{ such that } I_k^{\ell(m)} \text{ is contained in } S \text{ but } (k,\ell) \neq (k(m),\ell(m)) \text{ for all } m = 1, \ldots, M \text{ that is for all indices } (k,\ell) \text{ such that } I_k^{\ell(m)} \text{ is a non-mortar we require that:} \]

\[ \int_{I_k^{\ell(m)}} (\mathbf{v}_{\ell(k)} - \Phi) \cdot \mathbf{v}_k = 0 \ \forall \mathbf{v}_k \in A_i(I_k^V)^d \]

and that

\[ \mathbf{v}_{\ell(k)}(\mathbf{x}_q) = \Phi(\mathbf{x}_q), \quad \forall \mathbf{x}_q \in \mathcal{V} \cup \mathcal{V}. \]

The integral matching condition (15) represents a minimization of the jump of the functions at internal boundaries with respect to the $L^2$-norm and is the counterpart in the Mortar framework of condition TC1. The vertex condition (16) ensures exact continuity at cross-points. The Mortar spectral formulation is obtained by solving in each region $\Omega_k$ the elastodynamics variational problem (8) with homogeneous Neumann boundary conditions on $S$ ($\mathbf{g}(\mathbf{u}) \cdot \mathbf{n} = 0$ so that $\sum_{k=1}^K B(\mathbf{u}, \mathbf{v})_{\partial \Omega_k} \mathbf{n} \Omega_k$ is identically zero, i.e., TC2 is satisfied), and enforcing weak continuity of the displacement on $S$ with mortar condition (15).

Thus, the semi-discrete Mortar spectral formulation reads: $\forall t \in (0, T]$ find $(\mathbf{u}, \mathbf{v}_k)_{k=1}^M \in V_m$ such that

\[ \sum_{k=1}^K d_k(\mathbf{u}_{k+1}, \mathbf{v}_k)_{\partial \Omega_k} + A_k(\mathbf{u}_{k+1}, \mathbf{v}_k)_{I_k} = \sum_{k=1}^K \mathcal{L}(\mathbf{v}_k), \]

for all $(\mathbf{v}_1, \ldots, \mathbf{v}_K) \in V_m$ where

$V_m = (\mathbf{v}_1, \ldots, \mathbf{v}_K) \in V_m : \text{mortar condition } MC1 \text{ is satisfied}.$

The Mortar Element Method was originally proposed as a non-overlapping domain decomposition approach, however recently it has been generalized to the case of overlapping subdomains [39–41].

![Fig. 4. Non-conforming domain decomposition (left) and skeleton structure (right) showing a cross-point (•), a virtual vertex (□), the mortars (dark continuous lines) and the non-mortars (dark dashed lines).](image-url)
The overlapping version may be quite useful in elastodynamics modelling to treat subdomains with complex shaped boundaries (see an application in Section 8). Let us consider the two cases presented in Fig. 5.

On the one hand, the rectangular domain Ω is partitioned into two non-overlapping subdomains Ω1 and Ω2 and the skeleton S of the decomposition coincides with the separation surface between two different materials such as elastic waves propagate faster in Ω2 than in Ω1. By adapting the mesh size h2 in each subdomain Ωk according to the propagation velocity of the elastic waves in the subdomain, one reasonably selects h1 > h2. However, h2 has to be small enough to follow the shape of S and h1 cannot be too large otherwise some holes appear close to the surface S. As a consequence, h1 → h2 in a neighbourhood of S resulting in a large number of unknowns to consider in both subdomains. The mortar matching condition allows to transfer the displacement from the set of master interpolation points to the set of slave ones and both sets of points are located on the (d-1)-dimensional surface S. Note that numerical results are independent of the choice of the master and of the slave subdomains.

On the other hand, the rectangular domain Ω is partitioned into two overlapping subdomains, namely, Ω1 which is the bottom left-handed region under the dashed polyhedral surface AB and Ω2 the top right-handed region over the solid line S. These two subdomains overlap in the region between S and the surface AB. In this case, we can have h1 > h2 everywhere in Ωk and the mortar matching condition allows to transfer the displacement from the set of interpolation points of Ωk which are contained in the d-dimensional region bounded by the polyhedral surfaces AB and CD to the set of interpolation points of Ωj which are on the (d-1)-dimensional surface S. Indeed, in the overlapping case, the slave subdomain always covers the master one. Moreover, the slave subdomain is chosen as the one where the mesh best describes the surface S and the master subdomain contains the source of elastic waves. In the overlapping case, the matching condition reads:

\[ (\text{MCO1}) \text{ let } \Phi \text{ be a function that is equal to } v_{ijk} \text{ in the } d \text{-dimensional elements of master subdomain } \Omega_k \text{ containing a part of } S, \text{ and zero elsewhere. Then, for each slave subdomain } \Omega_j \text{ such that } \partial \Omega_j \cap S \neq \emptyset, \text{ we require that:} \]

\[ \int_S (v_{ijk} - \Phi) \cdot \chi = 0 \quad \forall \chi \in [A_{ijk}(S)]^d, \quad (18) \]

where \( A_{ijk}(S) \) is the space of the traces over S of functions belonging to \( \mathcal{X}_K(\Omega_k) \).

4. Algebraic formulations and time integration

We discuss here the algebraic formulations of the two non-conforming approaches presented in the previous sections. In particular we describe how to construct the linear system resulting from the Mortar or the DG discretizations and subsequently we introduce the time integration scheme employed for the numerical simulations.

4.1. Algebraic formulation of the problem

We consider the elastodynamics equation (1) in a bounded region \( \Omega \subset \mathbb{R}^2 \) with Dirichlet boundary condition, thus \( f_B \equiv 0 \). To ease the presentation let also suppose that \( \Omega \) is partitioned into \( K \) non-overlapping spectral elements \( \Omega_1, \ldots, \Omega_K \) so that \( \mathcal{S} = \bigcap_{k=1}^{K} \partial \Omega_k \setminus \partial D \). The more general case can be obtained in a similar manner.

We denote by \( D = \sum_{k=1}^{K} (N_k + 1)^2 \) the dimension of each component of \( V_k \) and we introduce a basis \( \{ \psi_i^j \}_{i,j=1}^{D} \) for the finite dimensional space \( V_k \), where \( \psi_1^j = (\eta^j, 0) \) and \( \psi_2^j = (0, \eta^j) \). Dropping the subscript \( k \), we write the trial functions \( u \in V_h \) as linear combination of basis functions

\[ u(x,t) = \sum_{j=1}^{D} \begin{bmatrix} \eta_j^1(x) \ 0 \end{bmatrix} U_j^1(t) + \sum_{j=1}^{D} \begin{bmatrix} 0 \ \psi_j^2(x) \end{bmatrix} U_j^2(t), \quad (19) \]

Next, we define \( a_k = 1 + \sum_{j=1}^{N_k} (N_j + 1)^2 \) and \( b_k = \sum_{j=1}^{N_k} (N_j + 1)^2 \) and we order the basis functions such that

\[ u_{ijk} = (\eta_i^j, \eta_i^j)^T = \left( \sum_{k'=1}^{b_k} \psi_{j_{k'}}^1 \eta_{i_{k'}}^1 \eta_{i_{k'}}^1 \right), \quad (20) \]

for \( k = 1, \ldots, K \). With the notation just introduced, we write the Eq. (8) for any test function \( \psi_i^j(x) \), for \( i = 1, 2 \), in the space \( V_h \) and we obtain the following set of discrete ordinary differential equations:

\[ \mathbf{M} \dot{\mathbf{U}} + \mathbf{A} \mathbf{U} + \mathbf{B} \mathbf{U} = \mathbf{F}_{\text{ext}}, \quad (21) \]

or equivalently

\[ \begin{bmatrix} \mathbf{M}^1 & 0 \\ 0 & \mathbf{M}^2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^1 \\ \mathbf{U}_2^1 \end{bmatrix} + \begin{bmatrix} \mathbf{A}^1 + \mathbf{B}^1 & \mathbf{A}^2 + \mathbf{B}^2 \\ \mathbf{A}^2 + \mathbf{B}^2 & \mathbf{A}^2 + \mathbf{B}^2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^2 \\ \mathbf{U}_2^2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\text{ext},1} \\ \mathbf{F}_{\text{ext},2} \end{bmatrix}, \quad (22) \]

where \( \mathbf{U} \) represents the vector of nodal accelerations and \( \mathbf{F}_{\text{ext}} \) the vector of externally applied loads. As a consequence of assumptions on the basis functions, the mass matrices \( \mathbf{M}_i \) and \( \mathbf{M}_j \) have a block diagonal structure \( \mathbf{M}_i = \text{diag}(\mathbf{M}_{i1}, \mathbf{M}_{i2}, \ldots, \mathbf{M}_{ik}) \), for \( i = 1, 2 \), where each block \( \mathbf{M}_{ik} \) is associated to the spectral element \( \Omega_k \) and

\[ \mathbf{M}_{ij}(i,j) = \langle \rho \psi_i^j, \psi_i^j \rangle_{\Omega_k}, \quad \text{for } i,j = a_k, \ldots, b_k. \quad (23) \]

The matrix \( \mathbf{A} \) associated to the bilinear form \( \mathbf{A}(\cdot, \cdot) \) defined in (9) takes the form

\[ \mathbf{A} = \begin{bmatrix} \mathbf{A}_1^1 & \mathbf{A}_2^1 \\ \mathbf{A}_3^1 & \mathbf{A}_4^1 \end{bmatrix}, \]

where the block diagonal matrices \( \mathbf{A}_i^j \), for \( i = 1, \ldots, 4 \) are equal to \( \mathbf{A}_i^j = \text{diag}(\mathbf{A}_i^j_1, \mathbf{A}_i^j_2, \ldots, \mathbf{A}_i^j_k) \). The elements of the matrices \( \mathbf{A}_i^j \), for \( i = 1, \ldots, 4 \) and \( k = 1, \ldots, K \) are defined by
\[ A^i(i,j) = A\left(\varphi^i_j, \varphi^i_j\right)_{\Omega_i}, \quad A^j(i,j) = A\left(\varphi^j_j, \varphi^j_j\right)_{\Omega_j}, \quad A^k(i,j) = A\left(\varphi^k_j, \varphi^k_j\right)_{\Omega_k}, \]

for \( i, j = a_0, \ldots, b_0 \). We remark that the matrices \( M \) and \( A \) are very similar to those resulting from the discretization of the elastodynamics equation (6) with conforming methods like the Spectral Element Method.\(^\text{[see [13],[14]]}\).

The matrix \( B \), associated to the bilinear form \( B(\cdot, \cdot) \) defined in (9), is the one that takes into account the discontinuity of the numerical solution across the skeleton \( S \). In the DG approach it is expressed by

\[
B = \begin{bmatrix}
    B^1 & B^2 \\
    B^3 & B^4
\end{bmatrix},
\]

where

\[
B^\ell = \begin{bmatrix}
    B^\ell_{i,1} & \ldots & B^\ell_{i,k} \\
    \vdots & \ddots & \vdots \\
    B^\ell_{k,1} & \ldots & B^\ell_{k,k}
\end{bmatrix}, \quad \text{for } \ell = 1, \ldots, 4.
\]

In particular the elements of each matrix \( B^\ell_{i,n} \) are defined by:

\[
B^\ell_{i,n}(i,j) = \sum_{j'} B\left(\varphi^i_j, \varphi^j_{j'}\right)_\ell = \sum_{j'} \left\{ A\left(\varphi^i_j\right)_\ell : \varphi^j_{j'} \right\} + \theta \sum_{j'} \left\{ \varphi^i_j : \varphi^j_{j'} \right\},
\]

for \( i = a_0, \ldots, b_0 \) and \( j = a_0, \ldots, b_0 \). The elements of the matrices \( B^\ell_{i,n} \), for \( \ell = 2, 3, 4 \) are defined in a similar way.

The situation is a little bit more complicated in the Mortar approach, since the weak continuity condition across the skeleton \( S \) does not appear explicitly in the variational equation but it is a constraint in the functional space \( V^\text{mortar} \); in fact, in the Mortar Variational Formulation, \( B(\cdot, \cdot) = 0 \) implies that \( B \) is a null matrix.

To account for \( M^\text{CI} \) we need to modify (22) as follows. Without loss of generality let us suppose that \( \gamma_m \) is a non-mortar edge contained in \( S \) and moreover that it is shared by two regions \( \Omega_n \) and \( \Omega_m \). We call master the side of \( \gamma_m \) belonging to \( \Omega_m \) and slave the other side. Thus, the mortar conditions \( M^\text{CI} \) can be recast as:

\begin{enumerate}
    \item \( \Phi = u_m \) on \( \gamma_m \),
    \item \( \int_{\gamma_m} (u_j - u_m) \cdot \Phi = 0 \quad \forall \Phi \in \left[ A_2(\gamma_m^+) \right]^d \).
\end{enumerate}

We remark that when interfaces do not match geometrically, i.e., \( \gamma_m \) is shared by \( M^\ast + 1 \) regions \( \Omega_n, \Omega_{n+1}, \ldots, \Omega_{m^\ast} \), condition (ii) reads as

\[
\sum_{n=1}^{M^\ast} \int_{\gamma_m \cap \Omega_{n+1}} (u_j - u_m) \cdot \Phi = 0 \quad \forall \Phi \in \left[ A_2(\gamma_m^+) \right]^d.
\]

Then the following arguments have to be intended for each integral in the above expression. Now, for the spectral element \( \Omega_n \) (resp. \( \Omega_m \)) we order first the \( N_n + 1 \) (resp. \( N_m + 1 \)) degrees of freedom (d.o.f.) associated to the spectral nodes \( p_l \) that live in \( \gamma_m \) and next the d.o.f. associated to the remaining spectral nodes \( p_l \). With these assumptions the restriction of the function \( u_j \) on \( \gamma_m \) is rewritten as

\[
u_{j,m} = \left( \sum_{l=1}^{N_n+1} \varphi^j_{m,n} \sum_{l=1}^{N_m+1} \varphi^j_{m,m} \right)^T,
\]

and the same for the function \( u_{j,m} \). Hence, by definition of scalar product, the mortar condition (ii) becomes

\[
\int_{\gamma_m} (u_j^m - u_{m}^m) \Phi^1 + \int_{\gamma_m} (u_n^m - u_{m}^m) \Phi^2 = 0 \quad \forall \Phi^1, \Phi^2 \in A_2(\gamma_m^+).
\]

Since the integrals in (25) concern separately the two components of the displacement, we focus the attention onto one of them, dropping the superscripts to ease the notation. The other one is treated in the same manner. For the slave side of the mortar we obtain

\[
\int_{\gamma_m} u_i \Phi_i = \sum_{j=1}^{N_n+1} U_{j,n} \int_{\gamma_m} \varphi^i_j \Phi_i = \sum_{j=1}^{N_n+1} R_{i,j} U_{j,n}, \quad \text{for } i = 1, \ldots, N_n - 1,
\]

where \( R_{i,j} = \int_{\gamma_m} \varphi^i_j \Phi_i \). For the master side, using the mortar condition (i), we have that

\[
\int_{\gamma_m} u_i \Phi_i = \sum_{j=1}^{N_m+1} U_{j,m} \int_{\gamma_m} \varphi^i_j \Phi_i = \sum_{j=1}^{N_m+1} P_{i,j} U_{j,m}, \quad \text{for } i = 1, \ldots, N_n - 1,
\]

with \( P_{i,j} = \int_{\gamma_m} \varphi^i_j \Phi_i \). One may use (26) and (27) to recast the mortar constraint \( M^\text{CI} \) in matrix notation

\[
\begin{bmatrix}
    U_{1,n} \\
    \vdots \\
    U_{N_n+1,n}
\end{bmatrix} = P
\begin{bmatrix}
    U_{1,m} \\
    \vdots \\
    U_{N_n+1,m}
\end{bmatrix}
\]

Now, to compute numerically the matrices \( P \) and \( P \) we use suitable quadrature formulas depending if we are on the slave or in the master side of the mortar. We choose \( N_n + 1 \) GLL nodes to evaluate the integrals \( \int_{\gamma_m} \varphi^i_j \Phi_i ds \) such that the matrix \( R \) takes a special structure. In fact, because of this choice the interior part \( R_{nn} \) is diagonal. The first and the last columns are full but they are concerned only with d.o.f. (namely, \( U_{1,n} \) and \( U_{N_n+1,n} \)) but do not depend on the matching conditions. We observe also that the matrix \( P \) is full. Then the local projection operator can be written in a matrix form as

\[
\begin{bmatrix}
    U_{2,n} \\
    \vdots \\
    U_{N_n,n}
\end{bmatrix} = R
\begin{bmatrix}
    P_{1,1} & \ldots & P_{1,N_n+1} & -R_{1,1} & -R_{1,N_n+1} \\
    \vdots & \ddots & \vdots & \vdots & \vdots \\
    P_{N_n-1,1} & \ldots & P_{N_n-1,N_n+1} & -R_{N_n-1,1} & -R_{N_n-1,N_n+1} \\
    U_{1,m} \\
    \vdots \\
    U_{N_n+1,m} \\
    U_{1,n} \\
    \vdots \\
    U_{N_n,n}
\end{bmatrix}
\]

Thanks to the projection operator \( Q \), we are then able to recover the slave unknowns in \( \gamma_m \) once we know the master ones. To obtain a global projection operator \( Q \) we proceed as follows. For each component of \( u \) we denote by \( U_{\text{slave}} \) the vector of unknowns associated to d.o.f. that lay on the slave side of \( S \) and by \( U_{\text{master}} \) the vector of unknowns associated to all the remaining d.o.f. Then, for each \( \gamma_m \) belonging to the skeleton \( S \) we build the local projection operator \( Q \), and we store it into the matrix \( Q \). In this way \( Q \) has a block structure of the form

\[
\bar{Q} = \begin{bmatrix}
    Q & \hat{Q} \\
    0 & \bar{Q}
\end{bmatrix},
\]

where \( \bar{Q} \) is a block diagonal matrix with a block equal to the identity and the other equal to the rectangular matrix \( Q \) containing all the local matrices \( Q \). Thus, we have that the global linear system can be expressed as

\[
Q^T M Q U_{\text{master}} + \bar{Q}^T A \hat{Q} U_{\text{master}} = \bar{Q}^T F^\text{M}.
\]
where the matrices \( \mathbf{M} \) and \( \mathbf{A} \) have columns and rows modified with respect to the ones of \( \mathbf{M} \) and \( \mathbf{A} \) according to latter assumptions on the unknowns reordering. All the terms appearing in the matrices of the two algebraic formulation are computed using Gauss–Lobatto quadrature rule in which the quadrature points coincide with the GLL points. We remark that since the term \( \mathbf{P}^k \mathbf{P}^k \in Q_{\text{int}} \), for some \( k \), while the Gauss–Lobatto–Lobatto with \( N_k \) points is exact for polynomials up to degree \( 2N_k - 1 \) in each variable, the spectral mass matrix \( \mathbf{M} \) is slightly under integrated. However, the final accuracy of spectral methods is maintained [13].

4.2. Structural damping

When using Eq. (5) to model viscoelastic materials, very useful for seismic applications, we must compute additional external forces:

\[
\mathbf{F}^{\text{visc}} = -\mathbf{C} \mathbf{U} - \mathbf{D} \mathbf{U},
\]

or equivalently

\[
\begin{bmatrix}
\mathbf{F}^{\text{visc},1} \\
\mathbf{F}^{\text{visc},2}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{C}^1 & \mathbf{C}^2 \\
\mathbf{C}^2 & \mathbf{C}^1
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}' \\
\mathbf{U}''
\end{bmatrix} - 
\begin{bmatrix}
\mathbf{D}^1 & \mathbf{D}^2 \\
\mathbf{D}^2 & \mathbf{D}^1
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}' \\
\mathbf{U}''
\end{bmatrix},
\]

where the matrices \( \mathbf{C}^\ell \) and \( \mathbf{D}^\ell \), for \( \ell = 1, 2 \) are block diagonal. Each block \( \mathbf{C}^\ell \) and \( \mathbf{D}^\ell \) is associated to the spectral element \( \mathcal{Q}_k \) and

\[
\begin{align*}
\mathbf{C}^\ell_{(i,j)} = (\rho \mathbf{C}^\ell_{(i,j)}), & \quad \mathbf{D}^\ell_{(i,j)} = (\rho \mathbf{D}^\ell_{(i,j)}), \quad & (30) \\
\text{respectively for } i, j = a_0, \ldots, b_k. &
\end{align*}
\]

Then the final discretized system becomes:

\[
\mathbf{M}\mathbf{U} + \mathbf{C}\mathbf{U} + (\mathbf{A} + \mathbf{B} + \mathbf{D})\mathbf{U} = \mathbf{F}^{\text{ext}},
\]

(31)

where the accelerations \( \mathbf{U} \) and the velocities \( \dot{\mathbf{U}} \) are approximated as described in the following section.

4.3. Time integration scheme

Let now subdivide the interval \([0, T] \) into \( N \) subinterval of amplitude \( \Delta t = T/N \): at every time level \( t_n = n \Delta t \), for \( n = 0, \ldots, N \), the time integration scheme is achieved with the second order Leap-Frog scheme (cf. [31]):

\[
\mathbf{M}\mathbf{U}(t_{n+1}) = [2\mathbf{M} - \Delta t^2(\mathbf{A} + \mathbf{B})]\mathbf{U}(t_n) - \mathbf{M}\mathbf{U}(t_{n-1}) + \Delta t^2\mathbf{F}^{\text{ext}}(t_n),
\]

(32)

or

\[
\mathbf{\tilde{M}}\mathbf{U}_{\text{master}}(t_{n+1}) = [2\mathbf{M} - \Delta t^2\mathbf{A}]\mathbf{U}(t_n) - \mathbf{M}\mathbf{U}_{\text{master}}(t_{n-1}) + \Delta t^2\mathbf{F}^{\text{ext}}(t_n),
\]

(33)

respectively for (22) and (29), with initial conditions \( \mathbf{U}(t_0) = \mathbf{U}_0 \) and \( \dot{\mathbf{U}}(t_0) = \mathbf{U}_1 \). In particular if a DGSEM is employed the iteration matrix \( \mathbf{M} \) in (32) can be solved at very low computational cost. In the MSEM the matrix \( \mathbf{\tilde{M}}\) is non-diagonal, but taking advantage of the structure of \( \mathbf{\tilde{M}} \) it is possible to split the linear system (33) as follows

\[
\begin{bmatrix}
\mathbf{U}_{\text{master}} \\
\mathbf{U}_{\text{slave}}
\end{bmatrix} =
\begin{bmatrix}
0 & \mathbf{b}^T \\
\mathbf{Q}^T & \mathbf{b}^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_{\text{master}}(t_{n+1}) \\
\mathbf{U}_{\text{master}}(t_{n-1})
\end{bmatrix},
\]

(34)

with \( \mathbf{b} = [2\mathbf{M} - \Delta t^2\mathbf{A}]\mathbf{U}(t_n) - \mathbf{M}\mathbf{U}_{\text{master}}(t_{n-1}) + \Delta t^2\mathbf{F}^{\text{ext}}(t_n) \). Here the superscripts \( \mathcal{I} \) and \( \mathcal{S} \) identify those unknowns belonging respectively to the interior or to the skeleton of the domain. Then at each time step we solve separately the two blocks of the linear system (34). In particular for the non-diagonal block we perform the LU-factorization (see [31]).

To ensure stability, the explicit time integration scheme must satisfy the usual Courant–Friedrichs–Levy (CFL) condition (see [28]) that imposes a restriction on \( \Delta t \). We see in Section 6 that this limitation is proportional to the minimal distance between two neighbouring spectral nodes of the numerical grid. Since this distance scales as \( h_N N_k^2 \) (\( h_k \) size of the spectral element \( \mathcal{Q}_k \)), the stability requirement on \( \Delta t \) may become too restrictive for very large polynomial degrees \( N_k \). For these cases an implicit time scheme is recommended.

5. Error estimates

In this section we introduce some notation and present a priori error estimates for the semi-discrete formulations (13) and (17) respectively. For the DG formulation (13), we show, in a suitable mesh-dependent energy norm, error estimates that are optimal with respect to the mesh size \( h \) and suboptimal with respect to the polynomial approximation degree \( N \). Such results are in agreement with those proved in [12,35,15] for a slightly different DG method.

For the MSEM formulation (17), in agreement with [8,19], we prove an optimal error bound in both \( h \) and \( N \), using the \( H^1 \)-broken norm.

At the end of this section we also state a priori error estimates for the fully discrete problems (32) and (34) respectively, obtained using the above results and standard techniques. To ease the presentation, we postpone the proofs of the convergence results and all the technical details to Section 9.

For the error analysis we consider the problem (6) defined in \( \Omega \subset \mathbb{R}^2 \) with \( \partial \Omega = \Gamma_N \cup \Gamma_D \). We suppose that its unique solution \( U \) is regular enough so that all the norms we introduce are well defined. Moreover, in the following, \( C \) will denote a positive constant that varies at each occurrence but is independent of the discretization parameters \( h \) and \( N \). We also assume \( \Omega \) to be partitioned into \( K \) non overlapping quadrilaterals \( \Omega_1, \ldots, \Omega_K \) and that \( S \) is subvided in \( M \) elementary components \( \gamma_1, \ldots, \gamma_M \) (resp. non-mortar edges \( \gamma_1, \ldots, \gamma_M \)) for the DGSEM case (resp. for the MSEM case).

The more general situation can be obtained using similar arguments.

For \( \Omega_k \subset \mathbb{R}^2 \) we denote by \( \| \cdot \|_{\Omega_k} \) (resp. \( \| \cdot \|_{\partial \Omega_k} \)) the \( H^1(\Omega_k) = [H^0(\Omega_k)]^d \) norm (resp. seminorm). When \( \Omega_k = \Omega \) we simply write \( \| \cdot \| = \| \cdot \|_{\Omega} \) (resp. \( \| \cdot \|_{\partial} = \| \cdot \|_{\partial \Omega} \)). Since we are dealing with time dependent functions, we take the standard approach of treating these as maps from a time interval into a Banach space \( X \) and set

\[
\| U \|_{L^p(0,T;X)} = \left( \int_0^T \| U \|_X^p \right)^{1/p}, \quad 0 \leq t \leq T, \quad 1 \leq p \leq \infty,
\]

with the obvious modifications when \( p = \infty \).

5.1. Semi-discrete error estimates - DGSEM

To analyze the DG formulation (13) we introduce the enriched space \( V(\delta) = V_s \oplus (H^1(\Omega) \cap H^1_0(\Omega)) \) and define the following mesh-dependent norms:

\[
\| U \|_{D} = \left( \sum_{k=1}^K \| \mathbf{D}^I \|_2^2 \| \mathbf{g}(x) \|_{L^2(\Omega_k)}^2 + \sum_{j=1}^M \eta_j \| U_j \|_{L^2(\partial \Omega_j)}^2 \right)^{1/2} \quad \forall U \in V_s,
\]

and

\[
\| V \|_{D} = \left( \sum_{k=1}^K \frac{h_k N_k^2}{N_k} \| \mathbf{v}^I \|_2^2 \right)^{1/2} \quad \forall V \in V(\delta).
\]
Notice that, when restricted to $V_{h}$, these two norms are uniformly equivalent, thanks to a local inverse inequality [10]. We also set
\[ A_{DC}(u, v) = \sum_{k=1}^{K} A_{k}(u, v)_{\partial \Omega} + \sum_{j=1}^{M} B_{j}(u, v)_{\eta_{j}} \quad \forall \ u, v \in V(\delta), \] (37)
\[ A'_{DC}(u, v) = A_{DC}(u, v) + \sum_{j=1}^{M} \eta_{j} \langle \partial_{n} u, [v] \rangle_{\eta_{j}} \quad \forall \ u, v \in V(\delta). \] (38)

Notice that formulation (38) is introduced in [12,15]. We will make use of the form (38) to prove optimal error estimate in $h$ and suboptimal in $N$ for the DG scheme. In the sequel we will use the results in [12,15] to complete the analysis. All the details are given in Section 9.

**Lemma 1.** There exist two positive constants $M$ and $\kappa$ such that:
\[ A_{DC}(u, v) \leq M \|u\|_{H^{1}(\Omega)} \|v\|_{H^{2}(\Omega)} \] (39)
\[ A_{DC}(u, u) \geq \kappa \|u\|_{H^{1}(\Omega)}^{2} \quad \forall u \in V_\delta. \] (40)

For $\theta = 0, -1$ the last inequality holds provided that $\eta_{\delta j}$ is chosen sufficiently large $\forall \eta_{j} \in \mathcal{F}_{\delta}$.

**Proof.** Inequality (39) follows from the definition of the $\| \cdot \|_{H^1}$ norm (35) by applying the Cauchy–Schwarz and trace inequalities. If $\theta = -1$ (40) holds with $\kappa = 1$ (and is indeed an equality). If $\theta = 1$ or 0 we observe that by the inverse-trace inequality we get $\forall \eta_{j} \in \mathcal{F}_{\delta}: \eta_{j}^{1/2} \in C_{0}(\Omega)$
\[ \|u\|_{H^1(\Omega)}^{2} \leq \frac{1}{\eta_{j}^{1/2}} \|u\|_{H^2(\Omega)} \quad \forall u \in V_\delta. \]

For all $0 < t < T$ see [10]. Setting $d_{\ast} = \min\{\delta + 2\mu, \eta_{\delta j}\}$ we deduce
\[ \sum_{j=1}^{M} \frac{\langle \mathcal{B}(u, v), [u] \rangle_{\eta_{j}}}{\|u\|_{H^1(\Omega)}^{\frac{1}{2}}} \leq \frac{C}{\delta^{1/2}} \|u\|_{H^2(\Omega)} \quad \forall u \in V_\delta. \]

Then, it holds
\[ A_{DC}(u, u) \geq \|u\|_{H^1(\Omega)}^{2} - 2 \sum_{j=1}^{M} \frac{\langle \mathcal{B}(u, v), [u] \rangle_{\eta_{j}}}{\|u\|_{H^1(\Omega)}^{\frac{1}{2}}} \geq (1 - 2\delta/\kappa) \|u\|_{H^2(\Omega)}^{2}. \]

Choosing $\delta$ sufficiently large such that $1 - 2\delta/\kappa \delta t$ is bounded away from zero we have (40).

Now, for all $0 < t < T$ we set $u_{DC} = u_{DC}(t)$ the unique solution in $V_\delta$ of the problem,
\[ d_{t}(\rho u_{DC}, v) + A_{DC}(u_{DC}, v) = \mathcal{L}(\psi) \quad \forall v \in V_\delta. \] (41)

From Lemma 1 and standard techniques, follows that the variational problem in (41) is well posed. For $u_{DC}$ we have the following convergence result. For the proof see Section 9.

**Theorem 1.** There exists a positive constant $C$ such that
\[ \sup_{t \in [0,T]} \|u(t) - u_{DC}(t)\|_{H^{2}(\Omega)} \leq C \left( \sum_{k=1}^{K} \frac{h_{k}^{2m_{k}-2}}{N_{k}^{1/2}} \|u_{H^{2}(\Omega_{k})}\|_{H^{2}(\Omega_{k})} \right)^{1/2}, \] (42)
where $N_{k} \geq 1$ and $m_{k} = \min\{N_{k} + 1, s_{k}\}$.

5.2 Semi-discrete error estimates - MSEM

We now move on the error analysis for the MSEM semi-discretization (17). Let $H^{1/2}(\partial \Omega_{k})$ be the trace space of $H^{1}(\Omega_{k})$ on $\partial \Omega_{k}$ endowed with the norm
\[ \|u\|_{1/2, \partial \Omega_{k}} = \left( \|u\|_{0, \partial \Omega_{k}}^{2} + \|u\|_{H^{1/2}(\partial \Omega_{k})}^{2} \right)^{1/2}, \]
with $|u|^{1/2, \partial \Omega_{k}} = \min \{u_{\partial \Omega_{k}}, -u_{\partial \Omega_{k}}^{1/2} \}$, and for any $\gamma \in \partial \Omega_{k}$ define the space $H^{1/2}(\gamma)$ as
\[ H^{1/2}(\gamma) = \left\{ u \in H^{1/2}(\gamma) : u \in H^{1/2}(\partial \Omega_{k}) \right\}, \]
where $u$ is the extension by zero of $u$ to $\partial \Omega_{k}$ see [30]. Moreover we introduce the mesh-dependent norm
\[ \|u\| = \left( \sum_{k=1}^{K} \|u\|_{H^{1/2}(\partial \Omega_{k})}^{2} \right)^{1/2} \quad \forall u \in V(\delta), \]
and we define the bilinear form $A_{M} (\cdot, \cdot)$ by
\[ A_{M}(u, v) = \sum_{k=1}^{K} A_{k}(u, v)_{\partial \Omega_{k}} \quad \forall u, v \in V(\delta). \]

We have the following properties for $A_{M} (\cdot, \cdot)$.

**Lemma 2.** There exists two positive constants $M$ and $\kappa$, independent of $\delta$ and $N$, such that
\[ A_{M}(u, v) \leq M \|u\| \|v\| \quad \forall u, v \in V(\delta), \] (43)
\[ A_{M}(u, u) \geq \kappa \|u\|_{H^{1}(\Omega)}^{2} \quad \forall u \in V_{\delta}^{\text{mortar}}. \] (44)

**Proof.** Inequality (43) is a direct consequence of the Cauchy–Schwarz inequality, while (44) is easily obtained using the generalized Korn’s first inequality and the Poincaré inequality for $u_{\delta} \in V_{\delta}^{\text{mortar}}$, see [42].

For all $0 < t < T$ let $u_{M} = u_{M}(t)$ be the solution of the variational problem (17) in $V_{\delta}^{\text{mortar}}$ or equivalently of
\[ (\rho du_{M}, v) + A_{M}(u_{M}, v) = \mathcal{L}(\psi) \quad \forall v \in V_{\delta}^{\text{mortar}}. \] (45)

For $u_{M}$ it holds the following convergence result. For the proof see Section 9.

**Theorem 2.** There exists a positive constant $C$ such that
\[ \sup_{t \in [0,T]} \|u(t) - u_{M}(t)\|_{H^{1}(\Omega)} \leq C \left( \sum_{k=1}^{K} \frac{h_{k}^{2m_{k}-2}}{N_{k}^{1/2}} \|u_{H^{1}(\Omega_{k})}\|_{H^{1}(\Omega_{k})} \right)^{1/2}, \] (46)
where $N_{k} \geq 1$ and $m_{k} = \min\{N_{k} + 1, s_{k}\}$.

5.3. Fully-discrete error estimates – DGSEM/MSEM

As mentioned at the beginning of this section, we sketch the proof of the fully discrete error estimates form the DGSEM and MSEM formulations. We recall that the discrete formulation of problems (13) and (17) is obtained by approximating the second order time derivative with the Leap-Frog scheme as in Section 4.3. Then, at each time step $t_{n} = n \Delta t, n \geq 2$, the problem reads: find $u_{h}(t_{n}) \in V_{h}$ such that
\[ \sum_{k=1}^{K} \frac{1}{\Delta t} \left( \rho(u_{h}(t_{n}) - 2u_{h}(t_{n-1}) + u_{h}(t_{n-2})) \right)_{\partial \Omega_{k}} = \sum_{k=1}^{K} A(u_{h}(t_{n-1}) - u_{h}(t_{n-2}))_{\partial \Omega_{k}} + \sum_{j=1}^{M} B(u_{h}(t_{n-1}), v)_{\eta_{j}} - \sum_{j=1}^{M} \mathcal{L}(v_{j}) \quad \forall v \in V_{h}, \] (47)
for the DGSEM case, and: find $u_{h}(t_{n}) \in V_{\delta}^{\text{mortar}}$ such that
\[ \sum_{k=1}^{K} \frac{1}{\Delta t} \left( \rho(u_{h}(t_{n}) - 2u_{h}(t_{n-1}) + u_{h}(t_{n-2})) \right)_{\partial \Omega_{k}} = \sum_{k=1}^{K} A(u_{h}(t_{n-1}) - u_{h}(t_{n-2}))_{\partial \Omega_{k}} + \sum_{k=1}^{K} \mathcal{L}(v_{k}) \quad \forall v \in V_{\delta}^{\text{mortar}}. \] (48)
for the MSEM case, respectively. Comparing the fully discrete solution (47), resp. (48), with the semi-discrete solution (13), resp. (17), using estimate (42), resp. (46), and standard techniques it is possible to prove the following result.

**Theorem 3.** Suppose that \( \mathbf{u}_0, \mathbf{u}_f, \mathbf{f}, \mathbf{t} \) and the solution \( \mathbf{u} \) of (1) are sufficiently smooth. Then, there exists a constant \( C = C(\mathbf{u}_0, \mathbf{u}_f, \mathbf{f}, \mathbf{t}, \mathbf{u}) \) such that \( \forall n \geq 2 \) it holds

\[
\|\rho^{1/2}(\mathbf{u}_n(t_n) - \frac{\partial}{\partial t} \mathbf{u}(t_n))\|_0 + \|\mathbf{u}(t_n) - \mathbf{u}_{DG}(t_n)\|_{DG} \leq C \left( \Delta t^2 + \sum_{k=1}^{K} h_k^{-3/2} \right),
\]

for the DGSEM, and

\[
\|\rho^{1/2}(\mathbf{u}_n(t_n) - \frac{\partial}{\partial t} \mathbf{u}(t_n))\|_0 + \|\mathbf{u}(t_n) - \mathbf{u}_{DG}(t_n)\| \leq C \left( \Delta t^2 + \sum_{k=1}^{K} h_k^{-3/2} \right),
\]

for the MSEM, where \( N_k \geq 1 \) and \( m_k = \min(N_k + 1, s_k) \).

### 6. Analysis of grid dispersion and stability

In this section we study in detail the MSEM and the DGSEM in the two dimensional case, doing the so called Von Neumann analysis, namely the analysis of grid dispersion and stability. The former criterion determines the lowest sampling ratio for the analysis, namely the analysis of grid dispersion and stability. The latter determines the largest time step \( \Delta t \) that we are allowed to use in the explicit time integration scheme, such that the solution remains bounded with respect to problem data. For the sake of simplicity, we present the dispersion and stability analysis in a two dimensional framework.

To start with, let us consider the wave equation (1) in an isotropic, elastic, unbounded domain \( \Omega \), with \( \mathbf{u}(\mathbf{x}, t) \to 0 \) for all \( t \) as \( |\mathbf{x}| \to \infty \). We also assume \( \mathbf{f} \equiv 0 \); this is not a limitation. These are standard assumptions when using the Von Neumann’s method (plane wave analysis), see [43,23,44,16,21,45–47].

At the discrete level we assume that \( \Omega \) is partitioned into non-overlapping spectral elements \( \Omega_k \) having uniform size \( h \). This partitioning is supposed to be periodic and made by squared elements with sides parallel to the coordinate axes (cf. Fig. 6). We also suppose the polynomial approximation degree equals \( N \) in each \( \Omega_k \).

#### 6.1. Grid dispersion – DGSEM

We report the analysis of grid dispersion for the DGSEM, see also [16] for the scalar case. Let identify by \( \Psi_{\ell}^{\Omega_k}, \ell = 1, 2 \) the basis functions with support in \( \Omega_k \) with \( f \in \{ T, B, L, R \} \) (cf. Fig. 6). Without loss of generality, we consider respectively test and trial functions of the following form

\[
\mathbf{\Psi}_{\ell}^{\Omega_k} = \begin{cases} \mathbf{\Psi}_{\ell}^{\Omega_k} & \text{in } \Omega_k, \\ \mathbf{0} & \text{otherwise.} \end{cases}
\]

and

\[
\mathbf{\Psi}_{\ell f} = \begin{cases} \mathbf{\Psi}_{\ell f} & \text{in } \Omega_k, \\ \mathbf{0} & \text{otherwise.} \end{cases}
\]

By rewriting Eq. (21), we obtain a rectangular linear system in the unknowns

\[
\mathbf{U} = \begin{bmatrix} \mathbf{U}_{\ell T}^{DG}, & \mathbf{U}_{\ell T}^{DG}, & \mathbf{U}_{\ell B}^{DG}, & \mathbf{U}_{\ell B}^{DG}, & \mathbf{U}_{\ell L}^{DG}, & \mathbf{U}_{\ell R}^{DG} \end{bmatrix}, \quad \ell = 1, 2.
\]

Clearly this system is underdetermined because the number of columns, \( 10(N + 1)^2 \), exceeds the number of rows, \( 2(N + 1)^2 \). To reduce it into a square linear system we make use of the following plane wave hypothesis.

Let us assume that the displacement is a plane wave, i.e., in \( \Omega_k \) we have

\[
\mathbf{U}_{\ell f}^{DG} = e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega \mathbf{t})}, \quad \ell = 1, 2,
\]

where \( \mathbf{k} = (k_x, k_y) \) is the wave vector, \( \mathbf{p}_i \) contains the \( j \)th node in Cartesian coordinates and \( \mathbf{c}_i \) are arbitrary constants. The above assumption implies that

\[
\mathbf{U}_{\ell f}^{DG} = e^{i(\mathbf{p}_i \cdot \mathbf{x} - \omega \mathbf{t})}, \quad \ell = 1, 2,
\]

with \( \mathbf{p}_i = \begin{bmatrix} -ik_x h, & ik_y h, & -ik_x h, & -ik_y h \end{bmatrix} \) and \( f = \{ T, B, L, R \} \), respectively. Substituting (52) in (50) we obtain the modified square linear system of size \( 2(N + 1)^2 \):

\[
\begin{bmatrix} \mathbf{M} & \mathbf{0} \\
\mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\ell T}^{DG} \\
\mathbf{U}_{\ell B}^{DG} \end{bmatrix} + \begin{bmatrix} \mathbf{A}^1 + \mathbf{B}^1 & \mathbf{A}^1 + \mathbf{B}^2 \\
\mathbf{A}^2 + \mathbf{B}^4 & \mathbf{A}^2 + \mathbf{B}^4 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\ell T}^{DG} \\
\mathbf{U}_{\ell B}^{DG} \end{bmatrix} = \mathbf{0},
\]

where \( \mathbf{M}, i = 1, 2, \) and \( \mathbf{A}_i, i = 1, \ldots, 4, \) are defined in (23) and (24) respectively. The matrices \( \mathbf{B} \), for \( \ell = 1, \ldots, 4 \), are defined taking into account the hypothesis of periodicity of the discretization and the plane wave assumption (51); for example \( \mathbf{B}^{1} \) is given by

\[
\mathbf{B}^{1}(i,j) = \begin{cases} 0 & \text{if } i = j, \\ \\
0 & \text{otherwise}, \end{cases}
\]

where

\[
\mathbf{B}^{1}(i,j) = \begin{cases} \int_{\Omega_k} \left[ \sigma(\mathbf{\Psi}_{\ell f}^{\Omega_k}) \right] \cdot \left[ \mathbf{\Psi}_{\ell f}^{\Omega_k} \right] & : \left[ \mathbf{\Psi}_{\ell f}^{\Omega_k} \right] \\
+ \theta \int_{\Omega_k} \left[ \sigma(\mathbf{\Psi}_{\ell f}^{\Omega_k}) \right] : \left[ \mathbf{\Psi}_{\ell f}^{\Omega_k} \right] + \eta_j \int_{\Omega_k} \left[ \sigma(\mathbf{\Psi}_{\ell f}^{\Omega_k}) \right] : \left[ \mathbf{\Psi}_{\ell f}^{\Omega_k} \right], \end{cases}
\]

and

\[
\mathbf{B}^{1}(i,j) = \begin{cases} \int_{\Omega_k} \left[ \sigma(\mathbf{\Psi}_{\ell f}^{\Omega_k}) \right] \cdot \left[ \mathbf{\Psi}_{\ell f}^{\Omega_k} \right] & : \left[ \mathbf{\Psi}_{\ell f}^{\Omega_k} \right] \\
+ \eta_j \int_{\Omega_k} \left[ \sigma(\mathbf{\Psi}_{\ell f}^{\Omega_k}) \right] : \left[ \mathbf{\Psi}_{\ell f}^{\Omega_k} \right], \end{cases}
\]

Similarly, we define the elements of the matrices \( \mathbf{B}^{\ell} \) for \( \ell = 2, 3, 4 \). Now, calculating the second derivative with respect to time of
\( \mathbf{K}\mathbf{U}^{\text{dc}} = \mathbf{A}
abla \mathbf{U}^{\text{dc}}, \) \( (54) \)

where \( A = \omega^2 I \), with \( \omega \) the angular frequency at which the wave travels in the grid. As observed in [17,16,48] the number of eigenvalues of problem (54) naturally exceeds the number of admissible physical modes. Then we need a strategy to select which eigenvalues correspond to the compressional (\( c_s \)) and the shear (\( c_s \)) wave velocities. We do this by computing all the velocities associated to the eigenvalues of (54) and then comparing them to the real \( c_p \) and \( c_s \) velocities defined in (4).

We denote by \( A_p \) and \( A_s \) the eigenvalues used to compute the best approximations of \( c_p \) and \( c_s \), namely \( c_{pA} \) and \( c_{sA} \).

Note that the system (54) for the NIPG method is not symmetric thus complex eigenvalues are possible. However in [16] it has been remarked that \( A_p \) and \( A_s \) are in fact always real numbers. Next we define the grid dispersion of pressure and shear waves as the ratio between the velocity at which the wave travels in the grid (numerical velocity) and the physical velocity. By definition the numerical shear velocity \( c_{pA} \) is given by \( c_{pA} = h \omega h(2\pi \delta) \), where \( \delta = h/(NL) \) is the sampling ratio (or equivalently \( \delta = 1 \) is the number of GLL points per wavelength), and \( L \) is the wavelength of the plane wave. We have that \( c_{pA} = h \sqrt{\lambda}/(2\pi \delta) \) and therefore the grid dispersion is the relative error in the velocity, given by \( \varepsilon_p = c_{pA}/c_p - 1 \). Analogously \( c_{sA} = h \sqrt{\lambda}/(2\pi \delta) \) and \( \varepsilon_s = c_{sA}/c_s - 1 \).

\[ \varepsilon_p = c_{pA}/c_p - 1, \quad \varepsilon_s = c_{sA}/c_s - 1. \]

6.2. Grid dispersion – MSEM

In order to carry out a dispersion analysis for the MSEM, we adopt a strategy similar to the one described for the DGSEM. The goal is the definition of a generalized eigenvalue problem associated only to the degrees of freedom belonging to \( \Omega_L \). Under the hypothesis of shape regularity and periodicity of the mesh, we observe that the skeleton of the partitioning is uniquely defined once the master and slave edges for the reference element \( \Omega_L \) are selected. Consider the configuration shown in Fig. 7: this is the unique, up to a rotation, possible combination of master and slave edges for \( \Omega_L \) that does not violate the hypothesis of grid periodicity.

We rewrite the ODE system (21) in the MSEM framework: using (49) we obtain

\[ \mathbf{MU}^{\text{dc}} + \mathbf{AU}^{\text{dc}} = \mathbf{0}, \]

where \( \mathbf{M} \) and \( \mathbf{A} \) are defined in (23) and (24), respectively. Next, we impose the mortar conditions MC1 concerning the slave unknowns at the interfaces \( \gamma_L \) and \( \gamma_R \). In particular, for \( \ell = 1,2 \) and \( f = (R,B) \), we have that

\[ \sum_{j \in \gamma_L} \mathbf{U}_j^{\text{dc}} \psi_j^{\text{dc}} \phi_i' = \sum_{j \in \gamma_R} \mathbf{U}_j^{\text{dc}} \psi_j^{\text{dc}} \phi_i' \quad \forall \phi_i' \in \tilde{A}_s(\gamma_L), \]

(56)

To recast these conditions in terms of unknowns and basis functions defined only on \( \Omega_L \) we simply notice that, by periodicity,

\[ \psi_j^{\ell \Omega_L} = \psi_j^{\ell \Omega_R} \quad \text{and} \quad \psi_j^{\ell \Omega_R} = \psi_j^{\ell \Omega_L}, \quad \ell = 1,2 \]

(57)

and by the plane wave assumption, Eq. (52) holds. Substituting (57) in (56) we obtain

\[ \sum_{j \in \gamma_L} \mathbf{U}_j^{\text{dc}} \psi_j^{\text{dc}} \phi_i' = \sum_{j \in \gamma_R} \mathbf{U}_j^{\text{dc}} \psi_j^{\text{dc}} \phi_i' \quad \forall \phi_i' \in \tilde{A}_s(\gamma_L), \]

(58)

for \( \ell = 1,2 \) and \( f = (R,B) \). We remark that Eq. (58) relates slave unknowns in \( \Omega_R \) with master unknowns still in \( \Omega_L \): this means that the matrix projection \( \mathbf{Q} \) refers only to the reference element \( \Omega_L \). We use this matrix to reduce the linear system in (55) to one for the master unknowns only

\[ \mathbf{Q}^{-1} \mathbf{M} \mathbf{Q} \mathbf{U}^{\text{dc}}_{\text{master}} + \mathbf{AQ} \mathbf{U}^{\text{dc}}_{\text{slaves}} = \mathbf{0}. \]

(59)

We notice that \( \mathbf{Q} \) has always a block diagonal structure like (28) where each block \( \mathbf{Q} \) is defined according to (58). Calculating the second derivative of the displacement with respect to time and defining \( \mathbf{K} = \mathbf{Q} \mathbf{A} \mathbf{Q} \), we finally obtain the generalized eigenvalue problem of size \( 2(N^2 + 3) \)

\[ \mathbf{K}\mathbf{U}^{\text{dc}}_{\text{master}} = \mathbf{A} \mathbf{Q} \mathbf{K}\mathbf{U}^{\text{dc}}_{\text{master}}, \]

(60)

where \( A = \omega^2 I \) as in the DGSEM case.

We remark that in the definition of \( \varepsilon_p \) and \( \varepsilon_s \) the sign of the error indicates if the numerical approximation causes a delay or an acceleration of the travelling waves. The grid dispersion error will depend on the sampling ratio \( \delta \), the wave vector \( \kappa \), the degree of the basis function \( N \) and on the velocities \( c_p \) and \( c_s \). For the DGSEM, on the stability parameter \( \eta \) too.

6.3. Grid dispersion – numerical results

Now, we analyze the grid dispersion error for both the MSEM and the DGSEM from three different points of view: (i) the convergence with respect to the polynomial degree \( N \), (ii) the convergence with respect to the sampling ratio \( \delta \) and (iii) the numerical anisotropy introduced by the grid dispersion. Finally we compare the results with the conforming SEM: in this case the grid dispersion analysis is obtained using a technique similar to the one employed in the MSEM and the results obtained are in agreement with [16,22].

In the first set of experiments, we fix the ratio between the velocities \( r = c_p/c_s = 2 \) (that is a very common choice in geophysical applications), the incidence angle \( \theta = \pi/4 \) and, for the DGSEM, we fix the parameter \( \eta = 2N^2/h \).

In Fig. 8 we show the grid dispersion errors with respect to the degree \( N \) of the basis functions, fixing \( \delta = 0.2 \) (namely 5 grid points per wavelength). All the non-conforming approaches reproduce the same spectral convergence of the SEM. The SIPC and the MSEM reach the threshold value \( \approx 10^{-13} \) for \( N = 6 \) while the NIPC for \( N = 9 \).

The grid dispersion as a function of sampling ratio \( \delta \) is shown in Figs. 9–12 for the degrees \( N = 2, \ldots, 5 \), respectively. The aim of this
analysis is to establish a relation between the absolute value $|e_S|$, resp. $|e_P|$, and the mesh size $h$ (i.e., determine $q$, resp. $q'$, such that $|e_S| = \mathcal{O}(h^q)$, resp. $|e_P| = \mathcal{O}(h^{q'})$). The order of convergence is estimated by the slope of these lines in Figs. 9–12.

From the results reported in Figs. 9–12 it seems that the SIPG converges with order $q = q' = \mathcal{O}(N + 1)$, as the SEM; whereas a suboptimal order $q = q' = \mathcal{O}(N)$ is observed for both NIPG and MSEM methods. These results are in agreement with [49].

Finally, in Figs. 13–16, we show the anisotropy (that is the ratio $c_{ij}/c_0$ and the ratio $c_{ij}/c_0$) introduced by the numerical schemes. We consider $N = 2, 3, 4$ and five points per wavelength. For $N > 4$ the anisotropy is very small for all the practical purposes. We
notice that, for $N=2$, in the SIPG and in the MSEM the waves are slightly delayed for all possible incident angles while in the NIPG the waves are accelerated. In Tables 1 and 2 we also report the maximum value $\max_{0 < \theta < 2\pi} |\xi|$ and $\max_{0 < \theta < 2\pi} |\psi|$ respectively.
From these results it can be inferred that all the methods perform in a very similar way.

6.4. Stability – DGSEM and MSEM

To derive the stability condition for the analyzed methods we start by considering the problem

$$c_M \in U + b_K U = 0; \quad (61)$$

where all the terms appearing in the above equation are defined on the reference element $X_C$ (we omit the superscripts to ease the notation). In the DG framework the matrices $b_K$ and $c_M$ are $A + eB$ and $M$ respectively, while in the Mortar approach they are equal to $eQ > eAeQ$ and $eQ > fMeQ$ respectively. Assuming that the solution is the plane wave given in (51), substituting this expression in (61) and approximating the second order time derivative with the Leap-Frog scheme (32) or (33), we obtain the following eigenvalue problem

$$b_K U = \lambda c_M U; \quad (62)$$

depending on the degrees of freedom inside the reference element $\Omega_C$ and where

$$\lambda = \frac{4}{\Delta t^2} \sin^2 \left( \frac{\omega h \Delta t}{2} \right).$$

In order to make explicit the dependence of $\lambda$ on both the mesh size $h$ and the polynomial approximation degree $N$ we rewrite (62) on $\Omega = (-1, 1)^2$. Collecting out the size of the elements it yields to

$$\hat{K}U = A \hat{M}U,$$

with $A' = (h/\Delta t)^2 \sin^2(\omega h / 2)$. Defining the stability parameter $q = c_F \Delta t / h$, we deduce the relation

$$q^2 A' = c_F^2 \sin^2 \left( \frac{\omega h \Delta t}{2} \right) \leq c_p^2,$$

or equivalently

$$\frac{\omega h \Delta t}{2} \leq c_p$$

Fig. 14. Anisotropy curves $c_P/h$ of the SEM (left) and MSEM (right): sampling ratio $\delta = 0.2$ for polynomial degrees $N = 2 (-)$, $N = 3$ ($\cdots$) and $N = 4$ ($\cdots$). For visualization purposes, the grid dispersion has been magnified by a factor 10.

Fig. 15. Anisotropy curves $c_S/h$ of the SIPG (left) and NIPG (right): sampling ratio $\delta = 0.2$ for polynomial degrees $N = 2 (-)$, $N = 3$ ($\cdots$) and $N = 4$ ($\cdots$). For visualization purposes, the grid dispersion has been magnified by a factor 20.
positive constant. Thus, by estimating matrix $\matr{K}$ wave vectors inequality (64) must be fulfilled for all the eigenvalues and all the equivalent to requiring that

$$q = \min_{k \in \mathbb{R}^N} c_{\mu}(A') = c_{\mu}(A').$$

As noted in [17], $c_{\mu}$ is a function of $A'$ and then depends implicitly on the wave vector $\vec{k}$ through the matrices $\matr{K}$ and $\matr{M}$. Moreover, inequality (64) must be fulfilled for all the eigenvalues and all the wave vectors $\vec{k} = 2\pi\vec{n}/h(\cos(\theta), \sin(\theta))$. Thus, the stability condition is given by

$$q \leq \frac{c_{\mu}(A')}{\sqrt{A_{\text{max}}}},$$

where $A_{\text{max}}$ is the largest eigenvalue of problem (63) and $c(\epsilon, \mu)$ is a positive constant. Thus, by estimating $A_{\text{max}}$ in terms of $h$ and $N$, it is possible to determine a bound for $q$.

In the DG approach, the bilinear form $\tilde{K}(\cdot, \cdot)$ associated to the matrix $\matr{K}$ in (62) takes the form

$$\tilde{K}(\vec{u}, \vec{v}) = \int_{\Omega} \vec{a}(\vec{u}) : \vec{v} - \sum_{f=\text{F.E.R.L}} \int_{\gamma_f} \vec{a}(\vec{u}) : \vec{v} \otimes \vec{n}$$

$$+ \theta \int_{\gamma_f} (\vec{u} - \vec{g}_f) \otimes \vec{n} : \vec{a}(\vec{v}) + \eta_l \int_{\gamma_f} (\vec{u} - \vec{g}_f) \otimes \vec{n} : \vec{v} \otimes \vec{n},$$

where the functions $\vec{u}, \vec{v} \in V^{\mu,\nu}_h$ are zero outside $\Delta \Omega$, and $\vec{n}$ is the normal unit vector pointing outside $\Omega_\ell$. According to the plane wave hypothesis made at the beginning of Section 6, we take $\vec{g}_f = e^{i\theta} \vec{u}$ for $f = \{ T, B, L, R \}$.

Following [50], it is easy to prove that

$$\tilde{K}(\vec{u}, \vec{u}) \leq c(\lambda, \mu, \chi) \frac{N^4}{h},$$

Thus, for the generalized eigenvalue problem (63), we can derive the estimate

$$A_{\text{max}} \leq c(\mu, \lambda, \chi) \frac{N^4}{h},$$

and consequently

$$A'_{\text{max}} \leq c(\mu, \lambda, \chi) N^4. \quad (66)$$

For the MSEM we observe that writing $A'_{\text{max}}$ by the generalized Rayleigh quotient yields

$$A'_{\text{max}} = \sup_{\forall \vec{v} \in \mathbb{R}^{2m}(\vec{0})} \frac{\left( \tilde{K}(\vec{v}, \vec{v}) \right)_{L_2}}{\left( \tilde{M}(\vec{v}, \vec{v}) \right)_{L_2}}$$

$$= \sup_{\forall \vec{v} \in \mathbb{R}^{2m}(\vec{0})} \frac{(\vec{Q}^T \matr{A} \vec{Q})_{\vec{v}}}{\left( \tilde{M}(\vec{v}, \vec{v}) \right)_{L_2}}$$

$$\leq \sup_{\forall \vec{v} \in \mathbb{R}^{2m}(\vec{0})} \frac{(\vec{Q}^T \matr{A} \vec{Q})_{\vec{v}}}{\left( \tilde{M}(\vec{v}, \vec{v}) \right)_{L_2}}, \quad (67)$$

where $m = (N^2 + 3)$ and $n = (N + 1)^2$. In this way we obtain an upper bound for the maximum eigenvalue of (63) when using MSEM approximation. In fact the last term in (67) is exactly the maximum eigenvalue of the SEM discretization for which the following estimate holds (cf. [51])

$$c_1 N^4 \leq A'_{\text{max}} \leq c_2 N^4,$$

for $c_1$ and $c_2$ positive constants. We remark that in agreement with [50], we notice that for $d = 2$ the estimate (66) does not depend on $h$. This behaviour is confirmed from the results in Table 4. Finally, we can resume the stability analysis in the following statement.

![Fig. 16. Anisotropy curves $c_{\mu}/c_{\nu}$ of the SIPG (left) and NIPG (right): sampling ratio $\delta = 0.2$ for polynomial degrees $N = 2$ (-), $N = 3$ (-) and $N = 4$ (-). For visualization purposes, the grid dispersion has been magnified by a factor 10.](image)
Proposition 1. For every \( \mu > 0, \lambda > 0 \) and \( \alpha > \alpha_{\text{min}} > 0 \), the CFL condition \((64)\) is satisfied for both MSEM and DGSEM if there exists a positive constant \( C(\lambda, \mu, \alpha) \) such that
\[
q < \frac{C(\lambda, \mu, \alpha)}{N^2}.
\]
Moreover for the MSEM and the NIPG it holds \( \alpha_{\text{min}} = 0 \) and \( C(\lambda, \mu, \alpha) = C(\lambda, \mu) \).

We remark that for the SIPG, the constant \( C(\lambda, \mu, \alpha) \) is proportional to \( \alpha^{-1/2} \) (cf. [50]), then a less restrictive bound for \( q \) in \((68)\) is achieved when \( \alpha = \alpha_{\text{min}} \). Moreover it is possible to determine exactly the threshold value \( \alpha_{\text{min}} \) (cf. [52] for the elliptic case), but this is not the objective of this study. For the following numerical simulations we choose \( \alpha = 1 \).

6.5. Stability – numerical results

To determine an upper bound for the stability parameter \( q \) we fix \( \delta = 0.2 \) and the ratio \( r = 1.414 \). This choice gives a more restrictive stability condition: higher values of \( r = c_P/c_L \) produce milder stability condition [17]. As for the grid dispersion analysis we have fixed \( h = 2N^2/\bar{h} \) for the SIPG and NIPG methods. In Table 3 are shown the estimated threshold values for \( q \) as it is shown in Fig. 18. In order to study the property of convergence of MSEM an DGSEM with respect to \( h = (h_1, h_2) \) and \( N = (N_1, N_2) \) we examine two different situations: the first corresponding to a Cartesian matching grid (Fig. 18, left) and the second to a Cartesian non-matching grid (Fig. 18, right), referred to as grid A and grid B, respectively. In Fig. 18 is shown the first level (L1) of refinement for both grids, corresponding to the initial mesh sizes \( h_1 \) and \( h_2 \) for \( \Omega_2 \) and \( \Omega_2 \). At each further step of refinement (for a maximum number of four steps), we consider a uniform refinement of the grids at the previous level, in particular for grid A, \( L_1 \) refers to \( h_1 = h_2 = 2^{-1} \) whereas for grid B, \( L_1 \) refers to \( h_1 = 2^{-1} \) and \( h_2 = 2(2/3)^{1/3} \). For the time integration we employ the second order explicit Leap-Frog scheme described in Section 4.3.

For SEM approximations we recall that, under suitable assumptions on the partition size \( h \) and on the polynomial degree \( N \) an a priori error bound of the following form holds [14]
\[
\|u - u_h\|_{L^2} \leq C \left( \Delta t^2 + \frac{1}{N^2} \sum_{k=1}^{K} h_{k}^{2l} \left\| u \right\|_{H^{2l}(\Omega_k)}^2 \right)^{1/2},
\]
for \( C \) positive constant. Here \( S_k \) represents the Sobolev regularity of \( u \) in \( \Omega_k, r_k = \min(N_1 + 1, s_k) \) and \( \Delta t \) the time step. Similar bounds can be expected for MSEM and DGSEM on the basis of the estimates in Theorem 3 in Section 5.

In particular, if the mesh size \( h \) is constant (i.e., \( h_1 = \cdots = h_K = h \)) we expect exponential convergence in \( N = \min(N_1, N_K) \) whereas if the spectral order of approximation \( N \) is fixed (i.e., \( N_1 = \cdots = N_K = N \)) we expect algebraic convergence in \( h = \max_i h_i \).

### Table 3

Computed upper bound for the stability parameter \( q \) using \( r = 1.414 \): rate of decay with respect to \( N \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>SEM</th>
<th>MSEM</th>
<th>SIPG</th>
<th>NIPG</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3376</td>
<td>0.3333</td>
<td>0.2621</td>
<td>0.2163</td>
</tr>
<tr>
<td>3</td>
<td>0.1967</td>
<td>0.1770</td>
<td>0.1368</td>
<td>0.1045</td>
</tr>
<tr>
<td>4</td>
<td>0.1206</td>
<td>0.1118</td>
<td>0.0795</td>
<td>0.0607</td>
</tr>
<tr>
<td>5</td>
<td>0.0827</td>
<td>0.0776</td>
<td>0.0530</td>
<td>0.0409</td>
</tr>
<tr>
<td>6</td>
<td>0.0596</td>
<td>0.0529</td>
<td>0.0374</td>
<td>0.0281</td>
</tr>
<tr>
<td>7</td>
<td>0.0449</td>
<td>0.0434</td>
<td>0.0280</td>
<td>0.0210</td>
</tr>
<tr>
<td>8</td>
<td>0.0351</td>
<td>0.0342</td>
<td>0.0216</td>
<td>0.0162</td>
</tr>
<tr>
<td>9</td>
<td>0.0281</td>
<td>0.0277</td>
<td>0.0172</td>
<td>0.0129</td>
</tr>
<tr>
<td>10</td>
<td>0.0231</td>
<td>0.0227</td>
<td>0.0140</td>
<td>0.0105</td>
</tr>
<tr>
<td>N-rate</td>
<td>(-1.8463)</td>
<td>(-1.8253)</td>
<td>(-1.5247)</td>
<td>(-1.9360)</td>
</tr>
</tbody>
</table>

### Table 4

Computed upper bound for the stability parameter \( q' \) using \( r = 1.414 \). Note that \( q' \) is proportional to \( N^2 \) thus constant for different choices of \( \Delta t \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>SEM</th>
<th>MSEM</th>
<th>SIPG</th>
<th>NIPG</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.6752</td>
<td>0.6667</td>
<td>0.5241</td>
<td>0.4326</td>
</tr>
<tr>
<td>3</td>
<td>0.7115</td>
<td>0.6403</td>
<td>0.4951</td>
<td>0.3782</td>
</tr>
<tr>
<td>4</td>
<td>0.6983</td>
<td>0.6474</td>
<td>0.4607</td>
<td>0.3516</td>
</tr>
<tr>
<td>5</td>
<td>0.7039</td>
<td>0.6808</td>
<td>0.4515</td>
<td>0.3409</td>
</tr>
<tr>
<td>6</td>
<td>0.7017</td>
<td>0.6712</td>
<td>0.4460</td>
<td>0.3315</td>
</tr>
<tr>
<td>7</td>
<td>0.7009</td>
<td>0.6769</td>
<td>0.4360</td>
<td>0.3273</td>
</tr>
<tr>
<td>8</td>
<td>0.7005</td>
<td>0.6819</td>
<td>0.4303</td>
<td>0.3228</td>
</tr>
<tr>
<td>9</td>
<td>0.6994</td>
<td>0.6878</td>
<td>0.4282</td>
<td>0.3206</td>
</tr>
<tr>
<td>10</td>
<td>0.6995</td>
<td>0.6871</td>
<td>0.4247</td>
<td>0.3180</td>
</tr>
</tbody>
</table>

Fig. 17. \( A_{\text{max}} \) versus the polynomial degree \( N \) for the generalized eigenvalue problem \((63)\).
In the family of proposed DGSEMs, we analyze in detail the SIPG method (i.e., $\theta = -1$ in (14)) because it exhibits better performances in term of grid dispersion and stability (see Section 6).

In Fig. 19 (resp. Fig. 21) we report the $L^2$-error using MSEM and SIPG with grid $A$ for different choices of $N$ (resp. d.o.f.). The estimated norm is computed at the time $t^* = 2$ using $\Delta t = 5 \cdot 10^{-4}$. All plots in Figs. 19–22 are displayed in semilogarithmic scale.

The results show that both methods have the same rate of convergence as the SEM one. In Fig. 20 (resp. Fig. 22) it is shown

![Fig. 18. First level of refinement (L1) for the grid $A$ (left) and $B$ (right). The end points of the skeleton $S$ are highlighted by two circles.](image1)

![Fig. 19. Computed errors versus the polynomial degree $N$: MSEM (left) and DGSEM (right) at the observation time $t^* = 2$ using $\Delta t = 5 \cdot 10^{-4}$. The results are obtained with the grid $A$ and the refinement level L2.](image2)

![Fig. 20. As in Fig. 19 with $\Delta t = 10^{-4}$.](image3)
the \(L^2\)-error using a different time step \(\Delta t = 10^{-4}\) for different choices of \(N\) (resp. d.o.f.).

The results confirm that MSEM and SIPG have both exponential convergence in \(N\), until the threshold value given by \(\approx \Delta t^2\) is reached.

Now, we fix \(N_1 = N_2\) and we study the accuracy of the two methods with respect to the mesh size \(h\). For each level of refinement we compute the error in \(L^2\)-norm obtained using grid A and grid B. The algebraic order of convergence \(\mathcal{O}(h^{s+1})\) is achieved in both cases for different choices of \(N\) and \(\Delta t\) (see Figs. 23 and 24).

Finally in Figs. 25 and 26 we show a qualitative analysis of stability of MSEM and SIPG applied to this test case. The results are in agreement with those obtained in Section 6 and confirm that the
8. An application of geophysical interest

In this section we analyze the seismic response of an alluvial basin. We consider the viscoelastic model (5) in the computational domain \((x, z) \in \Omega = (0.2 \cdot 10^4 \text{ m}) \times (-9.6 \cdot 10^2 \text{ m}, f(x))\) where \(f\) describes the top profile of the valley, see Fig. 27. The bottom and the lateral boundaries are set far enough from the point source so to avoid any interference of possible reflections from non-perfectly absorbing boundaries with the waves of interest that are reflected, transmitted, or converted at the material or free surfaces. We simulate a point source load of the form...
$$f(x, t) = g(x) h(t),$$

where $f$ is the external force introduced in (1). The function $g$ describes the space distribution of the source and is written in the form

$$g(x) = \delta(x - x_0) w,$$

where $\delta$ represents the Dirac distribution, $x_0$ the source location and $w$ the direction of the applied force (cf. [27]). Alternative source distributions can be expressed in terms of gradient or curl of suitable potential functions, giving rise to pure pressure and shear waves: more complex and realistic source mechanisms are based on tensorial models (cf. [1]). The source time history is given by a Ricker-type function with maximum frequency $v_{\text{max}} = 3$ Hz, defined as

$$h(t) = h_0 [1 - 2\beta(t - t_0)^2] \exp[-\beta(t - t_0)^2]$$

where $h_0$ is a scale factor, $t_0 = 2$ s is the time shift and $\beta = \pi^2 v_{\text{max}}^2 = 9.8696$ s$^{-1}$ is a parameter that determines the width of the wavelet (70). A significant property is the cut-off at both low and high frequencies: the spectrum of the signal is maximum at $v_0 = \sqrt{\beta}/\pi$ Hz and is practically negligible for frequencies higher than $v_0 = 3 v_0$.

In Figs. 27 and 28, we show the two different mesh configurations. Fig. 27 shows a regular, structured grid with a mesh spacing of $h \approx 40$ m. The mesh size is chosen small enough to describe with sufficient precision the physical profile of the valley. Fig. 28 shows an irregular, quasi-structured grid with overlap with a mesh spacing $h_1 \approx 40$ m for layer 1 (basin) and $h_2 \approx 120$ m for layer 2 (bedrock). The finest mesh is used to describe the physical boundary of the valley while the coarsest mesh the bedrock. This type of overlapping discretizations are handled by the Mortar technique described in Section 3.2.

We assign constant material properties within each region as described in Table 5.

This regular conforming grid in Fig. 27 is used with SEM discretization to produce a reference solution for the problem and provides a sufficiently accurate discretization, since further mesh refinements generates quasi-identical seismograms.

In Figs. 29 and 30 we compare the horizontal and vertical displacement recorded by receiver R1 placed on the free surface of the valley (cf. Figs. 27 and 28).

Table 5

<table>
<thead>
<tr>
<th>Layer</th>
<th>$c_P$ (m/s)</th>
<th>$c_S$ (m/s)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$\zeta$ (1/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>700</td>
<td>350</td>
<td>1900</td>
<td>0.03141</td>
</tr>
<tr>
<td>2</td>
<td>3500</td>
<td>1800</td>
<td>2200</td>
<td>0.06283</td>
</tr>
</tbody>
</table>

Fig. 27. Conforming, structured grid with a mesh spacing of $h \approx 40$ m at the interface between the two materials. Top: receiver R1 placed on the top of the valley and point source $x_0$ within the bedrock. Bottom: zoom of the valley profile.

Fig. 28. Non-conforming, quasi-structured grid with overlap with a mesh spacing $h_1 \approx 40$ m for layer 1 (basin) and $h_2 \approx 120$ m for layer 2 (bedrock). Top: receiver R1 placed on the top of the valley and point source $x_0$ within the bedrock. Bottom: zoom of the valley profile.
The high discontinuities between the mechanical properties of the materials produce high oscillations and perturbations on the wave front. All these complex phenomena are well captured by both SEM and MSEM using fourth order spectral elements. We
remark that with MSEM we reduce the computational effort for the generation of the grid as well as the problem complexity (from 61,385 spectral nodes with SEM to 48,091 spectral nodes with MSEM). In Fig. 31 we show the time histories of the seismograms recorded by some receivers on the free surface of the domain, obtained using MSEM. It can be observed that the wave which starts travelling from the point source remains trapped into the valley, where it is amplified and where phenomena of reflection and refraction arise. This phenomenon is relevant in some geophysical contexts, e.g. it has occurred in the Gubbio valley (in Italy) on the occasion of the earthquake of September 27, 1997. We refer to [53] for a detailed analysis.

9. Error analysis

In this section we report the proofs of the theorems stated in Section 5.

9.1. Semi-discrete error analysis – DGSEM

We recall from [54] the following approximation result. For any \( \Omega_k \subset \Omega \), \( \gamma_k \in \Gamma_r \) and \( u \in H^1(\Omega_k) \) there exists a sequence \( u_k \in Q_{Nk}(\Omega_k) \), for \( N_k = 1, 2, \ldots \), such that

\[
\| u - u_k \|_{0, \Omega_k} \leq C \left( \frac{h_k}{N_k^2} \right) \| u \|_{1, \Omega_k}, \quad 0 \leq q \leq s_k, \tag{71}
\]

\[
\| u - u_k \|_{0, \Omega_k} \leq C \left( \frac{h_k^{m_q-1/2}}{N_k} \right) \| u \|_{1, \Omega_k}, \quad s_k > 1/2, \tag{72}
\]

where \( m_q = \min(N_k + 1, s_k) \) and \( C \) is a positive constant independent of \( h_k \) and \( N_k \). For further use we also introduce the projection operator \( P_{\Gamma_r} : V \rightarrow V = V \) such that

\[
P_{\Gamma_r}(u, v) = A_{\Gamma_r}(u, v) \quad \forall v \in V. \tag{73}
\]

Notice that \( P_{\Gamma_r} u \) is well defined thanks to Lemma 1. Moreover, since \( P_{\Gamma_r} u \) is a projection, we clearly have

\[
\| u - P_{\Gamma_r} u \|_{0, \Omega_k} \leq \| u - u_k \|_{0, \Omega_k} \leq C \left( \frac{h_k^{m_q-1/2}}{N_k} \right) \| u \|_{1, \Omega_k}, \tag{74}
\]

for any positive constant \( C \). Therefore for \( \epsilon = k/2M \) we have

\[
\| u - P_{\Gamma_r} u \|_{0, \Omega_k} \leq C \left( \frac{h_k}{N_k^2} \right) \| u \|_{1, \Omega_k}. \tag{75}
\]

Similarly, by using \((71) \) and \((72) \), it is easy to see that \( P_{\Gamma_r} u \) satisfies the following approximation property.

**Lemma 3.** There exists a positive constant \( C \) such that

\[
\| u - P_{\Gamma_r} u \|_{0, \Omega_k} \leq C \left( \sum_{k=1}^{K} \frac{h_k^{m_q-2}}{N_k^2} \| u \|_{1, \Omega_k}^2 \right)^{1/2}. \tag{74}
\]

Moreover it holds

\[
\| u - P_{\Gamma_r} u \|_{0, \Omega_k} \leq C \left( \frac{h_k^{m_q-2}}{N_k^2} \| u \|_{1, \Omega_k}^2 \right)^{1/2}. \tag{75}
\]

where \( N_k \geq 1 \) and \( m_q = \min(N_k + 1, s_k) \).

**Proof.** We start showing inequality \((74) \). Let \( u = u_k \). Using the inverse-trace inequality and the interpolation estimates \((71) \) and \((72) \) it holds

\[
\| u - u_k \|_{0, \Omega_k} \leq C \left( \sum_{k=1}^{K} \frac{h_k^{m_q-2}}{N_k^2} \| u \|_{1, \Omega_k}^2 \right)^{1/2}. \tag{74}
\]

Now, for all \( 0 \leq t \leq T \) we set \( u_{0, \Gamma_r}(t) \) the unique solution in \( V \) of the problem

\[
d_{\Gamma_r}(u_{0, \Gamma_r}(t), v) + A_{\Gamma_r}(u_{0, \Gamma_r}(t), v) = C(t, v) \quad \forall v \in V, \tag{77}
\]

From Lemma 1 and standard techniques, follows that the variational problem in \((77) \) is well posed. From the results given in \([12,15]\), the estimates \((74) \) and \((75) \), we have

**Lemma 4.** There exists a positive constant \( C \) such that for all \( t \in [0, T] \)

\[
\| u - u_{0, \Gamma_r}(t) \|_{0, \Omega_k} \leq C \left( \sum_{k=1}^{K} \frac{h_k^{m_q-2}}{N_k^2} \| u \|_{1, \Omega_k}^2 \right)^{1/2}. \tag{78}
\]

where \( N_k \geq 1 \) and \( m_q = \min(N_k + 1, s_k) \).
Proof. Let \( H \) be defined as in (73). By the triangle inequality we have
\[
\|u - u_{\text{bc}}\|_{\mathcal{BC}} \leq \|u - H u\|_{\mathcal{BC}} + \|H u - u_{\text{bc}}\|_{\mathcal{BC}} = T_1 + T_2.
\]
Estimate (74) yields
\[
|T_1| \leq C \left( \sum_{k=1}^{K} \frac{h_k^{2m-2}}{N_k^2} \|u\|_{H^2}^2 \right)^{1/2}.
\] (79)
For the term \( T_2 \) we set \( \chi = u_{\text{bc}} - \Pi u \) and \( \xi = u - H u \). Denoting by \( \phi_t = \partial_t \phi \) the derivative of \( \phi \) with respect to time we have for \( t > 0 \)
\[
(\rho \chi_t) + A_{\text{bc}}(\chi, \chi) = (\rho \xi_t, \chi) + A_{\text{bc}}(\xi, \chi) \quad \forall \chi \in L^2(0, T; V_0).
\]
Denoting by \( J(u, v) = \sum_{i=1}^{M} h_i \eta_i(u, [v]) \) and recalling (73), the above equation reduces to
\[
(\rho \chi_{tt}) + A_{\text{bc}}(\chi, \chi) + J(\chi, \chi) = (\rho \xi_{tt}, \chi) + J(\xi, \chi).
\] (80)
By choosing \( \chi = \chi_0 \), the error equation (80) becomes
\[
(\rho \chi_{tt}) + A_{\text{bc}}(\chi, \chi) + J(\chi, \chi) = (\rho \xi_{tt}, \chi) + J(\xi, \chi).
\]
We can rewrite the above equation as follows
\[
\frac{1}{2} \frac{d}{dt} \|\rho^{1/2} \chi_t\|^2 + \frac{1}{2} \|\chi_t\|^2_{\mathcal{BC}} + J(\chi_t, \chi_t)
\]
\[
= (\rho \xi_{tt}, \chi_t) + J(\xi_t, \chi_t) + \sum_{i=1}^{M} \left( (\rho \chi_t, [\chi_i]) \right)_{\gamma_i} + \frac{\theta}{2} \sum_{i=1}^{M} \left( [\chi_i], [\chi_i] \right)_{\gamma_i}.
\] (81)
Therefore, integrating (81) in time between 0 and \( t \), noting that by definition \( \chi(0) = 0 \), we obtain:
\[
\frac{1}{2} \|\rho^{1/2} \chi(t)\|^2 + \frac{1}{2} \|\chi(t)\|^2_{\mathcal{BC}} + \int_0^t J(\chi_t, \chi_t)
\]
\[
= (\rho \xi_{tt}, \chi(t)) + J(\xi(t), \chi(t)) + \frac{\theta}{2} \sum_{i=1}^{M} \left( [\chi_i], [\chi_i] \right)_{\gamma_i} + \frac{1}{2} \|\rho^{1/2} \chi_0\|^2.
\] (82)
We now bound each of the terms on the right-hand side of (82) that involves integrals on \( \gamma_i \) using the trace inequality:
\[
\left| \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{M} \left( [\chi_i], [\chi_i] \right)_{\gamma_i} \right| \leq C \frac{1}{2T} \int_0^T \|\chi_t\|^2_{\mathcal{BC}} + \frac{\epsilon}{2} \int_0^t J(\chi_t, \chi_t).
\]
\[
\left| \frac{\theta}{2} \sum_{i=1}^{M} \left( [\chi_i], [\chi_i] \right)_{\gamma_i} \right| \leq \frac{\epsilon}{2} \|\chi\|_{\mathcal{BC}} + C \frac{1}{2T^2} \int_0^T \|\chi_t\|^2.
\]
\[ \forall \epsilon, \tilde{\epsilon} > 0. \] We also have
\[
\int_0^t (\rho \xi_{tt}, \chi_t) \leq \int_0^t \frac{1}{2} \|\rho^{1/2} \chi_t\|^2 + \int_0^t \frac{1}{2} \|\rho^{1/2} \xi_t\|^2.
\]
and
\[
\int_0^t J(\xi_t, \chi_t) \leq \frac{C}{2T} \int_0^T J(\xi_t, \xi_t) + \frac{\epsilon}{2} \int_0^t J(\chi_t, \chi_t).
\]
Then, Eq. (82) reduces to
\[
\frac{1}{2} \|\rho^{1/2} \chi(t)\|^2 + \frac{1}{2} \|\chi(t)\|^2_{\mathcal{BC}} + (1 - \epsilon) \int_0^t J(\chi_t, \chi_t) - \frac{C}{2T} \|\chi(t)\|^2
\]
\[
\leq \frac{1}{2} \|\rho^{1/2} \chi_0\|^2 + \frac{\epsilon}{2} \int_0^T \|\chi(t)\|^2_{\mathcal{BC}} + \int_0^t \frac{1}{2} \|\rho^{1/2} \chi_t\|^2 + \int_0^t \frac{1}{2} \|\rho^{1/2} \xi_t\|^2 + \frac{\epsilon}{2} \int_0^t J(\xi_t, \xi_t).
\] (83)
Taking \( \epsilon = 1/4 \), \( \tilde{\epsilon} = 1/2 \) and \( N \gg 4C \) we have
\[
\int_0^t \|\rho^{1/2} \chi(t)\|^2 + \|\chi(t)\|^2_{\mathcal{BC}} \leq C \left[ \int_0^t \left( \|\rho^{1/2} \chi(t)\|^2 + \|\chi(t)\|^2_{\mathcal{BC}} \right) + \|\rho^{1/2} \chi_0\|^2 \right]
\]
\[
+ \int_0^t \|\rho^{1/2} \xi_t\|^2 + \int_0^t J(\xi_t, \xi_t). \]
By applying the Gronwall’s lemma [28] we obtain
\[
\int_0^t \|\rho^{1/2} \chi(t)\|^2 + \|\chi(t)\|^2_{\mathcal{BC}} \leq \left[ \int_0^t \|\rho^{1/2} \chi_0\|^2_0 + \int_0^t \|\rho^{1/2} \xi_t\|^2 + \int_0^t J(\xi_t, \xi_t) \right].
\]
By the approximation property (74) it holds:
\[
\|\rho^{1/2} \chi_0\|^2 \leq C \sum_{k=1}^{K} \frac{h_k^{2m-2}}{N_k^2} \|u\|_{L^2(0, T; V_0)}^2
\]
\[
\int_0^t \|\rho^{1/2} \xi_t\|^2 \leq C \sum_{k=1}^{K} \frac{h_k^{2m-2}}{N_k^2} \|u\|_{L^2(0, T; V_0)}^2
\]
\[
\int_0^t J(\xi_t, \xi_t) \leq C \sum_{k=1}^{K} \frac{h_k^{2m-2}}{N_k^2} \|u\|_{L^2(0, T; V_0)}^2.
\]
Therefore
\[
|T_2| \leq C \left( \sum_{k=1}^{K} \frac{h_k^{2m-2}}{N_k^2} \left[ \|u\|_{L^2(0, T; V_0)}^2 + \|u\|_{L^2(0, T; V_0)}^2 \right] \right)^{1/2}.
\] (84)
Then, (78) follows by combining the estimate (79) and (84) and taking the supremum over all \( t \in [0, T] \). □

Now, we are ready to prove Theorem 1.

Proof. Let \( u_{\text{bc}} \in V_0 \) be the solution of (77). From Lemma 1 we can show that
\[
\|u_{\text{bc}} - u(t)\|_{\mathcal{BC}} \leq C \|u - u_{\text{bc}}(t)\|_{\mathcal{BC}} \quad \forall t \in [0, T].
\] (85)
In fact, it holds
\[
\|u_{\text{bc}} - u(t)\|_{\mathcal{BC}} \leq C \|u_{\text{bc}} - u(t)\|_{\mathcal{BC}} \leq C \frac{1}{K} A_{\text{bc}}(u_{\text{bc}} - u(t), u_{\text{bc}} - u(t))
\]
\[
= \frac{C}{K} A_{\text{bc}}(u - u_{\text{bc}}, u_{\text{bc}} - u(t))
\]
\[
\leq CM K \|u - u_{\text{bc}}\|_{\mathcal{BC}} \|u_{\text{bc}} - u(t)\|_{\mathcal{BC}}.
\]
Now, by the triangle inequality,
\[
\|u - u(t)\|_{\mathcal{BC}} \leq \|u - u_{\text{bc}}(t)\|_{\mathcal{BC}} + \|u_{\text{bc}} - u(t)\|_{\mathcal{BC}}.
\]
then, the desired result is obtained using (78)–(85) and taking the supremum over all \( t \in [0, T] \). □

9.2. Semi-discrete error analysis – MSEM

The crucial point of the MSEM error analysis relies on the construction of a modified elliptic projection operator \( H \) from \( V \) to \( V_{\text{mort}} \) satisfying optimal error estimate with respect to both \( h \) and \( N \). In order to define it we need some preliminary approximation results that we recall from [8,9,19].
For any non-mortar side $\gamma_n$ of $\mathcal{S}$ such that $\gamma_n = \gamma_m(\omega) \subset \partial \Omega_k$, we define the projection operator

$$
\pi_n : \{ L^2(\gamma_n) \}^2 \to \{ A_3(\gamma_n) \}^2 \cap \mathbf{H}_0^1(\gamma_n),
$$

by

$$
\int_{\gamma_n} (\mathbf{v} - \pi_n \mathbf{v}) \cdot \mathbf{n} = 0 \quad \forall \mathbf{n} \in \{ A_3(\gamma_n) \}^2.
$$

Then, see [9], for any non-negative real numbers $s$ and $q$ it holds

$$
\| \mathbf{v} - \pi_n \mathbf{v} \|_{2,q} \leq C \left( \frac{h}{N_k} \right)^{s+1} \| \mathbf{v} \|_{2,s} \quad \forall \mathbf{v} \in \mathbf{H}^2(\gamma_n).
$$

We now define a lifting operator $\mathbf{R}_n : \{ A_3(\gamma_n) \}^2 \cap \mathbf{H}_0^1(\gamma_n) \to \mathbf{X}_0(\Omega_k)$ such that $\mathbf{R}_n \mathbf{v} = \mathbf{v}$ on $\gamma_n$. $\mathbf{R}_n$ vanishes on each side of $\Omega_k$ except on $\gamma_n$ and satisfies (see [55])

$$
\| \mathbf{R}_n \mathbf{v} \|_{1,\gamma_n} \leq C \| \mathbf{v} \|_{1,\gamma_n} \quad \forall \mathbf{v} \in \{ A_3(\gamma_n) \}^2 \cap \mathbf{H}_0^1(\gamma_n).
$$

Moreover we introduce the operator $\mathcal{P} : V \to V^{\text{mortar}}$ defined by

$$
\mathcal{P} \mathbf{v} = \sum_{k=1}^K \left( \mathbf{u}_{t+k}^{(n)} + \sum_{j \in \partial \Omega_k} \mathbf{q}_j^{(n)} \right),
$$

where $\mathbf{u}_t$ satisfies (71) and (72) and

$$
\mathbf{q}_j^{(n)} = \begin{cases} 0, & \text{if } I_j^n \text{ is a mortar edge}, \\ \mathbf{R}_n \left( \pi_n(\mathbf{u}_t - \mathbf{u}_j) \right), & \text{if } I_j^n = I_{k,0}^{(n)} \text{ is a non-mortar edge}.
\end{cases}
$$

Finally we state the following approximation result (see [8] for the proof).

**Lemma 5.** There exists a positive constant $C$, independent of $h$ and $N$ such that for any $\mathbf{v} \in \mathbf{H}^2(\Omega_k)$ it holds

$$
\| \mathbf{v} - \mathcal{P} \mathbf{v} \|_{1,\gamma_n} \leq C \left( \frac{h^{s+1}}{N_k} \right) \| \mathbf{v} \|_{1,\gamma_n}, \quad s_k > 3/2,
$$

with $N_k \gg 1$ and $m_k = \min(N_k + 1, s_k)$.

We now define the modified elliptic projection $\Pi : V \to V^{\text{mortar}}$ as:

$$
\mathbf{A}_\mathcal{M}(\mathbf{u} - \Pi \mathbf{u}, \mathbf{v}) - \sum_{n=1}^M \left( \sigma(\mathbf{u}) : [\mathbf{v}] \right)_{\gamma_n} = 0 \quad \forall \mathbf{v} \in V^{\text{mortar}}.
$$

Note that $\Pi \mathbf{u} \in V^{\text{mortar}}$ is well defined since the bilinear form $\mathbf{A}_\mathcal{M} \cdot \cdot$ satisfies the coercive property (44).

**Lemma 6.** There exists a positive constant $C$ such that

$$
\| \mathbf{u} - \Pi \mathbf{u} \|_{1,\gamma_n} \leq C \left( \sum_{k=1}^K \frac{h^{s+1}}{N_k^{s+1}} \| \mathbf{u} \|_{1,\gamma_n} \right)^{1/2}, \quad s_k > 3/2,
$$

with $N_k \gg 1$ and $m_k = \min(N_k + 1, s_k)$. Moreover it holds

$$
\| \mathbf{u} - \Pi \mathbf{u} \|_{1,\gamma_n} \leq C \left( \sum_{k=1}^K \frac{h^{s+1}}{N_k^{s+1}} \| \mathbf{u} \|_{1,\gamma_n} \right)^{1/2}.
$$

**Proof.** Using the projection operator introduced in (87) we rewrite the Eq. (89) obtaining

$$
\mathbf{A}_\mathcal{M}(\mathbf{P} \mathbf{u} - \Pi \mathbf{u}, \mathbf{v}) = -\mathbf{A}(\mathbf{u} - \mathbf{P} \mathbf{u}, \mathbf{v}) + \sum_{n=1}^M \left( \sigma(\mathbf{u}) : [\mathbf{v}] \right)_{\gamma_n} = 0 \quad \forall \mathbf{v} \in V^{\text{mortar}}.
$$

Choosing $\mathbf{v} = \mathbf{P} \mathbf{u} - \Pi \mathbf{u}$ and using the boundedness and coercive property (43) and (44) respectively, we have

$$
\kappa \| \mathbf{v} \|^2 \leq M \| \mathbf{u} - \Pi \mathbf{u} \|, \| \mathbf{v} \|, + \sum_{n=1}^M \| \sigma(\mathbf{u}) : [\mathbf{v}] \|_{\gamma_n}.
$$

Now, from [9, Proposition 3.1] we have

$$
\sum_{n=1}^M \left( \sigma(\mathbf{u}) : [\mathbf{v}] \right)_{\gamma_n} \leq C \sum_{k=1}^K \frac{h^{m_k-1}}{N_k^{2s}} \| \mathbf{u} \|_{1,\gamma_n} \| \mathbf{v} \|_{1,\gamma_n}.
$$

Therefore,

$$
\| \mathbf{v} \| \leq C \left( \| \mathbf{u} - \Pi \mathbf{u} \| + \sum_{k=1}^K \frac{h^{m_k-1}}{N_k^{2s}} \| \mathbf{u} \|_{1,\gamma_n} \right).
$$

(90) is obtained combining the above inequality with the estimate (88) and using the triangle inequality

$$
\| \mathbf{u} - \Pi \mathbf{u} \|, \| \mathbf{u} - \mathbf{P} \mathbf{u} \|, + \| \mathbf{P} \mathbf{u} - \Pi \mathbf{u} \|.
$$

To prove (91) we set $\mathbf{v} = \mathbf{P} \mathbf{u} - \Pi \mathbf{u}$. By duality arguments (see proof of Lemma 3) and integrating by parts on each element yields:

$$
\| \mathbf{v} \|^2 = \left( - \nabla \cdot \sigma(\Phi), \mathbf{v} \right)_{\Omega_k} = \mathbf{A}(\Phi, \mathbf{v}) - \sum_{n=1}^M \left( \sigma(\Phi) : \mathbf{v} \right)_{\gamma_n},
$$

or equivalently,

$$
\| \mathbf{v} \|^2 = \mathbf{A}(\Phi - \Pi \Phi, \mathbf{v}) + \mathbf{A}(\Pi \Phi, \mathbf{v}) - \sum_{n=1}^M \left( \sigma(\Phi) : \mathbf{v} \right)_{\gamma_n}.
$$

Using the symmetry of $\sigma$ and the properties (89) and (86) we have

$$
\| \mathbf{v} \|^2 = \mathbf{A}(\Pi \Phi, \mathbf{v}) + \sum_{n=1}^M \left( \sigma(\Phi) : [\Pi \Phi] \right)_{\gamma_n} - \sum_{n=1}^M \left( \sigma(\Phi) : \Pi \mathbf{u} \right)_{\gamma_n}.
$$

We now bound the three terms on the right-hand side of the above equation. Using (43) we have

$$
|\mathbf{A}(\Pi \Phi, \mathbf{v})| \leq C \left( \frac{h \| \mathbf{v} \|_{1,\gamma_n} \| \mathbf{v} \|_{1,\gamma_n} \right).$$

for any $\gamma_n \subset \partial \Omega_k$. Therefore we conclude combining the above estimates with (90). □

Let $\mathbf{u}_M = \mathbf{u}_M(t)$ be the solution in $V^{\text{mortar}}$ of the variational problem (45). From the results given in [8,19], the estimate (90) and (91) we have the following

**Lemma 7.** There exists a positive constant $C$ such that for all $t \in [0,T]$ it holds

$$
\| (\Pi \mathbf{u} - \mathbf{u}_M)(t) \|_{1,\gamma_n} \leq C \left( \sum_{k=1}^K \frac{h^{m_k-2}}{N_k^{2s}} \| \mathbf{u} \|_{1,\gamma_n} \right)^{1/2}.
$$

where $N_k \gg 1$ and $m_k = \min(N_k + 1, s_k)$.

**Proof.** We introduce the modified elliptic projection $\Pi \mathbf{u}$ as in (89) and we set $\mathbf{v} = \mathbf{u} - \Pi \mathbf{u} \mathbf{v} \mathbf{u} - \Pi \mathbf{u} \mathbf{u}$. When multiplying the first line in (1) by a function $\mathbf{v} = \mathbf{v}(t) \in L^2(0,T;V^{\text{mortar}})$ and integrating by parts on each $\Omega_k$, we observe that
We rewrite it as follows
\[ (\rho \partial_t \mathbf{u}, \mathbf{v}) + A_m(\mathbf{u}, \mathbf{v}) + \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{v}])_{\gamma_n} = \mathcal{C}(\mathbf{v}). \]

Now subtracting (45) from the above equation we have, for any \( \mathbf{v} \in L^2(0, T; \mathbf{V}_{\text{mortar}}) \),
\[ (\rho \partial_t (\mathbf{u} - \mathbf{u}_h), \mathbf{v}) + A_m(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{v}])_{\gamma_n} = 0, \]
or equivalently,
\[ (\rho \mathbf{z}_{ht}, \mathbf{v}) + A_m(\mathbf{z}, \mathbf{v}) + \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{v}])_{\gamma_n}. \]
Choosing \( \mathbf{v} = \mathbf{z}_h \) and using the property (89) we obtain
\[ (\rho \mathbf{z}_{ht}, \mathbf{z}_h) + A_m(\mathbf{z}_h, \mathbf{z}_h) + 2 \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{z}_h])_{\gamma_n}. \]

We rewrite it as follows
\[
\begin{align*}
\frac{1}{2} d_n \| \rho^{1/2} \mathbf{z}_h(t) \|^2 &+ \frac{K}{2} \| \mathbf{z}(t) \|^2 \\
\leq & \int_0^t \left( (\rho \partial_t \mathbf{z}_h, \mathbf{z}_h) + 2 \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{z}_h])_{\gamma_n}(t) \\
& - 2 \int_0^t \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{z}_h])_{\gamma_n} + \frac{1}{2} \| \rho^{1/2} \mathbf{z}(0) \|^2. 
\end{align*}
\]

We now bound each of the terms in the right-hand side of (95) that involves integrals on \( \gamma_n \), using the trace inequality:
\[
\begin{align*}
2 \int_0^t \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{z}_h])_{\gamma_n} &\leq C \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \int_0^t \| \mathbf{u} \|_{1, \text{sk}}^2 + \frac{1}{2} \int_0^t \| \mathbf{z} \|^2, \\
2 \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{z}_h])_{\gamma_n} &\leq C 2 \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \| \mathbf{u} \|_{1, \text{sk}}^2 + \frac{\epsilon}{2} \| \mathbf{z} \|^2,
\end{align*}
\]
\( \forall \epsilon > 0 \). We also have
\[
\int_0^t (\rho \partial_t \mathbf{z}_h, \mathbf{z}_h) = \int_0^t \frac{1}{2} \| \rho^{1/2} \mathbf{z}_h \|^2 + \int_0^t \frac{1}{2} \| \rho^{1/2} \mathbf{z}_h \|^2.
\]
Then, inequality (95) yields
\[
\begin{align*}
\frac{1}{2} d_n \| \rho^{1/2} \mathbf{z}_h(t) \|^2 &+ \frac{K}{2} \| \mathbf{z}(t) \|^2 \\
\leq & \int_0^t \left( (\rho \partial_t \mathbf{z}_h, \mathbf{z}_h) + 2 \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{z}_h])_{\gamma_n}(t) \\
& - 2 \int_0^t \sum_{n=1}^{M} (\mathbf{g}(\mathbf{u}) : [\mathbf{z}_h])_{\gamma_n} + \frac{1}{2} \| \rho^{1/2} \mathbf{z}(0) \|^2, \\
& + C \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \| \mathbf{u} \|_{1, \text{sk}}^2 + \frac{1}{2} \| \rho^{1/2} \mathbf{z}(0) \|^2 \\
& + C 2 \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \| \mathbf{u} \|_{1, \text{sk}}^2 + \frac{\epsilon}{2} \| \mathbf{z} \|^2.
\end{align*}
\]
Choosing \( \epsilon \) such that \( \kappa - \epsilon \) is bounded away from 0 we obtain
\[
\| \rho^{1/2} \mathbf{z}_h(t) \|^2 + \| \mathbf{z}(t) \|^2
\leq C \left[ \int_0^t \left( (\rho^{1/2} \mathbf{z}_h, \mathbf{z}_h) + \frac{1}{2} \| \rho^{1/2} \mathbf{z}_h \|^2 \\
+ \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \| \mathbf{u} \|_{1, \text{sk}}^2 \right) + \int_0^t \| \rho^{1/2} \mathbf{z}(0) \|^2 \\
+ \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \| \mathbf{u} \|_{1, \text{sk}}^2 + \| \rho^{1/2} \mathbf{z}(0) \|^2 \right].
\]

By applying the Gronwall’s lemma [28] it holds
\[
\| \rho^{1/2} \mathbf{z}_h(t) \|^2 + \| \mathbf{z}(t) \|^2
\leq C \left[ \int_0^t \left( (\rho^{1/2} \mathbf{z}_h, \mathbf{z}_h) + \frac{1}{2} \| \rho^{1/2} \mathbf{z}_h \|^2 \\
+ \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \| \mathbf{u} \|_{1, \text{sk}}^2 \right) + \int_0^t \| \rho^{1/2} \mathbf{z}(0) \|^2 \right].
\]

Using the approximation properties (90) and (91) it follows that:
\[
\| \rho^{1/2} \mathbf{z}_h(t) \|^2 \leq C \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k} \| \mathbf{u} \|^2_{L^2(0,T;H^m(\Omega_k))}.
\]
\[
\| \rho^{1/2} \mathbf{z}(t) \|^2 \leq C \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k^2} \| \mathbf{u} \|^2_{L^2(0,T;H^m(\Omega_k))}.
\]

Therefore we have
\[
\| \mathbf{z}(t) \|^2 \leq \sum_{k=1}^{K} \frac{h^{2m-2}}{N_k^2} \| \mathbf{u} \|^2_{L^2(0,T;H^m(\Omega_k))}.
\]

Now the proof of Theorem 2 is obtained using the triangle inequality, estimates (90) and (91) and (93) and taking the supremum over all \( t \in [0, T] \).

10. Conclusions

In this paper we proposed, analyzed and compared two different domain decomposition non-conforming high order numerical techniques, namely the Mortar Spectral Element Method (MSEM) and the Discontinuous Galerkin Spectral Element Method (DGSEM), for the approximation of the elastic wave equation in heterogeneous media.

Both methods preserve the spectral accuracy typical of high order methods, allow geometrically non-conforming domain partitions where local meshes are independently generated and can handle variable local polynomial degrees. Note that the subdomain partition is constructed according to the information on the material properties.

Starting from a common weak formulation we described both approaches and highlighted their analogies and their differences. In particular, we gave special attention to the analysis of grid dispersion, stability, and accuracy, which represent the main important features determining the reliability of a numerical method to wave propagation problems. We numerically proved that the MSEM and the DGSEM do not suffer from grid dispersion. Indeed five points per wavelength with spectral element approximations of order four are sufficient to have negligible errors. For the stability analysis we derived a precise CFL bound for the Leap-Frog scheme when employed with the considered non-conforming approaches. The threshold values obtained for the DGSEM (resp. the MSEM) are around 70% (resp. 95%) of the ones typical of the Spectral Element Method (SEM). So, on the one hand, the symmetric version of the DGSEM yields optimal error decays in the grid dispersion as occurs with the SEM. On the other hand, the MSEM allows larger time step in the time advancing scheme. Finally, both non-conforming techniques are well suited for parallel computations.

Acknowledgements

The first author was partially supported by Italian MIUR PRIN 2008 grant “Analisi e sviluppo di metodi numerici avanzati per EDP”. The second and third authors acknowledge the financial support of the Italian MIUR PRIN 2009 grant “Modelli numerici per il calcolo scientifico e applicazioni avanzate”.

We thank Roberto Paolucci, Chiara Smerzini of the Department of Structural Engineering, Politecnico di Milano, and Marco Stupazzini of Munich RE for many interesting discussions, for providing input for the geophysical application addressed in Section 8 and their help in the analysis of numerical results.

References


