Transitions to high frequencies in Hamiltonian evolutions

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Introduction

Context

Consider a Hamiltonian system

$$\dot{u} = X_H(u)$$

in an infinite dimensional setting, e.g.

$$u \in L^2(M) \ , \ \omega(h_1, h_2) = \mathrm{Im} \int_M h_1 \overline{h}_2 \, d\mu \ ,$$

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and H = densely defined Hamiltonian.

Assume that the flow map is well-defined on Sobolev spaces $H^s(M) \ , \ s \geq s_0$.

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- Growth of high Sobolev norms ?
- Describe the mechanism (energy cascades ?)
- How generic is this phenomenon ?
- What does generic mean here ?

Example : the nonlinear Schrödinger equation

(M, g) complete Riemannian manifold

$$i\partial_t u + \Delta u = |u|^2 u \ ,$$
$$H[u] = \int_M \left[\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{4}|u(x)|^4\right] d\mu_g(x) \ .$$

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If $M = \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$, dispersion, Strichartz estimates, scattering theory (Ginibre–Velo 80',... , CKSTT 00').

Bourgain's conjecture

Question (Bourgain, 2000). If $M = \mathbb{T}^2$ and $s > 1$ prove

$$\exists u : \limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty? \qquad (*)$$

Three partial answers:

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i) Colliander-Keel-Staffilani-Takaoka-Tao, 2010.

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iii) Hani-Pausader-Tzvetkov-Visciglia, 2015.
Property (*) holds on $\mathbb{T}^2 \times \mathbb{R}$.

The Majda-McLaughlin-Tabak model (1997)

Cubic fractional NLS on $\mathbb{T}$

$$i\partial_t u = |D|^\alpha u + |u|^2 u$$
$$H[u] = (|D|^\alpha u|u)_{L^2} + \frac{1}{2}\|u\|_{L^4}^4$$

Globally wellposed on $H^s(\mathbb{T})$, $s \geq \frac{\alpha}{2}$, $\alpha > \frac{2}{3}$.

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Similar question : for $s > \frac{\alpha}{2}$, study

$$\exists u : \limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty ?$$

The cubic half-wave equation

Assume $\alpha = 1$ in MMT,

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Reformulation as a coupled system of transport equations

$$\begin{cases} i(\partial_t + \partial_x)u_+ = \Pi_+(|u|^2 u) & , \\ i(\partial_t - \partial_x)u_- = \Pi_-(|u|^2 u) & , \\ u = u_+ + u_- & . \end{cases}$$

The approximation result

Theorem (Grellier-PG, 2012)

Assume $s > 1$, $\|u(0)\|_{H^s} = \varepsilon$, $\varepsilon > 0$ *small enough, and* $u(0, x + \pi) = -u(0, x)$. *Let* $v_\pm$ *:*

$$\begin{cases} i(\partial_t + \partial_x)v_+ = \Pi_+(|v_+|^2 v_+)\ ,\ v_+(0, x) = \Pi_+ u_0(x) & , \\ i(\partial_t - \partial_x)v_- = \Pi_-(|v_-|^2 v_-)\ ,\ v_-(0, x) = \Pi_- u_0(x) & . \end{cases}$$

Then, for some $c_s > 0$,

$$\|u(t) - v_+(t) - v_-(t)\|_{H^s} = O(\varepsilon^2)\ ,\ |t| \le \frac{c_s}{\varepsilon^2} \log\left(\frac{1}{\varepsilon}\right)\ .$$

Sketch of the proof : averaging method

$$w(t) := \varepsilon^{-1} \mathrm{e}^{it|D|} u(t)$$

$$\hat{w}(t,k) = \hat{w}_0(k) - i\varepsilon^2 \sum_{k_1-k_2+k_3=k} I(k_1,k_2,k_3,k,t)$$

$$I(k_{1,2,3},k,t) := \int_0^t \mathrm{e}^{-i\tau\omega(k_1,k_2,k_3,k)} \hat{w}(k_1,\tau)\overline{\hat{w}(k_2,\tau)}\hat{w}(k_3,\tau)\, d\tau$$

$$\omega(k_1,k_2,k_3,k) := |k_1| - |k_2| + |k_3| - |k| \ .$$

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Non trivial resonant quartets : $\omega(k_1, k_2, k_3, k) = 0$ correspond to $\forall j \ , \ k_j > 0$ or $\forall j \ , \ k_j < 0$.

The cubic Szegő equation

We are led to study

$$i\partial_t u = \Pi_+(|u|^2 u)$$

on $H^s_+ := \Pi_+(H^s(\mathbb{T}))$ (Hardy space on the unit disc), $s \geq \frac{1}{2}$.

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Conservation laws:

$$\begin{aligned} M(u) &= (Du|u)_{L^2} = \sum_{k=0}^{\infty} k|\hat{u}(k)|^2 \\ H(u) &= \|u\|_{L^4}^4 \\ Q(u) &= \|u\|_{L^2}^2 \end{aligned}$$

Main theorem

Theorem (Grellier-PG, 2017)

Let $s > \frac{1}{2}$. *There exists a dense* G_δ *subset* G_s *of* H^s_+ *such that, for every* $u_0 \in G_s$, *the solution* u *of*

$$i\partial_t u = \Pi_+(|u|^2 u) \ , \ u(0) = u_0$$

satisfies

$$\begin{cases} \limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty \ . \\ \liminf_{t\to\infty} \|u(t)\|_{H^s} < +\infty \ . \end{cases}$$

Comments

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- The growth of Sobolev norms can be made superpolynomial,

$$\forall N > 0 \ , \ \limsup_{t \to \infty} \frac{\|u(t)\|_{H^s}}{|t|^N} = +\infty \ .$$

Almost optimal in view of the general estimate

$$\|u(t)\|_{H^s} \leq C_s \mathrm{e}^{C_s |t|} \ .$$

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- Large relative length of time intervals with big norm.

$$\limsup_{T \to +\infty} \frac{1}{T} \int_0^T \|u(t)\|_{H^1} \, dt = +\infty \ .$$

An application

Following Hani-Pausader-Tzvetkov-Visciglia's road map :

Theorem (H. Xu, 2017)

There exists solutions u of

$$i\partial_t u = -\partial_x^2 u + |D_y| u + |u|^2 u \ , \ (x, y) \in \mathbb{R} \times \mathbb{T}$$

with $u(t, x, y + \pi) = -u(t, x, y)$*, such that*

$$\forall N \geq 1 \ , \ \limsup_{t \to +\infty} \frac{\|u(t)\|_{L^2_x(\mathbb{R}, H^1_y(\mathbb{T}))}}{(\log t)^N} dt = +\infty,$$

and on the other hand

$$\liminf_{t \to \infty} \|u(t)\|_{L^2_x(\mathbb{R}, H^1_y(\mathbb{T}))} < +\infty.$$

Plan of the minicourse

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3. Related results.

1. The Lax pair structure and the inverse spectral theorem.

Hankel matrices

Given $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, consider $\Gamma_c : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$$\Gamma_c := \begin{pmatrix} c_0 & c_1 & c_2 & . & . \\ c_1 & c_2 & . & . & . \\ c_2 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix} = (c_{p+q})_{p,q\geq 0}$$

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$$\mathrm{Tr}(\Gamma_c \Gamma_c^*) = \sum_{p,q\geq 0} |c_{p+q}|^2 = \sum_{n=0}^{\infty} (n+1)|c_n|^2 < \infty$$

Fundamental property

Shift operator : $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$$S : (x_0, x_1, x_2, \dots) \longmapsto (0, x_0, x_1, x_2, \dots).$$

Antishift : $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$$S^* : (x_0, x_1, x_2, \dots) \longmapsto (x_1, x_2, x_3, \dots).$$

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$$\tilde{\Gamma}_c \tilde{\Gamma}_c^* = \Gamma_c \Gamma_c^* - (.|c)c$$

Hankel operators on the Hardy space

Fourier isometry

$$\mathscr{F} : L^2_+ = \Pi_+(L^2) \xrightarrow{\sim} \ell^2(\mathbb{N})$$
$$u \longmapsto (\hat{u}(n))_{n\geq 0},$$

Define $H_u : L^2_+ \to L^2_+$ by

$$H_u(h) = \Pi_+(u\bar{h}) \ , \ h \in L^2_+ \ .$$

Observe that

$$\widehat{H_u(h)} = \Gamma_{\hat{u}}(\bar{\hat{h}}) \quad , \quad \mathscr{F} H_u \mathscr{F}^{-1} = \Gamma_{\hat{u}} \circ \mathcal{C}$$
$$H_u^2 \; = \; \mathscr{F}^{-1}\Gamma_{\hat{u}}\Gamma_{\hat{u}}^*\mathscr{F} \ .$$

Toeplitz operators and shift on the Hardy space

Given $b \in L^\infty(\mathbb{T})$, define $T_b : L^2_+ \to L^2_+$ by

$$T_b(h) = \Pi_+(bh) \; , \; h \in L^2_+ \; .$$

Notice that $T_b^* = T_{\overline{b}}$ and

$$\mathscr{F}^{-1} S \mathscr{F} = T_{\mathrm{e}^{ix}}$$

still denoted by S, and

$$K_u := S^* H_u = H_u S = H_{S^* u} = \mathscr{F}^{-1} \tilde{\Gamma}_{\hat{u}} \mathscr{F}$$
$$K_u^2 = H_u^2 - (.|u)u \; .$$

The Lax pair structure

Theorem (Grellier-PG, 2010)

If u is a H^s_+ solution of $i\partial_t u = \Pi_+(|u|^2 u)$ *with* $s > \frac{1}{2}$, *then*

$$\frac{d}{dt} H_u = [B_u, H_u] \ , \ B_u := -iT_{|u|^2} + \frac{i}{2} H_u^2 \ .$$

Similarly

$$\frac{d}{dt} K_u = [C_u, K_u] \ , \ C_u := -iT_{|u|^2} + \frac{i}{2} K_u^2 \ .$$

Main ingredient of the proof

Lemma

Given $a, b, c \in L^\infty \cap L^2_+$,

$$H_{\Pi_+(a\bar{b}c)} = T_{a\bar{b}} H_c + H_a T_{b\bar{c}} - H_a H_b H_c \ .$$

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$$H_{\Pi_+(a\bar{b}c)}(h) = \Pi_+(\Pi_+(a\bar{b}c)\bar{h}) = \Pi_+(a\bar{b}c\bar{h} - \Pi_-(a\bar{b}c)\bar{h}) = \Pi_+(a\bar{b}c\bar{h})$$

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$$\begin{aligned} H_{\Pi_+(a\bar{b}c)}(h) &= \Pi_+(a\bar{b}c\bar{h}) = \Pi_+(a\bar{b}\Pi_+(c\bar{h})) + \Pi_+(a\bar{b}\Pi_-(c\bar{h})) \\ &= T_{a\bar{b}} H_c(h) + H_a(f) \ , \ \ f := \overline{b\Pi_-(c\bar{h})} \end{aligned}$$

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$$f = \Pi_+(f) = \Pi_+(b\bar{c}h) - \Pi_+(b\overline{\Pi_+(c\bar{h})}) = T_{b\bar{c}}(h) - H_b H_c(h),$$

$$\begin{aligned} H_{\Pi_+(a\bar{b}c)}(h) &= \Pi_+(a\bar{b}c\bar{h}) = \Pi_+(a\bar{b}\Pi_+(c\bar{h})) + \Pi_+(a\bar{b}\Pi_-(c\bar{h})) \\ &= T_{a\bar{b}}H_c(h) + H_a(f)\ ,\ \ f := \overline{b\Pi_-(c\bar{h})} \end{aligned}$$

Consequences

Corollary

Define $U = U(t)$, $V = V(t)$ *to be the solutions of the following linear ODEs on* $\mathcal{L}(L^2_+)$,

$$\frac{dU}{dt} = B_u U \ , \ \frac{dV}{dt} = C_u V \ , \ U(0) = V(0) = I \ .$$

Then $U(t), V(t)$ *are unitary operators and*

$$H_{u(t)} = U(t) H_{u(0)} U(t)^* \ , \ K_{u(t)} = V(t) K_{u(0)} V(t)^* \ .$$

In particular, the spectra of H_u^2 *and* K_u^2 *are conservation laws.*

A first application

Corollary

Assume $u(0) \in H^s_+$ *for some* $s > 1$. *Then*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty} < +\infty \ ,$$

$$\|u(t)\|_{H^s} \leq C_s e^{C_s|t|} \ , \ t \in \mathbb{R} \ .$$

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The first estimate comes from

$$\|u\|_{L^\infty} \le \sum_{n=0}^{\infty} |\hat{u}(n)| \le \mathrm{Tr}|H_u| + \mathrm{Tr}|K_u| \le C_s \|u\|_{H^s} \ , \ s > 1 \ .$$

The interlacement property

Lemma

For $u \in H_+^{1/2}$, let $(s_j^2)_{j\geq 1}$ and $(\tilde{s}_k^2)_{k\geq 1}$ be the decreasing sequence formed by the eigenvalues of H_u^2 and K_u^2 respectively, written with multiplicities. Then we have

$$s_1 \geq \tilde{s}_1 \geq s_2 \geq \tilde{s}_2 \geq \cdots \geq 0.$$

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$$K_u^2 = H_u^2 - (.|u)u$$

and apply the min-max formula.

The Kronecker theorem

Theorem (Kronecker, 1877)

The Hankel operator H_u has finite rank if and only if u is a rational function of e^{ix} with no pole in the closed unit disk.

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Precision. For every $d \geq 1$, the set

$$\mathcal{V}(d) := \{u : \mathrm{rank}(H_u) + \mathrm{rank}(K_u) = d\}$$

is a d-dimensional Kähler submanifold of dimension d.

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Examples.

$$\mathcal{V}(2) = \left\{ \frac{a}{1 - p\mathrm{e}^{ix}} \, , \; a \in \mathbb{C}^* \, , \; p \in \mathbb{D} \setminus \{0\} \right\}$$

$$\mathcal{V}(3) = \left\{ \frac{a\mathrm{e}^{ix} + b}{1 - p\mathrm{e}^{ix}} \, , \; a \in \mathbb{C}^* \, , \; b \in \mathbb{C} \, , \; p \in \mathbb{D} \right\}$$

Generic states

Denote by $\mathcal{V}(d)_{\rm gen} \subset \mathcal{V}(d)$ the subset corresponding to simple and distinct positive eigenvalues for H_u^2, K_u^2. This leads to d positive numbers

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Fact : $u \in \mathcal{V}(d)_{\rm gen}$ if and only if $H_u^{2k}(u), k = 0, \dots, \left[\frac{d-1}{2}\right]$ are linearly independent.

Consequently, $\mathcal{V}(d)_{\rm gen}$ is an open and dense subset of $\mathcal{V}(d)$.

Angles associated to a generic state

Let $u \in \mathcal{V}(d)_{\mathrm{gen}}$. Consider the lines

$$E_u(s_j) := \ker(H_u^2 - s_j^2 I)\ ,\ F_u(\tilde{s}_k) := \ker(K_u^2 - \tilde{s}_k^2 I)\ .$$

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Because $K_u^2 = H_u^2 - (.|u)u$, u is not orthogonal to these lines. Consider the orthogonal projections of u,

$$u_j \in E_u(s_j)\ ,\ \tilde{u}_k \in F_u(\tilde{s}_k)\ .$$

There exist $\psi_j, \tilde{\psi}_k \in \mathbb{T}$ such that

$$H_u(u_j) = s_j \mathrm{e}^{i\psi_j} u_j\ ,\ K_u(\tilde{u}_k) = \tilde{s}_k \mathrm{e}^{i\tilde{\psi}_k} \tilde{u}_k\ .$$

The inverse spectral theorem

Let $\Omega_d := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > \dots > x_d > 0\}$.

Theorem (Grellier-PG, 2012)

The mapping

$$u \in \mathcal{V}(d)_{\rm gen} \longmapsto ((s_1, \tilde{s}_1, s_2, \dots); (\psi_1, \tilde{\psi}_1, \psi_2, \dots)) \in \Omega_d \times \mathbb{T}^d$$

is a diffeomorphism.

The inverse spectral theorem, precision 1

In this coordinate system, the symplectic form and the Hamiltonian read

$$\begin{aligned} \omega &= \frac{1}{2}\sum_j d(s_j^2)\wedge d\psi_j - \frac{1}{2}\sum_k d(\tilde{s}_k^2)\wedge d\tilde{\psi}_k \\ \|u\|_{L^4}^4 &= \sum_j s_j^4 - \sum_k \tilde{s}_k^4 \ . \end{aligned}$$

In particular, the Hamiltonian evolution reads

$$\frac{d}{dt}s_j = 0\ ,\ \frac{d}{dt}\tilde{s}_k = 0\ ,\ \frac{d}{dt}\psi_j = s_j^2\ ,\ \frac{d}{dt}\tilde{\psi}_k = \tilde{s}_k^2\ .$$

The inverse spectral theorem, precision 2

The inverse of the diffeomorphism enjoys a simple algebraic expression. Set

$$n := \left[\frac{d+1}{2}\right] \;,\; \tilde{s}_n = 0 \text{ if } d \text{ is odd}$$

For every $z \in \overline{\mathbb{D}}$, the matrix

$$\mathscr{C}(z) := \left(\frac{s_j \mathrm{e}^{i\psi_j} - z\tilde{s}_k \mathrm{e}^{i\tilde{\psi}_k}}{s_j^2 - \tilde{s}_k^2}\right)_{1\le j,k\le n}$$

is invertible, and u is given by

$$u(x) = \langle \mathscr{C}(\mathrm{e}^{ix})^{-1}(\mathbf{1}_n), \mathbf{1}_n \rangle \;,$$

the sum of all the entries of $\mathscr{C}(\mathrm{e}^{ix})^{-1}$.

The inverse spectral theorem, extension

Similarly, denote by $H^{\frac{1}{2}}_{+,\mathrm{gen}}$ the set of $u \in H^{\frac{1}{2}}_{+}$ with infinitely many simple and distinct non zero eigenvalues for H_u^2, K_u^2.

This a dense G_δ subset of $H^{\frac{1}{2}}_{+}$.

Denote by Ω_∞ the subset of $\ell^2(\mathbb{N}_{\geq 1})$ made of decreasing sequences of positive numbers.

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$$u \in H^{\frac{1}{2}}_{+,\mathrm{gen}} \longmapsto ((s_1, \tilde{s}_1, s_2, \dots); (\psi_1, \tilde{\psi}_1, \psi_2, \dots)) \in \Omega_\infty \times \mathbb{T}^\infty$$

is a homeomorphism. Its inverse is given by

$$u(x) = \lim_{n\to\infty} \langle \mathscr{C}_n(\mathrm{e}^{ix})^{-1}(\mathbf{1}_n), \mathbf{1}_n \rangle \ ,$$

and the evolution laws of the coordinates are the same.

Intermezzo: Cauchy matrices

Given $2n$ numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$, such that the a_j are pairwise distinct, the b_k are pairwise distinct, and $\forall j, k, a_j + b_k \neq 0$, it is well known that the associated Cauchy matrix

$$\mathcal{C} := \left(\frac{1}{a_j + b_k} \right)_{1 \leq j,k \leq n}$$

is invertible, and that, with $D(\gamma) := \mathrm{diag}(\gamma_1, \ldots \gamma_n)$

$$\mathcal{C}^{-1} = D(\beta)\, \mathcal{C}^t\, D(\alpha) \,,$$

$$\beta_k := \frac{\prod_\ell (b_k + a_\ell)}{\prod_{\ell \neq k} (b_k - b_\ell)} \quad , \quad \alpha_j := \frac{\prod_i (a_j + b_i)}{\prod_{i \neq j} (a_j - a_i)} \,.$$

Furthermore, $\mathcal{C} D(\beta) \mathbf{1}_n = \mathbf{1}_n$.

The proof

$$\sum_{k=1}^{n} \frac{x_k}{a_j + b_k} = y_j \ , \ j = 1, \ldots, n \ .$$

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$$Q(a) := \prod_{k=1}^{n} (a + b_k) \ , \ \frac{P_x(a)}{Q(a)} := \sum_{k=1}^{n} \frac{x_k}{a + b_k} \ .$$

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Residue and Lagrange interpolation

$$x_k = \frac{P_x(-b_k)}{Q'(-b_k)} = \sum_{j=1}^{n} \frac{L_j(-b_k)}{Q'(-b_k)} Q(a_j) y_j$$

$$L_j(c) := \prod_{i \neq j} \frac{c - a_i}{a_j - a_i} \ .$$

Complex Cauchy matrices

Theorem

Let $a_1, \dots, a_n, b_1, \dots, b_n$ *be complex numbers such that* $|a_1|, \dots, |a_n|, |b_1|, \dots, |b_n|$ *are pairwise distinct. Then*

$$\mathscr{C} := \left(\frac{a_j - b_k}{|a_j|^2 - |b_k|^2} \right)_{1 \le j,k \le n}$$

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We give an independent and simpler proof
(PG-A.Pushnitski, 2017).

Proof of the invertibility

Assume

$$\sum_{k=1}^{n} \frac{(a_j - b_k)x_k}{|a_j|^2 - |b_k|^2} = 0 \ , \ j = 1, \ldots, n \ .$$

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...but also for $r = |b_1|^2, \dots, |b_n|^2$! Thus $P_1 = P_2 = 0, x = 0$.

More on complex Cauchy matrices

Theorem

Assume $|a_1| > |b_1| > |a_2| \cdots > |a_n| > |b_n|$. *For every* $z \in \overline{\mathbb{D}}$,

$$\mathscr{C}(z) := \left(\frac{a_j - b_k z}{|a_j|^2 - |b_k|^2} \right)_{1 \leq j,k \leq n}$$

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Proof. May assume $|z| < 1$. Write

$$\mathscr{C}(z) = D(a)\mathcal{T} - z\mathcal{T}D(b) \; , \; \mathcal{T} := \left(\frac{1}{|a_j|^2 - |b_k|^2} \right)_{1 \le j,k \le n}$$

Invertibility for $|z| < 1$

$$\mathcal{T} := \left(\frac{1}{|a_j|^2 - |b_k|^2}\right)_{1\leq j,k\leq n} \quad , \quad D(|a|^2)\mathcal{T} - \mathcal{T}D(|b|^2) = \mathbf{1}_n\mathbf{1}_n^t$$

$$\mathcal{T}^{-1} = D(\kappa)^2\,\mathcal{T}^t\,D(\tau)^2 \quad , \quad \mathcal{T}D(\kappa)^2\mathbf{1}_n = \mathbf{1}_n \ .$$

$$\tau_j^2 := \frac{\prod_i(|a_j|^2 - |b_i|^2)}{\prod_{i\neq j}(|a_j|^2 - |a_i|^2)} \quad , \quad \kappa_k^2 := \frac{\prod_\ell(|a_\ell|^2 - |b_k|^2)}{\prod_{\ell\neq k}(|b_\ell|^2 - |b_k|^2)} \ .$$

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$$MM^* = I - V\,\overline{V}^t \ , \ V := D(\kappa)\mathcal{T}^t D(\tau)^2 D(a)^{-1}\mathbf{1}_n \ .$$

Back to the Hankel inverse spectral problem

Let $u(x) = R(\mathrm{e}^{ix})$ where $R \in \mathbb{C}(z)$, with no pole in the closed unit disc.

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Here $n = \left[\frac{d+1}{2}\right]$.

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Given $((s_1, \tilde s_1, s_2, \dots); (\psi_1, \tilde\psi_1, \psi_2, \dots)) \in \Omega_d \times \mathbb{T}^d$, consider

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and let us prove that

$$H_u(\mathcal{U}) = D(a)\mathcal{U} \ , \ a_j := s_j \mathrm{e}^{i\psi_j}, b_k := \tilde s_k \mathrm{e}^{i\tilde\psi_k} \ .$$

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$$(\mathcal{T}^t D(\overline{a}) - S^* D(\overline{b}) \mathcal{T}^t) H_u \mathcal{U} = u \mathbf{1}_n \ .$$

This equation is also satisfied by $D(a)\mathcal{U}$!

Link with the cubic Szegő equation

Consider the equation

$$i\partial_t u = \Pi_+(|u|^2 u)$$

with the initial condition

$$u(0,x) = \langle \mathscr{C}[s_j, \tilde{s}_k; \psi_j, \tilde{\psi}_k](\mathrm{e}^{ix})^{-1}(\mathbf{1}_n), \mathbf{1}_n\rangle \ .$$

Then, from the Lax pair structure,

$$u(t,x) = \langle \mathscr{C}[s_j, \tilde{s}_k; \psi_j + ts_j^2, \tilde{\psi}_k + t\tilde{s}_k^2](\mathrm{e}^{ix})^{-1}(\mathbf{1}_n), \mathbf{1}_n\rangle \ .$$

2. Generic transition to high frequencies : sketch of the proof

Example : the daisy effect

If $u(0,x) = \mathrm{e}^{ix}$, check directly that $u(t,x) = \mathrm{e}^{i(x-t)}$.

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Now assume $u^\varepsilon(0,x) = \mathrm{e}^{ix} + \varepsilon$. Then

$$\Gamma_{\hat{u}_0^\varepsilon} := \begin{pmatrix} \varepsilon & 1 & 0 & . \\ 1 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}, \quad \begin{cases} s_1 = \frac{\varepsilon}{2} + \sqrt{1+\frac{\varepsilon^2}{4}}, & \psi_1 = 0 \\ \tilde{s}_1 = 1, & \tilde{\psi}_1 = 0 \\ s_2 = -\frac{\varepsilon}{2} + \sqrt{1+\frac{\varepsilon^2}{4}}, & \psi_2 = \pi \end{cases}$$

$$\mathscr{C}^\varepsilon(t,\mathrm{e}^{ix}) = \begin{pmatrix} \frac{s_1 \mathrm{e}^{its_1^2} - \mathrm{e}^{it}}{s_1^2 - 1} & \frac{\mathrm{e}^{its_1^2}}{s_1} \\ \frac{-s_2 \mathrm{e}^{its_2^2} - \mathrm{e}^{it}}{s_2^2 - 1} & -\frac{\mathrm{e}^{its_2^2}}{s_2} \end{pmatrix}$$

$$u^\varepsilon(t,x) = \langle \mathscr{C}^\varepsilon(t,\mathrm{e}^{ix})^{-1}(\mathbf{1}_2), \mathbf{1}_2 \rangle$$

The expression of $u(t)$

$$
\begin{aligned}
u^\varepsilon(t,x) &= \frac{a^\varepsilon(t)\mathrm{e}^{ix} + b^\varepsilon(t)}{1 - p^\varepsilon(t)\mathrm{e}^{ix}} \;, \; a^\varepsilon(t) = \mathrm{e}^{-it(1+\varepsilon^2)} \;, \\
b^\varepsilon(t) &= \mathrm{e}^{-it(1+\varepsilon^2/2)} \left(\varepsilon \cos(\omega^\varepsilon t) - i \frac{2+\varepsilon^2}{\sqrt{4+\varepsilon^2}} \sin(\omega^\varepsilon t) \right) \\
p^\varepsilon(t) &= -\frac{i}{\sqrt{1+\frac{\varepsilon^2}{4}}} \sin(\omega^\varepsilon t)\, \mathrm{e}^{-it\varepsilon^2/2} \;, \; \omega^\varepsilon := \varepsilon \sqrt{1 + \frac{\varepsilon^2}{4}} \;.
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$$p^\varepsilon(t) = -\frac{i}{\sqrt{1+\frac{\varepsilon^2}{4}}}\sin(\omega^\varepsilon t)\,\mathrm{e}^{-it\varepsilon^2/2}\ ,\ \omega^\varepsilon := \varepsilon\sqrt{1+\frac{\varepsilon^2}{4}}\ .$$

At time $t^\varepsilon = \frac{\pi}{2\omega^\varepsilon} \sim \frac{\pi}{2\varepsilon}$,

$$1 - |p^\varepsilon(t^\varepsilon)|^2 \sim \frac{\varepsilon^2}{4}\ ,\ \|u^\varepsilon(t^\varepsilon)\|_{H^s} \simeq \frac{1}{\varepsilon^{2s-1}}\ ,\ s > \frac{1}{2}\ .$$

Time variations of the momentum density

$$\mu^\varepsilon(t,k) := k\,|\hat{u}^\varepsilon(t,k)|^2 = |a^\varepsilon(t) + b^\varepsilon(t)p^\varepsilon(t)|^2\,|p^\varepsilon(t)|^{2(k-1)}\,,\ k \geq 1$$

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$$\begin{aligned}
&\sum_{k=1}^{\infty}\mu^\varepsilon(t,k) = 1\ ,\\
&\mu^\varepsilon(0,k) = \delta_{k=1}\ ,\\
&\sum_{\varepsilon^2 k\ll 1}\mu^\varepsilon(t^\varepsilon,k) + \sum_{\varepsilon^2 k\gg 1}\mu^\varepsilon(t^\varepsilon,k) \ll 1\ .
\end{aligned}$$

Transition to frequency $\frac{1}{\varepsilon^2}$ within time $t^\varepsilon \simeq \frac{1}{\varepsilon}$.

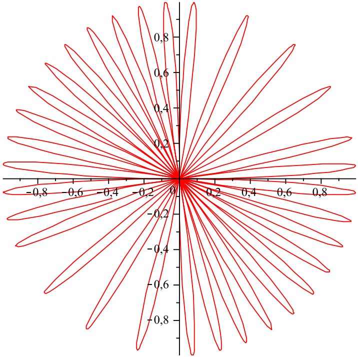

Figure: The quasi-periodic trajectory of $p^{\varepsilon}(t)$ for small ε.

Time averages of the H^1 norm

From

$$\begin{aligned} u^\varepsilon(t,x) &= \frac{a^\varepsilon(t)\mathrm{e}^{ix} + b^\varepsilon(t)}{1 - p^\varepsilon(t)\mathrm{e}^{ix}} \;,\; a^\varepsilon(t) = \mathrm{e}^{-it(1+\varepsilon^2)} \;, \\ b^\varepsilon(t) &= \mathrm{e}^{-it(1+\varepsilon^2/2)} \left(\varepsilon \cos(\omega^\varepsilon t) - i \frac{2+\varepsilon^2}{\sqrt{4+\varepsilon^2}} \sin(\omega^\varepsilon t) \right) \\ p^\varepsilon(t) &= -\frac{i}{\sqrt{1+\frac{\varepsilon^2}{4}}} \sin(\omega^\varepsilon t)\, \mathrm{e}^{-it\varepsilon^2/2} \;,\; \omega^\varepsilon := \varepsilon \sqrt{1 + \frac{\varepsilon^2}{4}} \;, \end{aligned}$$

one checks that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \|u^\varepsilon(t)\|_{H^1} \, dt \simeq \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(\varepsilon^2 + \cos^2\theta)^{\frac{1}{2}}} \simeq \log\left(\frac{1}{\varepsilon}\right) .$$

The transition to high frequencies

Theorem (Grellier, PG, 2017)

There exists a dense G_δ-subset of H^1_+, denoted by G, such that for all u solution of the Szegő equation, with $u(0) \in G$, we have

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \|u(t)\|_{H^1}\, dt = \infty,$$

and on the other hand

$$\liminf_{t\to\infty} \|u(t)\|_{H^1} \le \|u(0)\|_{H^1}.$$

Strategy

Baire's theorem combined with

Lemma

For any $u_0 \in H^1_+$, $\exists \{u_0^n\} \to u_0$ *in* H^1_+, *and* $T_n \to \infty$, $t_n \to \infty$ *such that, if* $u^n(0) = u_0^n$,

$$\frac{1}{T_n}\int_0^{T_n} \|u^n(t)\|_{H^1}\, dt \longrightarrow +\infty,$$

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Proof. Stick a small daisy to your initial data !

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May assume

$$u_0 : s_1 > \tilde{s}_1 > \dots ; \psi_1, \tilde{\psi}_1, \dots$$

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$$\frac{1}{T_n}\int_0^{T_n} \|u^n(t)\|_{H^1}\, dt \longrightarrow +\infty,$$

$$u^n(t_n) \longrightarrow u_0 \quad in\ H^1_+ \ .$$

Choose

$$u_0^n : s_1 > \tilde{s}_1 > \ldots > \delta_n(1+\varepsilon_n) > \delta_n > \delta_n(1-\varepsilon_n); \psi_1, \tilde{\psi}_1, \ldots, 0, 0, \pi$$

Open problems

Open problems

- The system of coordinates

$$u \in H^{\frac{1}{2}}_{+,\mathrm{gen}} \longmapsto ((s_1, \tilde{s}_1, s_2, \dots); (\psi_1, \tilde{\psi}_1, \psi_2, \dots)) \in \Omega_\infty \times \mathbb{T}^\infty$$

allows to describe the Szegő flow, but the high regularity is difficult to read, as it may strongly depend on angles. However, one can prove that, for $\delta > 0$ small enough, if $\tilde{s}_j \leq \delta s_j$ and $s_{k+1} \leq \delta \tilde{s}_k$, then u is holomorphic on a disc of radius $\rho > 1$ uniform. Give examples of angle sensitivity of the regularity in other cases ?

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- Appropriate probability measures ? Assume u has i.i.d. random Fourier coefficients. What is the image of the law of u through the above homeomorphism ? On the other hand : given a sequence $(s_1, \tilde{s}_1, s_2, \dots)$, a natural invariant measure is $\prod_{j=1}^{\infty} d\psi_j d\tilde{\psi}_j$. What is $\mathbb{E}(\|u\|^2_{H^s})$?

3. Related problems.

The cubic Szegő equation on the line

Consider

$$L^2_+(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subset [0, \infty)\}$$

and Π_+ the corresponding orthogonal projector. The equation

$$i\partial_t u = \Pi_+(|u|^2 u) \ , \ u(t, .) \in H^s_+(\mathbb{R}) := H^s(\mathbb{R}) \cap L^2_+(\mathbb{R})$$

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Furthermore, $H_u : h \in L^2_+ \mapsto \Pi_+(u\overline{h}) \in L^2_+$ satisfies the same Lax pair identity.
Long time behaviour of solutions ?

Solitons

Solutions of the form $u(t,x) = \mathrm{e}^{-i\omega t} u_0(x - ct)$, $\omega, c \in \mathbb{R}$, where

$$\omega u_0 - ic\partial_x u_0 = \Pi_+(|u_0|^2 u_0) .$$

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Theorem (O. Pocovnicu, 2012)

Solitons of the cubic Szegő equation on the line are given by

$$u_0(x) = \frac{a}{x+p} \ , \ a \in \mathbb{C}^* \ , \ \mathrm{Im}(p) > 0 \ .$$

Two-soliton solutions

Theorem (O. Pocovnicu, 2013)

Set $Q(x) := (x + \frac{i}{2})^{-1}$. *Assume*

$$u(0,x) = \alpha_1^0 Q\left(\frac{x - x_1^0}{\kappa_1^0}\right) + \alpha_2^0 Q\left(\frac{x - x_2^0}{\kappa_2^0}\right) .$$

Then the solution is exactly

$$u(t,x) = \alpha_1(t) Q\left(\frac{x - x_1(t)}{\kappa_1(t)}\right) + \alpha_2(t) Q\left(\frac{x - x_2(t)}{\kappa_2(t)}\right) ,$$

where $\alpha_{1,2}, \kappa_{1,2}$ *satisfy some ODE system. Furthermore, under some condition on the data, one of the* $\kappa_j(t)$ *is* $\simeq |t|^{-2}$ *at infinity, and* $\|u(t)\|_{H^1} \simeq |t|$ *as* $t \to \infty$.

Two-soliton interaction for focusing half-wave

Theorem (PG, O. Pocovnicu, E. Lenzmann, P. Raphaël, 2016)

Given $\delta \ll 1$, $K \gg 1$, *there exists* $T = T(\delta, K)$ *and a solution* u *of*

$$i\partial_t u = |D|u - |u|^2 u$$

such that

$$\|u(0)\|_{H^1} = \delta \ , \ \forall t \geq T \ , \ \ \|u(t)\|_{H^1} \geq K \ .$$

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Principle : construct a two soliton interaction for the half wave equation by modulation theory from soliton solutions to the half wave close to Q.

Open problems

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- Is it possible to make $\|u(t)\|_{H^1}$ unbounded in the half-wave equation ? Defocusing case ? Periodic case ?

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- Is it possible to make $\|u(t)\|_{H^1}$ unbounded in the half-wave equation ? Defocusing case ? Periodic case ?
- Genericity of such phenomena ? Weak turbulence approach to the half-wave equation ?

The half-wave maps equation (work in progress)

Phase space $\{\vec{S} : \mathbb{R} \to \mathbb{S}^2 : E[\vec{S}] < \infty\}$ with

$$E[\vec{S}] = \frac{1}{2} \int \vec{S} \cdot |D| \vec{S} \, dx = c \int\int_{\mathbb{R}\times\mathbb{R}} \frac{|\vec{S}(x) - \vec{S}(y)|^2}{|x-y|^2} \, dx \, dy \ .$$

Consider the corresponding Hamiltonian system

$$\partial_t \vec{S} = \vec{S} \wedge |D| \vec{S} \ .$$

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Continuum limit of some spin chain system.

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Continuum limit of some spin chain system.
Quasilinear hyperbolic system. Global existence of weak solutions — compensated compactness. Local existence of smooth solutions. Energy critical geometric wave equation.

Pauli matrices

Given $\vec{X} \in \mathbb{R}^3$, define

$$\mathbf{X} = \vec{X} \cdot \sigma = \sum_{j=1}^{3} X_j \sigma_j = \begin{pmatrix} X_3 & X_1 - \mathrm{i} X_2 \\ X_1 + \mathrm{i} X_2 & -X_3 \end{pmatrix}.$$

so that

$$(\vec{X} \cdot \sigma)(\vec{Y} \cdot \sigma) = (\vec{X} \cdot \vec{Y})\mathbf{1} + \mathrm{i}(\vec{X} \wedge \vec{Y}) \cdot \sigma.$$

and the equation reads

$$\partial_t \mathbf{S} = -\frac{\mathrm{i}}{2}[\mathbf{S}, |D|\mathbf{S}]$$

The Lax pair

Consider $\mu_{\mathbf{S}} : \varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mapsto \mathbf{S}\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$ and

$$L_{\mathbf{S}} = [H, \mu_{\mathbf{S}}]$$

where $H = -\mathrm{i}\,\mathrm{sgn}(D)$ is the Hilbert transform.

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Theorem (PG-Lenzmann, 2017)

If $\vec{S}$ is a smooth solution of the half-wave maps equation, then

$$\frac{d}{dt} L_{\mathbf{S}} = [B_{\mathbf{S}}, L_{\mathbf{S}}],$$

with $B_{\mathbf{S}} := -\frac{\mathrm{i}}{2}\left(\mu_{\mathbf{S}}|D| + |D|\mu_{\mathbf{S}}\right) + \frac{\mathrm{i}}{2}\mu_{|D|\mathbf{S}}$.

Link with Hankel operators

If $|D|\vec{S} \in H^s(\mathbb{R}, \mathbb{R}^3)$ for s big enough,

$$\vec{S}(x) = \vec{S}_0 + \vec{S}_+ + \vec{S}_- \ , \ \vec{S}_0 \in \mathbb{S}^2 \ , \ \vec{S}_\pm \in H_\pm^{1+s}(\mathbb{R}, \mathbb{C}^3) \ .$$

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Then, for $\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$,

$$L_{\mathbf{S}}(\varphi) = -2\mathrm{i}\Pi_+(\mathbf{S}_+\Pi_-\varphi) + 2\mathrm{i}\Pi_-(\mathbf{S}_-\Pi_+\varphi) \ .$$

Consequence 1

Corollary

Let $\vec{S} = \vec{S}(t,x)$ solve the half-wave maps equation for every t in an interval I containing 0. If $\vec{S}(0,x)$ is a rational function of x, then so is $\vec{S}(t,x)$ for every $t \in I$.

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Let $\vec{S} = \vec{S}(t,x)$ *solve the half-wave maps equation for every* t *in an interval* I *containing 0. If* $\vec{S}(0,x)$ *is a rational function of* x*, then so is* $\vec{S}(t,x)$ *for every* $t \in I$*.*

Just apply Kronecker's theorem : finite rank Hankel operators correspond to rational functions.

Consequence 2

Corollary

Let $\vec{S} = \vec{S}(t,x)$ solve the half-wave maps equation for every t in an interval I containing 0. If $\vec{S}(0,x) - \vec{S}_0$ is smooth and decaying enough at infinity — in $B^1_{1,1}(\mathbb{R},\mathbb{R}^3)$—, then $D\vec{S}$, $|D|\vec{S}, \partial_t \vec{S}$ are bounded in $L^1(\mathbb{R},\mathbb{R}^3)$ for $t \in I$.

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Apply Peller's theorem : $\mathrm{Tr}(|L_{\mathbf{S}}|) \simeq \|\vec{S} - \vec{S}_0\|_{B^1_{1,1}}$

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Application. An information about finite time blow-up : no jump discontinuity can occur.

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Application. An information about finite time blow-up : no jump discontinuity can occur.

Indeed, the weak limit μ of $D\vec{S}$ at the blow up time would be a measure satisfying $H\mu$ is a measure. Appeal to Riesz brothers' theorem !!

Open questions

- Finite time blow up ? Long time behaviour ? Study the system of ODE in the rational case.
- Same questions in the periodic case.
- Same questions if $\mathbb{S}^2$ is replaced by the hyperbolic plane ?