Effet tunnel et observation approchée pour des opérateurs hypoelliptiques

Camille Laurent,
CNRS, LJLL, Paris 6
avec Matthieu Léautaud, CMLS, Ecole Polytechnique

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Degenerate hypoelliptic operators

We are interested in some class of degenerate operators $\mathcal{L}u = div(A(x)\nabla u) + b\cdot\nabla u$ with $A(x) \geq 0$

$$\mathcal{L} = -\sum_{i=1}^{m} X_i^* X_i \quad (+X_0).$$

where X_i are C^∞ first-order differential operators.

Assumption (Hörmander hypothesis)

There exists $k \geq 1$ so that for any $x \in \mathcal{M}$,
$\mathrm{Lie}^k(X_0, \cdots, X_m)(x) = T_x\mathcal{M}$.

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This implies $\mathcal{L}$ hypoelliptic (Hörmander).

Question of approximate observability/controlability

We are interested in the quantification of the unique continuation property

$$u = 0 \text{ on } ([0,T]\times)\omega \Rightarrow u = 0.$$

for some $\omega \subset \mathcal{M}$ for some u solution of either

$$\partial_t^2 u - \mathcal{L}u = 0 \qquad \text{wave like equation}$$

$$\Downarrow$$

$$-\mathcal{L}u = \lambda u \qquad \text{eigenfunctions}$$

$$\partial_t u - \mathcal{L}u = 0 \qquad \text{heat like equation}$$

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Dual property : exact or approximate controlability

Classical Unique Continuation Theorems

Holmgren, John

- analytic coefficients
- Φ non characteristic for $P : p(x, \nabla\Phi) \neq 0$

Tataru, Robbiano-Zuily, Hörmander

- partially analytic coefficients in some variable x_a
- Φ pseudoconvex in $\{\xi_a = 0\}$

Carleman, Hörmander

- C^∞ (even C^1) coefficients
- Φ pseudoconvex : $\{p, \{p, \Phi\}\} > 0$ sufficient if real order 2

Theorem (Tataru (95,99), with improvements by Robbiano-Zuily (98), Hörmander (97))

Let $x_0 \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$. P with smooth coefficients, analytic in the x_a variable. P analytically principally normal in $\{\xi_a = 0\}$ (OK if elliptic or real invariant in x_a).
$S = \{\Phi = 0\}$ oriented hypersurface pseudoconvex in $\{\xi_a = 0\}$.

If u solution of $Pu = 0$ near x_0 and $u = 0$ in $\{\Phi \geq 0\}$, then $u = 0$ in a small neighborhood of x_0.

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The main tool is the Carleman estimate "with pseudodifferential weight"

$$\tau \left\| e^{-\frac{\varepsilon}{2\tau}|D_a|^2} e^{\tau\psi} u \right\|_{m-1,\tau}^2 \leq C \left(\left\| e^{-\frac{\varepsilon}{2\tau}|D_a|^2} e^{\tau\psi} P u \right\|_0^2 + e^{-\tau d} \left\| e^{\tau\psi} u \right\|_{m-1,\tau}^2 \right)$$

Theorem (Quantification of the unique continuation with partial analyticity)

In the above geometric setting, P be a differential operator of order m, analytically principally normal operator on Ω in $\{\xi_a = 0\}$.
Assume also that, for any $\varepsilon \in [0, 1+\eta)$, the oriented surfaces $S_\varepsilon = \{\phi_\varepsilon = 0\}$ with $\phi_\varepsilon(x', x_n) := G_\varepsilon(x') - x_n$ are strictly pseudoconvex in $\{\xi_a = 0\}$ for P on the whole S_ε.
Then, for any open neighborhood $\omega \subset \Omega$ of S_0, there are constants $\kappa, C, \mu_0 > 0$ such that for all $\mu \geq \mu_0$ and $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|u\|_{L^2(K)} \leq Ce^{\kappa\mu} \left(\|u\|_{H_b^{m-1}(\omega)} + \|Pu\|_{L^2(\Omega)} \right) + \frac{C}{\mu^{m-1}} \|u\|_{H^{m-1}(\Omega)},$$

where we have denoted $\|u\|_{H_b^{m-1}(\omega)} = \sum_{|\beta| \leq m-1} \left\| D_b^\beta u \right\|_{L^2(\omega)}$.

Theorem (Waves)

Let $\mathcal{M}$ be a compact Riemannian manifold with boundary. Let ω be a non empty open subset of $\mathcal{M}$, for any $T > T_{UC} = 2\sup_{x\in\mathcal{M}} d(x,\omega)$, there exist $C, c > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and associated solution u of

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 & \text{in } [0,T]\times\mathcal{M}, \\ u_{|\partial\mathcal{M}} = 0 & \text{in } [0,T]\times\partial\mathcal{M}, \\ (u,\partial_t u)_{|t=0} = (u_0,u_1) & \text{in } \mathcal{M}, \end{cases} \tag{1}$$

we have,

$$\|(u_0,u_1)\|_{H^1\times L^2} \leq Ce^{c\Lambda}\|u\|_{L^2(]0,T[\times\omega)}$$

with $\Lambda = \frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|(u_0,u_1)\|_{L^2\times H^{-1}}}$ the typical frequency of the solution.

Previous results : Lebeau (92) analytic case, Robbiano (95), Phung (02). See also Bosi-Lassas-Kurylev (16)

Approximate controllability for the wave

Theorem (Cost of boundary approximate control)

For any $T > T_{UC}$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $g \in L^2((0,T) \times \omega)$ with

$$\|g\|_{L^2((0,T)\times\omega)} \leq Ce^{\frac{c}{\varepsilon}} \|(u_0, u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})},$$

such that the solution of

$$\begin{cases} (\partial_t^2 - \Delta)u = g & \text{in } (0,T)\times\mathcal{M}, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1), & \text{in } \mathcal{M}, \end{cases}$$

satisfies $\left\|(u, \partial_t u)_{|t=T}\right\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} \leq \varepsilon \|(u_0, u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}$.

Up to now, we will make the following assumptions :

$\mathcal{M}$ is a compact manifold without boundary (except in Grushin type case)

$$\mathcal{L} = -\sum_{i=1}^{m} X_i^* X_i.$$

Assumption

*The manifold $\mathcal{M}$, the density ds, and the vector fields X_i are **real-analytic**.*

k is the same as in Hörmander hypothesis.

General estimates for the wave

Theorem

Assume that ω *is a non empty open set of* $\mathcal{M}$ *and let* $T > \sup_{x\in\mathcal{M}} d_{\mathcal{L}}(x,\omega)$. *Then, there exist* $C > 0$ *such that we have*

$$\|(u_0,u_1)\|_{\mathcal{H}^1_{\mathcal{L}}\times L^2} \leq Ce^{c\Lambda^k} \|u\|_{L^2(]-T,T[\times\omega)}, \qquad \text{with } \Lambda = \frac{\|(u_0,u_1)\|_{\mathcal{H}^1_{\mathcal{L}}\times L^2}}{\|(u_0,u_1)\|_{L^2\times\mathcal{H}^{-1}_{\mathcal{L}}}},$$

for any $(u_0,u_1) \in \mathcal{H}^1_{\mathcal{L}} \times L^2$, *and associated solution u solution*

$$\begin{cases} (\partial_t^2 - \mathcal{L})u = 1_\omega g & \text{in } (0,T)\times\mathcal{M}, \\ (u,\partial_t u)|_{t=0} = (u_0,u_1), & \text{in } \mathcal{M}, \end{cases}$$

$d_{\mathcal{L}}$ is the natural distance induced by the sub-Riemannian structure coming from the control problem

Eigenfunction tunneling

Theorem

Let ω be a nonempty open subset of $\mathcal{M}$. Then, there is $C, c > 0$ such that every eigenfunction φ_i of $\mathcal{L}$ associated to the eigenvalue λ_i satisfies

$$\|\varphi_j\|_{L^2(\mathcal{M})} \leq Ce^{c\lambda_j^{k/2}} \|\varphi_j\|_{L^2(\omega)}.$$

Optimal for the example of Grushin type (see Beauchard-Cannarsa-Guglielmi (14)).

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Proof : Theorem on the wave easily implies eigenfunction tunneling with the solution $u(t,x) = \cos(\sqrt{\lambda_j}t)\varphi_j$.

Idea of the proof for the wave

Construct some appropriate noncharacteristic hypersurfaces so that we can apply our Theorem 2 in the Holmgren case. After this construction, this gives some estimates of the form

$$\|u\|_{L^2(]-\varepsilon,\varepsilon[\times\mathcal{M})} \leq Ce^{\kappa\mu}\|u\|_{L^2(]-T,T[\times\omega)} + \frac{C}{\mu}\|u\|_{H^1(]-T,T[\times\mathcal{M})},$$

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Then, we use the subelliptic estimate of Rothschild and Stein(76)

$$\|u\|^2_{H^{\frac{2}{k}}(\mathcal{M})} \leq C\|\mathcal{L}u\|^2_{L^2(\mathcal{M})} + C\|u\|^2_{L^2(\mathcal{M})}.$$

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It gives after energy estimates
$\left\|u\right\|_{H^1(]-T,T[\times\mathcal{M})} \leq C\left\|(u_0,u_1)\right\|_{\mathcal{H}^k_{\mathcal{L}}\times\mathcal{H}^{k-1}_{\mathcal{L}}}$ and then

$$\left\|(u_0,u_1)\right\|_{L^2\times\mathcal{H}^{-1}_{\mathcal{L}}} \leq Ce^{\kappa\mu}\left\|u\right\|_{L^2(]-T,T[\times\omega)} + \frac{C}{\mu}\left\|(u_0,u_1)\right\|_{\mathcal{H}^k_{\mathcal{L}}\times\mathcal{H}^{k-1}_{\mathcal{L}}},$$

Previous results on hypoelliptic operators : unique continuation

- positive results : Bony (69) using Holmgren, Garofalo (93) Grushin like operators...
- negative results : Bahouri (86) large class of counterexamples for $\mathcal{L} + V$ with $\mathcal{L}$ like Heisenberg...

Previous results of control of hypoelliptic heat-like operators

- Type I ($X_0 = 0$) : Grushin (see after)
  Beauchard-Cannarsa-Guglielmi (14), Beauchard-Miller-Morancey (15), Koenig (17)
  Heisenberg : Beauchard-Cannarsa (17)
- Type II : Kolmogorov : Beauchard-Zuazua (09), Le Rousseau-Moyano (16), Beauchard-Helffer-Henry-Robbiano (15)
  Ornstein-Uhlenbeck operators : Beauchard-Pravda-Starov (16)

General estimates for the heat

Theorem

For all $T > 0$, there exist $C, c > 0$ such that for any $u_0 \in \mathcal{H}^1_{\mathcal{L}}$ and associated solution u of $\partial_t u - \mathcal{L}u = 0$, we have

$$\|u_0\|^2_{L^2} \leq Ce^{c\Lambda^k} \int_0^T \int_\omega |u(t,x)|^2 \, dx \, dt, \qquad \Lambda = \frac{\|u_0\|_{\mathcal{H}^1_{\mathcal{L}}}}{\|u_0\|_{L^2}}, \tag{2}$$

Elliptic case : Fernandez-Cara-Zuazua (00), Phung (04)

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Corollary (Exponential cost of approximate null control)

For any $T > 0$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $u_0 \in L^2(\mathcal{M}), u_1 \in L^2(\mathcal{M})$, there exists $g \in L^2((0,T) \times \omega)$ with

$$\|g\|_{L^2((0,T)\times\omega)} \leq Ce^{\frac{c}{\varepsilon^k}} \left\|e^{-T\mathcal{L}}u_0 - u_1\right\|_{L^2(\mathcal{M})},$$

such that the solution of $\partial_t u - \mathcal{L}u = g$ issued from u_0 satisfies

$$\|u(T) - u_1\|_{\mathcal{H}^{-1}_{\mathcal{L}}} \leq \varepsilon \left\|e^{-T\mathcal{L}}u_0 - u_1\right\|_{L^2(\mathcal{M})}.$$

Previous results on heat-like Grushin type operators

$$\|u(T)\|^2_{L^2(\mathcal{M})} \leq C\int_0^T \int_\omega |u(t,x)|^2 \, dtdx, \quad u \text{ solution of heat} \qquad (3)$$

Theorem (Beauchard, Cannarsa and Guglielmi (14))

Assume $\mathcal{L} = \partial_x^2 + x^{2\gamma}\partial_y^2$ *with Dirichlet on* $[-1,1]_x \times [0,1]_y$.

1. *If* $\gamma \in [0,1[$, *then the observability inequality* (3) *holds true for any nonempty open set* $\omega \subset \mathcal{M}$ *in any time* $T > 0$.
2. *If* $\gamma = 1$ *and if* $\omega =]a,b[\times]0,1[$ *where* $0 < a < b < 1$, *then there exists* $T^* \geq a^2/2$ *such that*
   - *for every* $T > T^*$ *the observability inequality* (3) *holds true,*
   - *for every* $T < T^*$ *the observability inequality* (3) *is false.*
3. *If* $\gamma > 1$ *and* $\omega \subset (0,1)\times(0,1)$, *then* (3) *is false, in any* $T > 0$.

Theorem (Koenig (17))

$\gamma = 1$. *Assume that there is* $0 < c < d < 1$ *such that* $\omega \cap (]-1,1[\times]c,d[) = \emptyset$. *Then, for any* $T > 0$, (3) *is false.*

The specific case $k = 2$

Theorem

Assume that $k = 2$. There exist $T^* > 0$ such that for all $T > T^*$ and all $\varepsilon > 0$, we have for any solution u to $\partial_t u - \mathcal{L}u = 0$,

$$\|u(T)\|_{L^2}^2 \leq \frac{1}{\varepsilon^\beta} \int_0^T \int_\omega |u(t,x)|^2 \, dt \, dx + \varepsilon \|u(0)\|_{L^2}^2 . \tag{4}$$

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Corollary (Polynomial cost of approximate null control if $k = 2$)

For any $u_0 \in L^2$, there exists $g \in L^2((0,T) \times \omega)$ with

$$\|g\|_{L^2((0,T)\times\omega)} \leq \frac{C}{\varepsilon^\beta} \|u_0\|_{L^2} ,$$

such that the associated solution u of $\partial_t u - \mathcal{L}u = g$ satisfies

$$\|u(T)\|_{L^2(\mathcal{M})} \leq \varepsilon \|u_0\|_{L^2} ,$$

Structure of the proof

Theorem on the wave implies Theorem on the heat using some variant of the transmutation method as done by Ervedoza-Zuazua (11) :
There exists some kernel $k_T(t,s)$ compactly supported in t such that if y is solution of the heat, $u(s) = \int_0^T k_T(t,s)y(t)dt$ is solution of the wave.

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The case $k = 2$ is the limit case where the cost of the control $\approx e^{c\lambda^{k/2}} = e^{c\lambda}$ is of the same order of the dissipation $e^{-\lambda T}$ of the heat.
That is why we need T large to get a polynomial cost.
Lebeau-Robbiano in one shot.

Some "Lebeau-Robbiano like" estimates

Lemma

There exist $C, \gamma > 0$ such that for any $T > 0, \lambda \geq 0$, for every $y_0 \in E_\lambda$ and associated solution y to $\partial_t y - \mathcal{L} y = 0$, we have

$$\|y(T)\|_{L^2}^2 \leq \frac{C}{T} e^{\left(2\gamma\lambda^{k/2} + \frac{C}{T}\right)} \int_0^T \int_\omega |y(t,x)|^2 \, dt \, dx.$$

Rk : $k = 1$: elliptic Lebeau-Robbiano, $k = 2$ (Grushin) behaves like half-Laplacian

Further resultats we have obtained

- polynomial cost in some Gevrey type spaces
- some Grushin type cases where we only need partial analyticity (with respect to y only) and allow a boundary
- $\Lambda = \frac{\|(u_0,u_1)\|_{\mathcal{H}^1_{\mathcal{L}} \times L^2}}{\|(u_0,u_1)\|_{L^2 \times \mathcal{H}^{-1}_{\mathcal{L}}}}$ may be changed by $\Lambda_s^{1/s}$ with $\Lambda_s = \frac{\|(u_0,u_1)\|_{\mathcal{H}^s_{\mathcal{L}} \times \mathcal{H}^{s-1}_{\mathcal{L}}}}{\|(u_0,u_1)\|_{L^2 \times \mathcal{H}^{-1}_{\mathcal{L}}}}$ for any $s > 0$

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Further open problems

- in the case $k=2$, find the right condition to get from exact controllability to approximate controlability with polynomial cost
- understand more generally the case when drift X_0 is necessary

MERCI DE VOTRE ATTENTION ! ! ! ! !