Anisotropic MHD & Alfvén Wave Turbulence

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Contents

MHD approximation

- crucial role of the magnetic field in geophysical and astrophysical fluid dynamics (stellar or solar wind, convective zone of stars, accretion discs, magnetic field generation by dynamo effect, ...) leads to explore properties of the magnetohydrodynamic (MHD), namely the interaction between an electrically conducting fluid and a magnetic field

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- crucial role of the magnetic field in geophysical and astrophysical fluid dynamics (stellar or solar wind, convective zone of stars, accretion discs, magnetic field generation by dynamo effect, ...) leads to explore properties of the magnetohydrodynamic (MHD), namely the interaction between an electrically conducting fluid and a magnetic field
- MHD is a fluid approximation: does not describe the detailed processes of plasma physics which require description of individual motions of particles (plasma frequencies, frequencies and lenght scales for both ions and electrons, ..)

- MHD approximation is valid for some plasmas and liquid metals
  * quasi-neutral property ($\nabla \cdot \mathbf{E} \simeq 0$, with $\mathbf{E}$ the electric field)
  * fluid approximation: electrical conduction of the medium by electrons alone
  * non relastivistic limit (typical velocity $U \ll c$)
  * collisional plasma/fluid: conductivity is independent of U (time evolution of the fluid $\gg$ time between 2 collisions ions/electrons, fluid elements contain many ions & electons)
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- MHD description is limited when
  * non-fluid or kinetic effects are important (dissipation in the turbulent solar wind, magnetic reconnection, small-scales dynamics in Earth's magnetosphere)
  * plasmas are fully ionized (solar photosphere/chromosphere, molecular clouds, protoplanetary disks, Earth's ionosphere, some laboratory plasmas)

MHD equations

- Maxwell's equations
  Faraday's law $\nabla \times \mathbf{E} + \partial \boldsymbol{B}/\partial t = 0$
  Ampère's law $\nabla \times \boldsymbol{B} = \mu_0[\boldsymbol{j} + \cancel{\epsilon_0 \partial \mathbf{E}/\partial t}]$ with $c^2 = (\mu_0 \epsilon_0)^{-1}$
  Coulomb's gauge $\nabla \cdot \boldsymbol{B} = 0$
  where $\boldsymbol{j} \equiv$ current density, $\mu_0 \equiv$ magnetic permeability of free space, $\epsilon_0 \equiv$ permittivity of free space, and displacement current is neglected
- add Ohm's law $\boldsymbol{j} = \sigma(\mathbf{E} + \boldsymbol{u} \times \boldsymbol{B})$
  where $\sigma \equiv$ electrical conductivity of the medium
- equations combine to yield an evolution equation for the magnetic field or the so-called induction equation

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B}$$

  where $\eta = (\mu_0 \sigma)^{-1}$ is the magnetic resistivity of the medium
- note that $\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$ represents both advection and stretching of the field $\boldsymbol{B}$ (with $\nabla \cdot \boldsymbol{u} = 0 = \nabla \cdot \boldsymbol{B}$)

- Incompressible MHD equations

$$\partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p/\rho + \nu\Delta\boldsymbol{u} + (\boldsymbol{j}\times\boldsymbol{B})/\rho + \mathbf{F}$$
$$\partial \boldsymbol{B}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} + \eta\Delta\boldsymbol{B}$$
$$\nabla\cdot\boldsymbol{u} = 0 = \nabla\cdot\boldsymbol{B} \quad \text{and} \quad B.C.$$

where $\boldsymbol{j}\times\boldsymbol{B} = (\nabla\times\boldsymbol{B})\times\boldsymbol{B}/\mu_0$ is the Lorentz force and $\mathbf{F}$ a body force (friction, gravity, Coriolis force,..). $\boldsymbol{B}$ can be replaced by the scaled magnetic field $\boxed{\boldsymbol{b} = \boldsymbol{B}/(\rho\mu_0)^{1/2} = \boldsymbol{v_a}}$ which has dimension of a velocity and is called the Alfvén velocity although $\boldsymbol{B}$ is a pseudovector [1]

[1] $(\mathbf{i},\mathbf{j},\mathbf{k}) \to (\mathbf{i},\mathbf{j},-\mathbf{k})$, $\mathbf{A}$ vector : $(A_x, A_y, A_z) \to (A_x, A_y, -A_z)$ & $\mathbf{B}$ pseudovector : $(B_x, B_y, B_z) \to (-B_x, -B_y, B_z)$

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- dimensionless parameters
  * kinetic Reynolds number = advection / dissipation = $R_e = UL_0/\nu$
  * magnetic Reynolds number = induction / diffusion = $R_M = UL_0/\eta$
  * magnetic Prandtl numer = $P_M = \nu/\eta = R_M/R_e$

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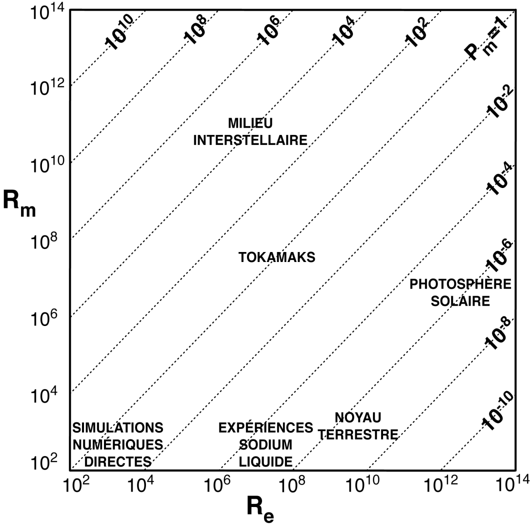

Estimates of R_M and R_e Reynolds numbers for different media.Diagonals correspond to constant magnetic Prandtl numbers $P_M = \frac{kinematic\ \ viscosity}{magnetic\ \ diffusivity}$.

another important dimensionless parameter β

using $\nabla(\boldsymbol{B}^2)/2 = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \boldsymbol{B}\times(\nabla\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \boldsymbol{B}\times\mu_0\boldsymbol{j}$

thus, from Ampère's law, the Lorentz force can be written as

$$\boldsymbol{j}\times\boldsymbol{B} = \underbrace{\frac{(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}}{\mu_0}}_{magnetic\ \ tension} - \nabla \underbrace{\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right)}_{magnetic\ \ pressure}$$

and the parameter β is defined as

$$\boxed{\beta = \frac{plasma\ \ pressure}{magnetic\ \ pressure} = \frac{p}{\boldsymbol{B}^2/(2\mu_0)}}$$

- if $\beta \ll 1$, the magnetic field dominates;
  solar corona, tokamaks ($\beta \lessapprox 0.1$)
- if $\beta \gg 1$, plasma pressure forces dominate; stellar interiors
- $\beta \sim 1$, pressure/magnetic forces are both important;
  solar chromosphere, parts of solar wind & interstellar medium, some laboratory plasma experiments

MHD equations in Elsässer variables

The velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{b}$ can be combined into the Elsässer fields $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$ (remember $\boldsymbol{b} = \boldsymbol{B}/(\rho\mu_0)^{1/2}$), to obtain the following symmetric equations :

$$(\partial_t + \boldsymbol{z}^{\mp} \cdot \nabla)\boldsymbol{z}^{\pm} = \nu_1 \Delta \boldsymbol{z}^{\pm} + \nu_2 \Delta \boldsymbol{z}^{\mp} - \nabla P_* + \boldsymbol{f}^{\pm}$$

where $\nabla \cdot \boldsymbol{z}^{\pm} = 0$, and $P_* = (p/\rho + \boldsymbol{b}^2/2)$ is the total pressure, $\nu_1 = \frac{1}{2}(\nu + \eta)$, $\nu_2 = \frac{1}{2}(\nu - \eta)$, and, from now on, boundary conditions are assumed periodic in space.

Note that

- the $\boldsymbol{z}^+$ fluctuations are advected by the $\boldsymbol{z}^-$ ones and conversely
- the total pressure writes $\Delta P_* = -\nabla \cdot (\boldsymbol{z}^{-s} \cdot \nabla \boldsymbol{z}^s)$ with $s = \pm$
- $\boldsymbol{u} = (\boldsymbol{z}^+ + \boldsymbol{z}^-)/2$, $\boldsymbol{b} = (\boldsymbol{z}^+ - \boldsymbol{z}^-)/2$
- $\boldsymbol{u} \cdot \boldsymbol{b} = ({\boldsymbol{z}^+}^2 - {\boldsymbol{z}^-}^2)/4$, $\boldsymbol{u}^2 - \boldsymbol{b}^2 = \boldsymbol{z}^+ \cdot \boldsymbol{z}^-$
- $\boldsymbol{\omega}^{\pm} = \nabla \times \boldsymbol{z}^{\pm} = \boldsymbol{\omega} \pm \boldsymbol{j}$

Ideal invariants in homogeneous MHD turbulence

- $\partial_t E^T = -\nu < \boldsymbol{\omega}^2 > -\eta < \boldsymbol{j}^2 >$, for $\nu = \eta = 0 \ \rightarrow$ total energy (kinetic + magnetic) is a conserved quantity

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  for $\nu = \eta = 0 = \nu_1 = \nu_2 \ \rightarrow$ conservation of the $\boldsymbol{z}^+$ and the $\boldsymbol{z}^-$ energies. Note that $H^C = (E^+ - E^-)/2$

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- note that: 1) helicities are pseudo-scalars, 2) in a perfect MHD fluid, the mutual topologies of tubes are conserved

Alfvén waves

- Linearization of incompressible MHD eqs around a uniform magnetic field $\boldsymbol{b}_0$ with $\rho_0 = cst, p_0 = cst, \boldsymbol{u}_0 = 0$ (ν and η neglected) leads to :

$$\partial_t \boldsymbol{z}^+ - (\boldsymbol{b}_0 \cdot \nabla)\boldsymbol{z}^+ = 0$$
$$\partial_t \boldsymbol{z}^- + (\boldsymbol{b}_0 \cdot \nabla)\boldsymbol{z}^- = 0$$

Looking for a solution of plane-wave type for perturbations

$$\boldsymbol{z}^\pm = \boldsymbol{z}_k{}^\pm \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \bar{\omega}^\pm t)}$$

gives: $\bar{\omega}^+ = -(\boldsymbol{b}_0 \cdot \boldsymbol{k})$ and $\bar{\omega}^- = +(\boldsymbol{b}_0 \cdot \boldsymbol{k})$ with $\boldsymbol{k} \cdot \boldsymbol{z}_k^+ = 0$ and $\boldsymbol{k} \cdot \boldsymbol{z}_k^- = 0$ (incompressibility).

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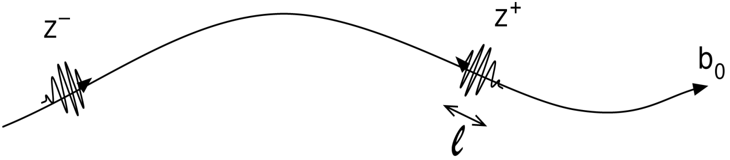
- $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ are the so-called Alfvén waves : transverse waves ($\boldsymbol{z}_k^\pm \perp \boldsymbol{k}$) with group velocity $v_g = \mp b_0$ and phase velocity $v_\phi = \mp b_0 k_\parallel / k$, where $k_\parallel$ is the component of wavevector $\boldsymbol{k} \parallel \boldsymbol{b}_0$ (non-dispersive).

- oppositely travelling waves: $\boldsymbol{z}^-$ travels in the $\boldsymbol{b}_0$-direction while $\boldsymbol{z}^+$ is backward travelling, with group velocity $\boldsymbol{b}_0$, the so-called Alfvén velocity denoted $\boldsymbol{v_a}$



Two Alfvén disturbances/wavepackets, with typical lenght ℓ, travelling along an ambient magnetic field $\boldsymbol{b}_0$.

Concept

* A uniform magnetic field $\boldsymbol{b}_0$ (or might be a local one at scale much larger than a given ℓ of interest, or at the largest scale) has a significant dynamical effect for energy transfers : $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ blob disturbances (wavepackets) only interact when they collide $\longrightarrow$ weakening of the transfer of energy between scales (in that sense, we refer to weak nonlinearity)

* Multiple collisions are needed to pass energy in the blobs to smaller scales

* This is the basic idea of "IK" phenomenology (Iroshnikov 63, Kraichnan 65): interplay between turbulent eddies and Alfvén waves travelling along a mean magnetic field $\longrightarrow$ crucial difference between hydrodynamic and conducting fluids

* Alfvén waves and correlation between $\boldsymbol{u}$ and $\boldsymbol{b}$ fields (cross helicity) - e.g. dynamical alignment - are crucial and lead to a lack of universality for inertial MHD spectra

Turbulent flows

* Assume a turbulent flow characterized with
  - energy injected at large scales and dissipated at small scales
  - in between, conservative mean rate of energy transfer per unit mass, ϵ, (finite & non zero), due to nonlinear interactions among scales $\longrightarrow$ concept of inertial range and energy cascade
  - as the Reynolds numbers $\to \infty$, scale statistics depend only on scale $\ell \sim 1/k$ ($k = |\boldsymbol{k}|$) and ϵ within the inertial range (Kolmogorov)

* Statistical invariances (useful here)

From a theoretical point of view, define ensemble average as average over a very large number $\mathcal{N}$ of identical experiments of a given flow

* homogeneity: statistical properties of the turbulence (e.g. ensemble average quantities) are invariant under abitrary spatial translations

* stationarity: time translation invariance. For stationary turbulence, ensemble average and time average do yield the same results (ergodicity)

In incompressible MHD, fully turbulence can occur only if there is enough flux of counter-propagating disturbances/waves.

IK phenomenology

Let's take $P_M \sim 1$ from now on.
Write $\boldsymbol{B} = \boldsymbol{b}_0 + \boldsymbol{b}$ with $|\boldsymbol{b}| \ll |\boldsymbol{b}_0|$, the IK phenomenology is based on weak nonlinear interactions and many collisions, say N, between $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ wavepackets of similar size ℓ, are needed to pass energy to smaller scales. For simplicity, ignore anisotropy ($\ell_\parallel \sim \ell_\perp \sim \ell$) for a while, and, suppose $H^C \sim 0$ i.e. energy balance $E \sim E^\pm$ or $z_\ell^+ \sim z_\ell^- \sim z_\ell$.

What is the energy transfer time ?

Disturbances are sheared by an amount

$$\delta z_\ell \sim (z_\ell z_\ell/\ell)(\ell/b_0) \longrightarrow \delta z_\ell/z_\ell \sim z_\ell/b_0$$

- $t_a \sim \ell/b_0 \equiv \ell/v_a$ is the interaction time for one collision (Alfvén time) at scale ℓ, i.e. characteristic time for wave propagation over distance ℓ

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- this gives the energy transfer time at scale ℓ; $\boxed{t_{tr} \sim N(\ell/v_a) \sim t_\ell^2/t_a}$
- t_ℓ is the advection (nonlinear) time at scale ℓ; $t_\ell \sim \ell/z_\ell$

We still suppose $P_M \sim 1$, zero cross helicity ($H^c \sim 0$), thus $z^+_\ell \sim z^-_\ell \sim z_\ell$ and a dynamical behaviour dominated by Alfvén waves

- if $b_0 \gg z_\ell$, $t_a \ll t_\ell$
  energy transfer time $t_{tr} \sim t^2_\ell / t_a \sim (\ell_\perp / z_\ell)^2 / (\ell_\| / b_0) \sim (k_\| b_0)/(k^2_\perp z^2_\ell)$
  energy flux down through the inertial range
  $\epsilon^+_\ell \sim \epsilon^-_\ell \sim \epsilon_\ell \sim \epsilon \sim z^2_\ell / t_{tr} \sim k^2_\perp z^4_\ell / k_\| b_0 \longrightarrow z_\ell \sim (\epsilon k_\| b_0 / k^2_\perp)^{1/4}$
  which leads to $K^\pm_\ell \sim z^2_\ell \sim k_\| k_\perp E(k_\perp, k_\|)$ and
  $\boxed{E(k_\perp, k_\|) \sim (\epsilon b_0)^{1/2} k_\|^{-1/2} k_\perp^{-2}}$ (Ng & Bhattacharjee, 1997)
- if $b_0 \gg \varepsilon z_\ell$, or $t_a \ll \varepsilon t_\ell$, asymptotic analytical result within Alfvén waves turbulence theory with no energy transfer along $\boldsymbol{b}_0$
  $\boxed{E(k_\|, k_\perp) \sim C_k f(k_\|) {k_\perp}^{-2}}$ $(k_\| \neq 0,\ k_\perp \gg k_\|)$ (Galtier et al., 2000)

Remark: If $H^c \not\sim 0$ ($E^+ \not\sim E^-$), defining $E^\pm \sim k_\perp^{n_\pm}$ and using $t^\pm_\ell \sim \ell_\perp / z^\mp_\ell$ & $\epsilon^\pm \sim {z^\pm_\ell}^2 / {t_{tr}}^\pm \longrightarrow$ family of solutions such as
$\boxed{n_+ + n_- = -4}$

What is wave turbulence ?

Wave turbulence is generally defined as out-of-equilibrium statistical mechanics of random nonlinear waves

- non-equilibrium : initial conditions/forcing and dissipation are key so equipartition of energy has limited relevance
- statistical : many degrees of freedom are active
- wave interaction : nonlinearity cannot be neglected

Three-wave resonance condition

It is possible to take into account anisotropy within *weak turbulence theory* (weak nonlinearity) using resonant triad waves interactions theory; waves satisfy conditions:
$\boldsymbol{k}^{(1)} + \boldsymbol{k}^{(2)} = \boldsymbol{k}^{(3)}$, $\bar{\omega}^{(1)} + \bar{\omega}^{(2)} = \bar{\omega}^{(3)}$, with dispersion relationship $\bar{\omega} = \pm v_a k_\parallel$
As only oppositely travelling waves interact, the 3 waves must satisfy
$k_\parallel^{(1)} + k_\parallel^{(2)} = k_\parallel^{(3)}$ and $v_a k_\parallel^{(1)} - v_a k_\parallel^{(2)} = \pm v_a k_\parallel^{(3)}$,
the only possibilities are

$$k_\parallel^{(1)} = k_\parallel^{(3)}, \quad k_\parallel^{(2)} = 0, \quad \bar{\omega}^{(2)} = 0$$

$$k_\parallel^{(2)} = k_\parallel^{(3)}, \quad k_\parallel^{(1)} = 0, \quad \bar{\omega}^{(1)} = 0$$

- modes $k_\parallel = 0$, $\bar{\omega} = 0$ are not really waves but rather quasi-2D fluctuations highly elongated along $\boldsymbol{b}_0$
- wave (1), for example, interacts with a quasi-static quasi-2D disturbance and the generated wave (3) has $k_\parallel \sim k_\parallel^{(1)}$, so a negligible change in $\ell_\parallel$ as a result of the collision

Triadic interactions

Writing the inviscid and ideal incompressible MHD equations ($\nu = 0$ and $\eta = 0$) in Elsässer variables $\boldsymbol{z}^s$, with $s = \pm$, leads to :

$$\partial_t \boldsymbol{z}^s - s\boldsymbol{b}_0 \cdot \nabla \boldsymbol{z}^s = -\boldsymbol{z}^{-s} \cdot \nabla \boldsymbol{z}^s - \nabla P_*$$

whith parallel direction taken along the applied magnetic field $\boldsymbol{b}_0 = b_0 \hat{\mathbf{e}}_{\|}$
The Fourier transform of $\boldsymbol{z}^s$ (supposed to be defined) writes :

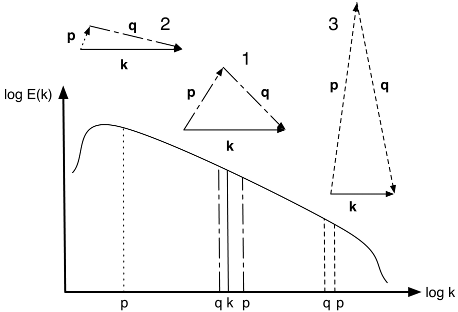
$$z_j^s(\boldsymbol{x}, t) \equiv \iiint \hat{z}_j^s(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{k} = \iiint \left[\varepsilon a_j^s(\boldsymbol{k}, t) e^{is\omega_k t}\right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d\boldsymbol{k}$$

with $\omega_k \equiv k_{\|} b_0$, wavevector $\boldsymbol{k} = (\boldsymbol{k}_\perp, k_{\|})$ with $k_{\|} = b_0 \cdot \hat{\mathbf{e}}_{\|}$, $\boldsymbol{k}_\perp = \boldsymbol{k} - k_{\|} \cdot \hat{\mathbf{e}}_{\|}$ and z^s amplitudes proportional to ε ($\ll 1$), to follow slow time evolution of Alfvén wavepackets because of weak nonlinearities (consider a strong $\boldsymbol{b}_0$ compared to fluctuations)

After Fourier transform, the MHD equations are :

$$\partial_t a_j^s(\boldsymbol{k}, t) = -i\varepsilon k_m P_{jn} \int_{R^6} a_m^{-s}(\boldsymbol{q}, t) a_n^s(\boldsymbol{p}, t) e^{-is(\omega_k - \omega_p + \omega_q)t} \delta(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q}) \, d\boldsymbol{p} \, d\boldsymbol{q}$$

with the Leray projection $P_{jn}(k) = \delta_{jn} - k_j k_n / |\boldsymbol{k}|^2$ to keep the $\hat{\boldsymbol{z}}^s(\boldsymbol{k}, t)$ fields transverse to $\boldsymbol{k}$ (incompressibility). $\longrightarrow$ slow evolution of wavepacket amplitude at $\boldsymbol{k}$ due to quadratic nonlinear interactions between 2 counter-propagating wavepackets with triadic interactions $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$ (infinite number of triangles).



cartoon of local (1), nonlocal (2)and (3) triadic interactions at $\ell \sim 1/k$

Strategy

- Non-dispersive Alfvén waves are special waves: nonlinear interaction coefficient for co-propagating waves is null whereas counter-propagating wavepackets pass throught each other in a finite time & exchange only small amounts of energy over the waveperiod time scale ($t_a \sim (b_0 k_\parallel)^{-1}$)
  $\longrightarrow$ weak turbulence approach is applicable
- Consider the second order correlation function of particular interest since its trace provides the energy spectrum.
- For homogeneous turbulence, a spectral energy tensor reads (with $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$)

$$
\begin{aligned}
< \hat{u}_i{}^*(\boldsymbol{k}_1, t)\hat{u}_j(\boldsymbol{k}_2, t) > &= \int\int < u_i(\boldsymbol{x})u_j(\boldsymbol{x}') > e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} e^{-i\boldsymbol{k}_2\cdot\boldsymbol{x}'} d\boldsymbol{x}\, d\boldsymbol{x}' \\
&= \int\int < u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r}) > e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} e^{-i\boldsymbol{k}_2\cdot(\boldsymbol{x}+\boldsymbol{r})} d\boldsymbol{x}\, d\boldsymbol{r} \\
&= \int R_{ij}(\boldsymbol{r}, t) e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}} d\boldsymbol{r} \int e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}} d\boldsymbol{x} \\
&= \Phi_{ij}(\boldsymbol{k}_2, t)\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2) \equiv n(\boldsymbol{k}_2, t)\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)
\end{aligned}
$$

Equations of motion $\partial_t n(\boldsymbol{k}, t)$ lead to a closure problem

- write moment hierarchy in terms of cumulants and solve pertubatively in ε
- higher order cumulants decay as $t \to \infty$ (asymptotic closure) but resonant triads lead to secular terms

$$n(\boldsymbol{k}, t) = n^{(0)}(\boldsymbol{k}, t) + \varepsilon n^{(1)}(\boldsymbol{k}, t) + \varepsilon^2 \left[t\, n^{(2)}_{sec}(\boldsymbol{k}, t) + n^{(2)}_{nonsec}(\boldsymbol{k}, t) \right] + \cdots$$

- allow slow variation of lower order $n^{(0)}(\boldsymbol{k}, t, \varepsilon^2 t)$
- choose dependence on $\varepsilon^2 t$ to cancel secular terms

Asymptotic closure

Start from the typical equation written as :

$$\partial_t a_j(\boldsymbol{k},t) = \varepsilon \int_{R^6} \mathcal{H}_{jmn}^{kpq} a_m(\boldsymbol{p},t) a_n(\boldsymbol{q},t) e^{i\Omega_{k,pq}t} \delta_{k,pq}\, d\boldsymbol{p}\, d\boldsymbol{q}$$

where $\partial_t f \equiv \partial f/\partial_t$, $\delta_{k,pq} = \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})$ and $\Omega_{k,pq} = \omega_k - \omega_p - \omega_q$. Note symmetries $\mathcal{H}_{jmn}^{kpq} = (\mathcal{H}_{jmn}^{-k-p-q})^*$, $\mathcal{H}_{jmn}^{kpq}$ symmetric in $(\boldsymbol{p},\boldsymbol{q})$ and (m,n), $\mathcal{H}_{jmn}^{0pq} = 0$

Define the spectral energy tensor for homogeneous turbulence $q_{jj'}(\boldsymbol{k}')\delta(\boldsymbol{k}+\boldsymbol{k}') =< a_j(\boldsymbol{k}) a_{j'}(\boldsymbol{k}') >$ whose trace is the energy spectrum

From eq. for $\partial_t a_j(\boldsymbol{k})$, one finds

$$\begin{aligned}\partial_t q_{jj'}(\boldsymbol{k}')\delta(\boldsymbol{k}+\boldsymbol{k}') &= < a_{j'}(\boldsymbol{k}')\partial_t a_j(\boldsymbol{k}) > + < a_j(\boldsymbol{k})\partial_t a_{j'}(\boldsymbol{k}') > \\ &= \varepsilon \int_{R^6} \mathcal{H}_{jmn}^{kpq} < a_m(\boldsymbol{p}) a_n(\boldsymbol{q}) a_{j'}(\boldsymbol{k}') > e^{i\Omega_{k,pq}t} \delta_{k,pq}\, d\boldsymbol{p}\, d\boldsymbol{q} \\ &+ \varepsilon \int_{R^6} \mathcal{H}_{jmn}^{k'pq} < a_m(\boldsymbol{p}) a_n(\boldsymbol{q}) a_j(\boldsymbol{k}) > e^{i\Omega_{k',pq}t} \delta_{k',pq}\, d\boldsymbol{p}\, d\boldsymbol{q}\end{aligned}$$

A hierarchy of equations thus appears, i.e. we need to write

> the equation for the 3rd-order moments (= 3rd-order cumulants) in terms of the 4th-order moments

> the equation for the 4th-order moment in terms of 4th-order cumulant plus products of second ones. For $t \to \infty$, $\varepsilon^2 t$ finite, several terms disappear, in particular the 4th-order cumulant, wich is not a resonant term, and several second-order products

$\rightarrow$ basically, because of quadratic interactions, the nonlinear regeneration of the 3rd-order moments depends essentialy on products of 2nd-order moments (leading to an asymptotic closure)

> time integration of the equation for the 3rd-order moment $\partial_t < a_j(\boldsymbol{k}) a_{j'}(\boldsymbol{k}') a_{j''}(\boldsymbol{k}'') >$, followed by integration over wavevectors $\boldsymbol{p}$ and $\boldsymbol{q}$ and simplifications finally lead to :

$$< a_j(\boldsymbol{k}) a_{j'}(\boldsymbol{k}') a_{j''}(\boldsymbol{k}'') >= 2\varepsilon \Delta(\Omega_{k,k'k''}) \delta_{k,k'k''} [\mathcal{H}_{jmn}^{k-k'-k''} q_{mj'}(\boldsymbol{k}') q_{nj''}(\boldsymbol{k}'')$$

$$+ \mathcal{H}_{j'mn}^{k'-k-k''} q_{mj}(\boldsymbol{k}) q_{nj''}(\boldsymbol{k}'') + \mathcal{H}_{j''mn}^{k''-k-k'} q_{mj}(\boldsymbol{k}) q_{nj\prime}(\boldsymbol{k}')]$$

(with $\Delta(\Omega_{k,pq}) = \int_0^t e^{i\Omega_{k,pq} t'}\, dt' = \frac{e^{i\Omega_{k,pq} t} - 1}{i\Omega_{k,pq}}$)

> inserting this expression into the equation for the energy spectral tensor and taking the long time limit (for which $\Delta(x) \to \pi\delta(x) + i\mathcal{P}(1/x)$) give the asymptotical equations for weak wave turbulence:

$$\partial_t q_{jj'}(\boldsymbol{k}) = 4\pi\varepsilon^2 \int_{R^6} \delta_{k,pq} \delta(\Omega_{k,pq}) \mathcal{H}_{jmn}^{kpq}$$

$$[\mathcal{H}_{mrs}^{p-q-k} q_{rn}(\boldsymbol{q}) q_{j's}(\boldsymbol{k}) + \mathcal{H}_{nrs}^{q-p-k} q_{rm}(\boldsymbol{p}) q_{j's}(\boldsymbol{k}) + \mathcal{H}_{j'rs}^{-k-p-q} q_{rm}(\boldsymbol{p}) q_{sn}(\boldsymbol{q})] d\boldsymbol{p}\, d\boldsymbol{q}$$

Energy spectrum

- note that the δ function in the integral arises because of the three-wave resonance condition : in any interacting wave triad $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ there is always one wave with $q_{||} = 0$ (purely 2D motion) and the two others have $k_{||} = p_{||}$ $\longrightarrow$ decoupling of the dynamic at each level of $k_{||}$ $\longrightarrow$ no energy transfer along the $k_{||}$-direction $\longrightarrow$ two-dimensionalisation of the energy spectrum in each plane at $k_{||} = cst$

Energy spectrum

- note that the δ function in the integral arises because of the three-wave resonance condition : in any interacting wave triad $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ there is always one wave with $q_{\|} = 0$ (purely 2D motion) and the two others have $k_{\|} = p_{\|} \longrightarrow$ decoupling of the dynamic at each level of $k_{\|} \longrightarrow$ no energy transfer along the $k_{\|}$-direction $\longrightarrow$ two-dimensionalisation of the energy spectrum in each plane at $k_{\|} = cst$
- in incompressible MHD (within $k_{\perp} \gg k_{\|}$) the asymptotic equation of weak Alfvén wave turbulence is

$$\partial_t E^s(k_\perp) = \frac{\pi \varepsilon^2}{b_0} \int_\Delta cos^2\phi sin\theta \frac{k_\perp}{q_\perp} E^{-s}(q_\perp)[k_\perp E^s(p_\perp) - p_\perp E^s(k_\perp)] dp_\perp dq_\perp$$

where Δ means $\boldsymbol{k}_\perp$ must be kept equal to $\boldsymbol{p}_\perp + \boldsymbol{q}_\perp$ during integration, with angles $\phi = (\widehat{\boldsymbol{k}_\perp, \boldsymbol{p}_\perp})$ and $\theta = (\widehat{\boldsymbol{k}_\perp, \boldsymbol{q}_\perp})$.
Note that the parallel wavenumber $k_{\|}$ enters only as an external parameter (no energy transfer along $\boldsymbol{b}_0$), thus $E^s(k_\perp, k_{\|}) = f^s(k_{\|}) E^s(k_\perp)$ with $f^s(k_{\|})$ an arbitrary regular function of $k_{\|}$ (whose form depends essentially on I.C.)

- using the Zakharov transform, a conformal transformation, the 2D problem (i.e. the 2D coupled integro-differential eqs assuming isotropy in wavevector planes normal to the applied magnetic file $\boldsymbol{b}_0$) can be solved and exact stationary solutions (constant finite flux solutions) can be obtained in power laws : the so-called Kolmogorov Zakharov (KZ) spectra

$$E^s(k_\perp) \sim {k_\perp}^{n_s}$$

It can be shown that convergence of all integrals is guaranteed for $\boxed{-3 \leq n_s \leq -1}$ and that $\boxed{n_+ + n_- = -4}$ for constant finite (but $\neq 0$) flux solutions.

- in absence of cross helicity $H^C =< \boldsymbol{u} \cdot \boldsymbol{b} >\sim 0$, or case of energy balance $E \sim E^{\pm} \longrightarrow n_+ = n_- = -2$
$\boxed{E(k_\parallel, k_\perp) \sim C_k f(k_\parallel) {k_\perp}^{-2}}$ $(k_\parallel \neq 0,\ k_\perp \gg k_\parallel)$ asymptotic (long time) analytical result with **no** energy transfer along $\boldsymbol{b}_0$ assumed
(Galtier et al. 2000, 2001)

Remarks

clean KZ (Kolmogorov Zakharov) spectra are rare in experimental observations

- insufficient inertial range
- crossover between different scaling regimes
- more than one cascade leading to mixing
- a spectrally broad forcing can contaminate the inertial range
- KZ spectrum can break down at scales smaller then a given scale ; WT is valid for $\ell_\perp \gg \ell_{crit}$ ($t_a \ll t_\ell$). In astrophysical turbulence, scales of interest frequently fall below $\ell_{crit} \longrightarrow$ look for a theory of strong MHD turbulence (strong nonlinear collisions)
- coherence structures with their own scaling do exist

some open questions

- is it possible to find mathematically rigorous a prori conditions on the governing equation or its satistical hierarchy that guarantees that wave turbulence theory will obtain ?
- is broken spatial homogeneity a potential problem for all turbulence theories?

- K41a, $b_0 \sim b_{rms}$
  "strong" turbulence regime, i.e. strong non-linear collisions of $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ propagating waves to pass energy to smaller scales, with the so called *critical balance* assumption $t_\ell \sim t_a$, i.e. equilibrium between inertial forces and Maxwell stresses (Goldreich & Sridhar, 1995)
  * nonlinear interaction time = interaction time of 2 oppositely travelling waves (as only 1 collision is needed): $t_a \sim \ell_{||}/b_0$
  * flux of energy though inertial rang: $\epsilon_\ell \sim z_\ell^2/t_a \sim z_\ell^2/t_\ell \sim z_\ell^3/\ell_\perp$
  * this yields $z_\ell^2 \sim \epsilon^{2/3}\ell_\perp^{2/3} \rightarrow z_\ell^2 \sim k_\perp E(k_\perp) \sim \epsilon^{2/3}{k_\perp}^{-2/3}$ and thus

  $$\boxed{E(k_\perp) \sim \epsilon^{2/3}{k_\perp}^{-5/3}}$$

  Remarks:
  - $\ell_{||} \sim b_0\ell_\perp/z_\ell \sim (b_0/\epsilon^{1/3})\ell_\perp^{2/3}$
  - $z_\ell^2 \sim \epsilon^{2/3}\ell_\perp^{2/3} \sim \epsilon\ell_{||}/b_0 \longrightarrow E(k_{||}) \sim (\epsilon/b_0)k_{||}^{-2}$

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  - $z_\ell^2 \sim \epsilon^{2/3}\ell_\perp^{2/3} \sim \epsilon\ell_{||}/b_0 \longrightarrow E(k_{||}) \sim (\epsilon/b_0)k_{||}^{-2}$
- IKa, $t_a \ll t_\ell$, $b_0 \gg b_{rms}$, assuming $E(k_\perp, k_{||}) \sim k_\perp^{-\alpha}k_{||}^{-\beta}$, it can be show that $3\alpha + 2\beta = 7$, thus $\alpha = 5/3$, $\beta = 1$ for K41a & $\alpha = 2$, $\beta = 1/2$ for IKa, and, if $t_a(\ell_{||})/t_\ell(\ell_\perp) \sim cst$, $\ell_{||} \sim (b_0/\epsilon_{IKa}^{1/3})\ell_\perp^{2/3}$ (Galtier et al., 2005)

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