Semi-classical propagation of singularity for Stokes system

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- Introduction
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high-frequency eigenvalue problem for Stokes operator

$\Omega \subset \mathbb{R}^d$ be a smooth bounded, open domain and the eigenvalue problem can be written as

$$\begin{cases} -\Delta u_k + \nabla P_k = \lambda_k^2 u_k, \mathrm{in}\Omega \\ h \ \mathrm{div} \ u_k = 0, \mathrm{in} \ \Omega \\ u_k|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

$u_k \in H^2(\Omega) \cap V, \|u_k\|_{L^2} = 1$, the $\mathbb{R}^d$−valued normalized eigenfunction where

$$V = \{u \in (H_0^1(\Omega))^d : \mathrm{div} \ u = 0\}$$

high-frequency eigenvalue problem for Stokes operator

Several facts:

- u_k forms a orthonormal basis of

$$H = \{u \in (L^2(\Omega))^d : \ \mathrm{div}\ u = 0, u \cdot \nu|_{\partial\Omega} = 0\}$$

  $\Pi : (L^2(\Omega))^d \to H$ canonical orthogonal projector

- The pressure $P_k \in L^2(\Omega)/_{\mathbb{R}}$ in the sense that $\int_\Omega P_k = 0$.
- $\|\nabla u_k\|_{L^2}^2 = \lambda_k^2, \|u_k\|_{H^2} \leq C\lambda_k^2, \|\nabla P_k\|_{L^2} \leq C\lambda_k^2, \|P_k\|_{L^2} \leq C\lambda_k^2$.
- $u_k \rightharpoonup 0$ weakly in $L^2(\Omega)$ since for any $\varphi \in (L^2(\Omega))^d$

$$\lim_{k\to\infty} \int_\Omega u_k(x) \cdot \varphi(x) dx = \lim_{k\to\infty} \int_\Omega u_k(x) \cdot \Pi\varphi(x) dx = 0.$$

high-frequency eigenvalue problem for Stokes operator

Natural question: What can we say about the high frequency limit?
Semi-Classical reduction: Take $h_k = \lambda_k^{-1}$ and $q_k = \lambda_k^{-1} P_k$, drop the sub-index, we have the following $h-$dependence system (quasi-mode)

$$\begin{cases} -h^2 \Delta u - u + h\nabla q = f, \text{in } \Omega \\ h \operatorname{div} u = 0, \text{in } \Omega \\ u \in H^2(\Omega) \cap V \end{cases} \tag{2}$$

with the following conditions:

$$\|u\|_{L^2} = 1, \|h\nabla u\|_{L^2} = O(1), \|h^2\nabla^2 u\|_{L^2} = O(1),$$

$$\|h\nabla q\|_{L^2} = O(1), f \in H, \|f\|_{L^2} = o(h).$$

Outline

Interior

One way to capture the high-frequency limit behaviour is semi-classical defect measure. We will use it to describe the singularity.

In the interior, we use the usual $h-$pseudo-differential calculus for symbols $a \in C_c^\infty(T^*\Omega)$.

Proposition

There exists a Hermitian matrix-valued positive definite Radon measure μ_i on $T^*\Omega$ and a subsequence (h_k, u_k) such that

$$\lim_{k\to\infty} (a(x, h_k D_x)u_k | u_k)_{L^2(\Omega)} = \int_{T^*\Omega} tr\ (a d\mu_i)$$

for any $a \in C_c^\infty(T^*\Omega)$.

near the boundary: coordinate system

Near a point $x_0 \in \partial\Omega$, we consider a geodesic coordinate system

$$\Omega = \{(y, x') : 0 \leq y \leq \epsilon_0, x' \in \mathbb{R}^{d-1}\}$$

with the metric on $T\overline{\Omega}$ and $T^*\overline{\Omega}$ respectively

$$g = \begin{pmatrix} 1 & 0 \\ 0 & M(y,x') \end{pmatrix}, g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha(y,x') \end{pmatrix},$$

In this coordinate, we can write the operator $h^2\Delta + 1$ acting on $u = (u_\|, u_\perp)$ as

$$h^2\Delta_\| + 1 = h^2\partial_y^2 - \Lambda^2(y, x', hD_{x'}) + 1 + hM_\|(y, x', hD_x') + hM_1(y, x')h\partial_y,$$

$$h^2\Delta_g + 1 = h^2\partial_y^2 - \Lambda^2(y, x', hD_{x'}) + 1 + hM_\perp(y, x', hD_x') + hN_1(y, x')h\partial_y,$$

near the boundary: geometric objects

The principal symbol $p(x,\xi) = |\xi|^2 - 1$ is scalar and can be written as

$$p(y, x', \eta, \xi') = \eta^2 + |\xi'|^2_{g^{-1}(y)} - 1$$

near the boundary. We denote $r(y, x', \xi') = 1 - |\xi'|^2_{g^{-1}(y)}$.
Several geometric objects:

- ${}^bT\overline{\Omega}$: rank d vector bundle whose sections are the vector fields tangent to $\partial\Omega$ and ${}^bT^*\overline{\Omega}$ be its dual bundle.
- Canonical map: $j : T^*\overline{\Omega} \rightarrow^b T^*\overline{\Omega}$can be written in our coordinate system as $j(y, x', \eta, \xi') = (y, x', y\eta, \xi')$.
- characteristic manifold and its projection: $Car(P) = \{(x,\xi) \in T^*\mathbb{R}^d|_{\overline{\Omega}}; p(x,\xi) = 0\}, Z = j(Car(P))$.

near the boundary: geometric objects

Usual decomposition of $T^*\partial\Omega$ corresponding to the values of $r_0 = r|_{y=0}$: $T^*\partial\Omega = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G}$

- $\mathcal{E} = \{r_0 < 0\}$
- $\mathcal{H} = \{r_0 > 0\}$
- $\mathcal{G} = \{r_0 = 0\}$

Denote $r_1 = \partial_y r|_{y=0}$. We can further distinguish sets of certain order of contact in $\mathcal{G}$:$\mathcal{G}^2 := \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}$,

$$\mathcal{G}^{2,\pm} := \{(x', \xi') : r_0(x', \xi') = 0, \pm r_1(x', \xi') > 0\}$$

$$\mathcal{G}^{k+3} = \{(x', \xi') : r_0(x', \xi') = 0, H^j_{r_0}(r_1) = 0, \forall j \leq k; H^{k+1}_{r_0}(r_1) \neq 0\}$$

near the boundary: geometric objects

We say that $\partial\Omega$ has no infinite order of contact if

$$\mathcal{G} = \bigcup_{k\geq 2} \mathcal{G}^k.$$

Under this assumption, for any given $\rho_0 \in^b T^*\overline{\Omega}$, the unique generalized characteristic ray passing ρ_0 exists and this defines a flow $\gamma(s,\cdot)$ on ${}^bT^*\overline{\Omega}$ (Melrose-Sjöstrand flow)
Briefly speaking, generalized characteristic rays are

- bicharacteristics of $p = |\xi|^2 - 1$ in the interior $T^*\Omega$ or while touching certain point in $\mathcal{G}^{2,+}$.
- broken bicharacteristics while touching certain point in $\mathcal{H}$.
- gliding rays of the form $exp(sH_{-r_0})(\rho)$ in $\mathcal{G} \setminus \mathcal{G}^{2,+}$.

tangential symbol

Consider functions of the form $a = a_i + a_\partial$ with $a_i \in C_c^\infty(\Omega \times \mathbb{R}^d)$ which can be viewed as a symbol in S^0, and $a_\partial \in C_c^\infty(U \times \mathbb{R}^{d-1})$ can be viewed as a symbol in $S^0_{\xi'}$. We quantize a via the formula (in local coordinate)

$$\begin{aligned}\mathrm{Op}_h(a)f(y,x') =& \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^{2d}} e^{\frac{i(x-z)\xi}{h}} a_i(x,\xi)f(z)dzd\xi \\ &+ \frac{1}{(2\pi h)^{d-1}}\int_{\mathbb{R}^{2(d-1)}} e^{\frac{i(x'-z')\xi'}{h}} a_\partial(y,x',\xi')f(y,z')dz'd\xi'.\end{aligned}$$

The class of operators $\mathrm{Op}_h(a)$ is uniquely defined up to some operators with norms of order $O(h)$. For such a function a, we associated $\kappa(a) \in C^0(Z)$ by setting $\kappa(a) = a(j^{-1}(\rho))$.

The definition of the measure when we are working near the boundary is guaranteed thanks to

Proposition

There exists a Hermitian matrix-valued positive definite Radon measure and a subsequence (u_k, h_k) such that

$$\lim_{k\to\infty} (Op_{h_k}(a)u_k|u_k)_{L^2} = \langle \mu, \kappa(a)\rangle.$$

Attention: To make the measure be well-defined, we need verify that if $a_\partial \in C_c^\infty(U \times \mathbb{R}^{d-1})$ vanishing near Z (i.e. a_∂ is supported in the elliptic region for all y small)implies that

$$(\mathrm{Op}_{h_k}(a_\partial)u_k|u_k)_{L^2} \to 0.$$

This can be ensued by the analysis of boundary value problem in the elliptic region.

main result

The main result is as follows:

Theorem (C.Sun 2017)

Assume that Ω is a smooth, bounded domain with no infinite order of contact on the boundary. Suppose μ is the semi-classical defect measure associated to the pair (u_k, h_k) where (u_k) is a sequence of solutions to the quasi-mode problem (2) (with $h = h_k, f = f_k$ there) which are weakly convergence to 0 in $L^2(\Omega)$. Suppose that $\|f_k\|_{L^2(\Omega)} = o(h_k)$. Then the support of μ is invariant under the Melrose-Sjöstrand flow.

Outline

Application

Consider the following hyperbolic counterpart of Stokes equation:

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = 0, (t,x) \in [0,T] \times \Omega \\ u(t,.)|_{\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = (u_0, v_0) \in V \times H \end{cases} \tag{3}$$

We are interested in the stabilization of this system when adding a damping term $a(x)\partial_t u$:

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p + a(x)\partial_t u = 0, (t,x) \in [0,T] \times \Omega \\ u(t,.)|_{\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = (u_0, v_0) \in V \times H \end{cases} \tag{4}$$

Application

Define the energy

$$E[u](t) = \frac{1}{2}\int_{\Omega}(|\partial_t u|^2 + |\nabla u|^2)dx.$$

It satisfies $\frac{dE}{dt} = -\int_{\Omega} a(x)|\partial_t u|^2 dx \leq 0$.

Theorem (C.Sun,F.W.Chaves-Silva, 2017)

Suppose $\Omega \subset \mathbb{R}^d$ be a bounded open set with no infinite order of contact. Suppose $a \in C(\overline{\Omega})$ is a non-negative function and its support satisfies geometric control condition. Then there exist some uniform constants C_0, α, such that for any $u_0 \in V$, the corresponding solution $u(t)$ to (4) decays exponentially in time in the following sense:

$$E[u](t) \leq C_0 E[u](0)e^{-\alpha t}, \forall t \geq 0. \quad (5)$$

Outline

Sketch of the Proof

Part III: Sketch of the Proof

Let μ be a semi-classical defect measure associated to certain subsequence of $u(h)$. We will analyse the propagation property along a generalized ray in several geometric situations:

- In the interior
- Near a point in $\mathcal{E}$
- Near a point in $\mathcal{H}$
- Near a point in $\mathcal{G}$

a priori information

Denote $q_0 = q|_{y=0}$ and we have

Lemma

$\|q_0\|_{H^{1/2}(\partial\Omega)} = O(h^{-1})$.

Proof.

First we note that $h^2\Delta q = 0$ in Ω and $\|h\nabla q\|_{L^2(\Omega)} = O(1), \|hq\|_{L^2(\Omega)} = O(1)$. From trace theorem and $\|\nabla q\|_{L^2(\Omega)} = O(h^{-1})$ we have $\|q_0\|_{H^{1/2}(\partial\Omega)} = O(h^{-1})$. □

Lemma

$h\partial_{\mathbf{n}} u|_{\partial\Omega} = (h\partial_{\mathbf{n}} u_{\|}, 0)$, and $\|h\partial_{\mathbf{n}} u|_{\partial\Omega}\|_{L^2(\partial\Omega)} = O(1)$.

a priori information

Sketch of the proof:

The first assertion follows from h div $u = 0$ and Dirichlet boundary condition. For the second assertion, we apply a multiplier method. From the geometric assumption, we can find a vector field $L \in C^1(\overline{\Omega}))$ such that $L|_{\partial\Omega} = \nu$. In the global coordinate system, we write

$$L = \sum_{j=1}^{d} L_j(x) \frac{\partial}{\partial x_j}.$$

Multiplying to the equation, we have

$$\int_\Omega Lu \cdot f dx = \int_\Omega Lu \cdot (-h^2 \Delta u - u + h \nabla q) dx.$$

□

a priori information

Sketch of the proof.

Integrating by part and using the equation, the output of the right hand side is

$$\frac{1}{2}\int_{\partial\Omega}|h\partial_{\mathbf{n}}u|^2 d\sigma + O(1)$$

while the left hand side is $o(1)$. □

Lemma

After extracting to subsequences, we have $h\nabla q \rightharpoonup 0$ weakly in $L^2(\Omega)$, thus from Rellich theorem, we have $hq \to 0$, strongly in $H^{1/2}(\Omega)$ and $hq_0 \to 0$, strongly in $L^2(\partial\Omega)$.

Proof.

We may assume that $h\nabla q \rightharpoonup r$, weakly in $L^2(\Omega)$, and Rellich theorem implies that $hq \to P$, strongly in $L^2(\Omega)$, and thus $\nabla P = r$, with the property $\int_\Omega P = 0$. Now we claim that $\Delta P = 0$. in Ω. Indeed, take any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla P \cdot \nabla \varphi = \lim_{h \to 0} \int_\Omega h\nabla q \cdot \nabla \varphi = 0.$$

Now since the sequence $(h^2\nabla^2 u)$ is bounded in L^2, then up to a subsequence, $h^2\nabla^2 u \rightharpoonup W$, weakly in L^2. From Rellich theorem, the sequence $(h^2 u)$ converges strongly in L^2. The strong limit must be 0 since $u \rightharpoonup 0$ weakly in L^2. Thus $W = 0$ and this implies that $\nabla P = 0$. Finally, we must have $P = 0$. The last assertion follows from trace theorem and Rellich theorem. □

Information about q

Let us introduce some notations: for $\chi \in C^\infty(\mathbb{R}_+)$ with $\chi|_{s\geq 1} \equiv 1, \chi|_{(0,1/2)} \equiv 0$,

- $\lambda(y, x', \xi') = |\xi'|_{g^{-1}(y)}$
- $e^{-\frac{y\Lambda}{h}} = \mathrm{Op}_h(e^{-\frac{y\lambda}{h}}\chi\left(\frac{\lambda}{\delta}\right))$

The relevant information of the scale $\frac{1}{h}$ for the pressure q can be given by the following

- $\mathrm{Op}_h\left(\chi\left(\frac{\lambda}{\delta}\right)\right) q = e^{-\frac{y\Lambda}{h}} q_0 + O_{L^2_{y,x'}}(h)$
- $\mathrm{Op}_h\left(\chi\left(\frac{\lambda}{\delta}\right)\right) h\partial_y q = \mathrm{Op}_h\left(\chi\left(\frac{\lambda}{\delta}\right)\lambda\right) e^{-\frac{y\Lambda}{h}} q_0 + O_{L^2_{y,x'}}(h)$

Information about q

From the parametrix above, we can deduce a quantified non-concentration estimate as following

Lemma

For $0 < y_0 \ll \epsilon_0$, we have

$$\int_{y_0}^{\epsilon_0} \| Op_h \left(\chi \left(\frac{\lambda}{\delta} \right) \right) q \|_{L^2_{x'}}^2 dy \leq C_\delta \left(e^{-\frac{c y_0}{h}} + h^2 \right),$$

where the constant C_{δ_0} only depends on δ_0 and independent for small y_0, h.

This Lemma will play an important role in our analysis.

Outline

Interior propagation

In the interior, we have the precise invariance of defect measure. This is a consequence that the pressure can only have impact on the measure of (u_k) on the boundary. First we asserts that the singularity can only concentrate on the characteristic set.

Proposition

Take $a_i \in C_c^\infty(\Omega \times \mathbb{R}^d)$, *equal to* 0 *near* $Car(P)$, *then we have*

$$(Op_{h_k}(a_i)u_k|u_k)_{L^2} \to 0.$$

Interior propagation

Proof.

$b(x,\xi) = \frac{a_i(x,\xi)}{|\xi|^2-1} \in S^0$ is well-defined. Then from symbolic calculus, we have

$$\mathrm{Op}_{h_k}(a_i) = B_{h_k}(-h_k^2\Delta - 1) + O_{L^2\to L^2}(h_k).$$

We now calculate

$$\begin{aligned}(B_{h_k}(-h_k^2\Delta - 1)u_k|u_k)_{L^2} =& (B_{h_k} f_k|u_k)_{L^2} - (B_{h_k} h_k\nabla q_k|u_k)_{L^2}\\ =& o(1) + ([h_k\nabla, B_{h_k}]q_k|u_k)_{L^2}\\ &-(h_k\nabla B_{h_k} q_k|u_k)_{L^2}\\ =& o(1),\end{aligned}$$

where in the last line we have used the symbolic calculus,

Interior propagation

Proposition

For any $a \in C_c^\infty(\Omega \times \mathbb{R}^d)$,*we have*

$$\frac{d}{dt}\langle \mu, a \circ \phi(t) \rangle = 0,$$

where $\phi(t)$ *be the Hamiltonian flow corresponding to the symbol* $p = |\xi|^2 - 1$.

Interior propagation

Proof.

We may assume that the symbol $a(x,\xi)$ vanish near $\xi = 0$. Denote $A = \mathrm{Op}_h(a)$, $P = -h^2\Delta - 1$ and we apply equation to calculate

$$\begin{aligned}
\frac{i}{h}([P,A]u|u)_{L^2} &= \frac{i}{h}(Au|Pu)_{L^2} - \frac{i}{h}(APu|u)_{L^2} \\
&= \frac{i}{h}(Au|f - h\nabla q)_{L^2} - \frac{i}{h}(A(f - h\nabla q)|u)_{L^2} \\
&= -\frac{i}{h}(Au|h\nabla q)_{L^2} + \frac{i}{h}(Ah\nabla q|u)_{L^2} + o(1) \\
&= -\frac{i}{h}([A, h\mathrm{div}\ ]u|q)_{L^2} + \frac{i}{h}([A, h\nabla]q|u)_{L^2} + o(1) \\
&= i(\mathrm{Op}_h(\nabla a)\cdot u|q)_{L^2} - i(\mathrm{Op}_h(\nabla a)q|u)_{L^2} + o(1) \\
&= i(u|\mathrm{Op}_h(\nabla \bar{a})q)_{L^2} - i(\mathrm{Op}_h(\nabla a)q|u)_{L^2} + o(1).
\end{aligned}$$

Interior propagation

From $h\nabla q = O_{L^2}(1)$, we know that micro-locally away from $\xi = 0$, $q = O_{L^2}(1)$. On the other hand, $h^2\Delta(\mathrm{Op}_h(\nabla a)q) = O_{L^2}(h)$ and this implies that $\mathrm{Op}_h(\nabla a)q = o_{L^2}(1)$ since the symbol of $h^2\Delta\mathrm{Op}_h(\nabla a)$ vanishes away near $\xi = 0$ as well as x near the boundary. This completes the proof.

Outline

Elliptic Region

Near a point $\rho_0 \in \mathcal{E} = \{r_0(x', \xi') = 1 - \lambda(0, x', \xi') < 0\}$, we have

Proposition

$\mu \mathbf{1}_{\mathcal{E}} = 0$. *If we denote* ν *be the semi-classical defect measure of the sequence* $(h_k \partial_{\mathbf{n}} u_k|_{\partial\Omega}, h_k)$, *then* $\nu \mathbf{1}_{\mathcal{E}} = 0$.

Instead of giving rigorous proof, we only consider a particular case that the equation is homogeneous (i.e.$f = 0$) and the metric is flat: $\Omega = \{(y, x') : y > 0\}$. Write $u = (u_{\|}, u_{\perp}), v = h\partial_y u_{\|}|_{y=0}$, Denote the semi-classical Fourier transform

$$\mathcal{F}_h(f)(y, \xi') = \frac{1}{(2\pi h)^{d-1}} \int e^{-\frac{ix'\xi'}{h}} f(y, x') dx'$$

$w = \mathcal{F}_h(u)\mathbf{1}_{y \geq 0}, \kappa = \mathcal{F}_h(v)$ and $\theta = \mathcal{F}_h(q_0)$.

Elliptic Region

By taking semi-classical Fourier transform in x' of (2),

$$\begin{cases} -h^2\partial_y^2 w_{\|} + 2h\kappa \otimes \delta_{y=0} + (|\xi'|^2-1)w_{\|} + i\xi' e^{-\frac{y|\xi'|}{h}}\theta \mathbf{1}_{y\geq 0} = 0, \\ -h^2\partial_y^2 w_{\perp} + (|\xi'|^2-1)w_{\perp} - |\xi'|e^{-\frac{y|\xi'|}{h}}\theta \mathbf{1}_{y\geq 0} = 0. \end{cases} \tag{7}$$

where $\mu = \sqrt{|\xi'|^2 - 1}$. It can be solves explicitly

$$w_{\|} = \frac{1}{|\xi'|^2-1}\kappa(\xi')e^{-\frac{\mu y}{h}} + \frac{i\xi'\theta(\xi')}{2}\left(\frac{e^{-\frac{\mu y}{h}}}{(\mu(\mu-|\xi'|)} + 2e^{-\frac{y|\xi'|}{h}}\right),$$

$$w_{\perp} = \frac{|\xi'|\theta(\xi')}{2}\left(\frac{e^{-\frac{\mu y}{h}}}{\mu(\mu-|\xi'|)} + 2e^{-\frac{y|\xi'|}{h}}\right)$$

Elliptic region

Observe that

$$
\begin{aligned}
&-\left(\frac{e^{-\frac{\mu y}{h}}}{\mu(\mu-|\xi'|)}+2e^{-\frac{y|\xi'|}{h}}\right)\\
=&\left(e^{-\frac{y|\xi'|}{h}}\left(\frac{1}{\mu(|\xi'|-\mu)}-2\right)+\frac{e^{-\frac{\mu y}{h}}-e^{-\frac{|\xi'|y}{h}}}{\mu(|\xi'|-\mu)}\right)\\
>&0,
\end{aligned}
$$

This means the system $(v, q_0) \mapsto (u_{\|}\mathbf{1}_{y\geq 0}, u_{\perp}\mathbf{1}_{y\geq 0})$ is elliptic.

Outline

Hyperbolic Region: Heuristic

We again consider the flat metric on half space. Denote

$$\mathcal{H}_I := \{0 < |\xi'|^2 < 1\},$$

and $\omega^2 = 1 - |\xi'|^2$. We want to solve

$$\begin{cases} -h^2\partial_y^2 w_{||} - \omega^2 w_{||} + i\xi'\theta e^{-\frac{y|\xi'|}{h}} = 0, y > 0, \\ -h^2\partial_y^2 w_{\perp} - \omega^2 w_{\perp} - |\xi'|\theta e^{-\frac{y|\xi'|}{h}} = 0, y > 0, \\ h\partial_y w_{\perp} + i\xi' \cdot w_{||} = 0, \end{cases} \tag{8}$$

where $w = (w_{||}, w_{\perp}) = \mathcal{F}_h(u), \theta = \mathcal{F}_h(q_0)$ It is easy to verify that the solution can be written as

$$\begin{aligned} w_{||} &= i\xi'\theta e^{-\frac{y|\xi'|}{h}} + \alpha_+ e^{\frac{i\omega y}{h}} + \alpha_- e^{-\frac{i\omega y}{h}}, \\ w_{\perp} &= -|\xi'|\theta e^{-\frac{y|\xi'|}{h}} + \beta_+ e^{\frac{i\omega y}{h}} + \beta_- e^{-\frac{i\omega y}{h}}. \end{aligned} \tag{9}$$

Hyperbolic Region: Heuristic

Boundary condition gives:

$$
\begin{aligned}
&|\xi'|(\alpha_+ + \alpha_-) = -i\xi'|\xi'|\theta, \\
&- i\xi'|\xi'|\theta + i\omega(\alpha_+ - \alpha_-) = v, \\
&\beta_+ + \beta_- = |\xi'|\theta, \\
&i\omega(\beta_+ - \beta_-) = -|\xi'|^2\theta.
\end{aligned} \tag{10}
$$

where $v = \mathcal{F}_h(h\partial_y u|_{y=0})$.

Hyperbolic Region: Heuristic

We can determine the amplitudes

$$\begin{aligned}
\alpha_+ &= \frac{1}{2}\left(\frac{|\xi'|\xi'}{\omega}\theta - i\left(\xi'\theta + \frac{v}{\omega}\right)\right), \\
\alpha_- &= -\frac{1}{2}\left(\frac{|\xi'|\xi'}{\omega}\theta + i\left(\xi'\theta - \frac{v}{\omega}\right)\right), \\
\beta_+ &= \frac{1}{2}\left(|\xi'| - \frac{|\xi'|^2}{i\omega}\right)\theta, \\
\beta_- &= \frac{1}{2}\left(|\xi'| + \frac{|\xi'|^2}{i\omega}\right)\theta.
\end{aligned} \tag{11}$$

Hyperbolic Region: Heuristic

We write $A_+ = (\alpha_+, \beta_+), A_- = (\alpha_-, \beta_-)$, and

$$w_+ := A_+ e^{\frac{i\omega y}{h}}, w_- := A_- e^{-\frac{i\omega y}{h}},$$

can be interpreted as incoming wave and out-coming wave, thus

$$w = (i\xi', -|\xi'|)\widehat{\theta} e^{-\frac{y|\xi'|}{h}} + w_+ + w_-.$$

Hyperbolic Region: Heuristic

Notice also that if the singularity along the incoming (out-coming) wave is zero, i.e.$A_+ = 0 (A_- = 0)$, then we have $\theta(\xi') = 0$,, $v(\xi') = 0$, and $A_- = 0 (A_+ = 0)$.
These observations will motive the rigorous proof of the singularity propagation in the hyperbolic region.

Outline

Strategy of the proof

Near a point in $\bigcup_{k\geq 2}\mathcal{G}^k$, we will follow the argument of V.Ivrii(in the diffractive case) and Melrose-Sjöstrand(in other situations). We assume that some point ρ_0 in a generalized ray (or gliding ray) does not in the supp(μ) and we want to deduce that along such a ray, after some short time, it will not intersect with supp(μ). The strategy is to construct some test-operators A and calculate the quantity $\frac{1}{h}\Im([P,A]+(R-R^*)Au|u)_\Omega$ in two ways. One way is symbolic calculus, the other way is the equation satisfied by u. The output of this procedure is an inequality where the left hand side is a quantity of the form $\|Bu\|_{L^2}^2$ and the right hand side is of the form $\left|\frac{1}{h}(Au|h\nabla q)_{L^2}\right|+\left|\frac{1}{h}(Ah\nabla q|u)_{L^2}\right|+o(1)$.

Strategy of the proof

Compared with the quasi-mode problem of Laplace operator, there are two additional new terms involving pressure for the same problem of Stokes operator.

To overcome this difficulty, we pick a bump function $\varphi \in C_c^\infty((-2,2))$ which equals 1 on $(-1,1)$. We take $\chi_\epsilon(y,x,hD_x)$ with principal symbol $\chi_\epsilon(y,x,\xi) = \chi(y,x,\xi)\varphi\left(\frac{r(y,x,\xi)}{\epsilon}\right)$. We calculate

$$\begin{aligned}\frac{1}{h}(Ah\nabla q|u)_{L^2} =& \frac{1}{h}([A,h\nabla]q|u)_{L^2} \\ =& \frac{1}{h}([A,h\nabla]q|\mathrm{Op}_h(\chi_\epsilon)u)_{L^2} \\ &+ \frac{1}{h}([A,h\nabla]q|\mathrm{Op}_h(\chi-\chi_\epsilon)u)_{L^2} \\ =:& I_{h,\epsilon} + II_{h,\epsilon}\end{aligned} \tag{12}$$

Diffractive case

Geometrically, $I_{h,\epsilon}$ is related to the micro-localization close enough to $r = 0$. We have

$$\limsup_{h \to 0} \| hD_y Op_h(\chi_\epsilon) u \|_{L^2_{x,y}} \leq C\epsilon^{1/2},$$

$$\int_0^{y_0} \| Op_h(\chi_\epsilon) u \|^2_{L^2_x} dy \leq \frac{Cy_0^2}{h^2} \| hD_y Op_h(\chi_\epsilon) u \|^2_{L^2_{x,y}}.$$

Essentially, the first inequality comes from the fact that on the charactersitic $\eta^2 = r \leq \epsilon$. The second one is a simple consequence of the first one.

Diffractive case

Take $\theta \in (0, 1/2)$, and choose $y_0 = h\epsilon^{-\theta}$, we estimate

$$|I_{h,\epsilon}| \leq \int_0^{y_0} + \int_{y_0}^{\epsilon_0} \frac{1}{h} |([A, h\nabla]q | Op_h(\chi_\epsilon)u)_{L^2_x}| dy$$
$$\leq C \frac{1}{\epsilon^{2\theta}} (\|hD_y \mathrm{Op}_h(\chi_\epsilon)u\|^2_{L^2_{x,y}} + O(h)) + Ce^{-\frac{c}{\epsilon^\theta}},$$

here we have used the fact that $q = O_{L^2_{x,y}}(1)$, micro-locally far from $\xi = 0$ since the L^2 norm of q and $h\nabla q$ becomes comparable there. The term $e^{-\frac{c}{\epsilon^\theta}}$ comes from the concentration property of pressure. In summary we have

$$\limsup_{h \to 0} |I_{h,\epsilon}| \leq C(\epsilon^{1-2\theta} + e^{-\frac{c}{\epsilon^\theta}}).$$

Strategy of the proof

We need geometric argument to deal with $II_{h,\epsilon}$. Since on the support of $\chi - \chi_\epsilon$, we have $r \geq \epsilon$. We need to prove that each ray entering the set $S_\epsilon := \{r \geq \epsilon\} \cap \{y \leq \epsilon\} \cap \mathrm{supp}\ (\chi)$ must be well-controlled in the sense that

- it can be connected by a generalized ray issued in a neighborhood of ρ_0.
- it can only touch some point $\rho \in \mathcal{H} \cup \mathcal{G}^k$ and we have already established the propagation property for points in $\mathcal{G}^k$.

Once this geometric property is justified, we have

$$\limsup_{h \to 0} |II_{h,\epsilon}| = 0$$

for any fixed $\epsilon > 0$.

Diffractive case

In this section we will establish

Proposition

Suppose $\rho \in \mathcal{G}^{2,+}$, ρ_0 *be a phase point with* $y(\rho_0) > 0, \frac{\partial r}{\partial y}(\rho_0) \geq \frac{1}{2}\frac{\partial r}{\partial y}(\rho)$. *Denote* $\gamma_- = [\rho_0, \rho]$ *be a backward ray issued from* ρ_0 *to* ρ *(the trajectory under the canonical projection is tangent to the boundary at* ρ*). Suppose* $\rho_0 \notin supp\mu$. *Then we have* $\rho \notin supp\mu$.

Diffractive case

For $a^j(y,x,\xi), j=0,1$ tangential symbols, we denote A_h^j be with principal symbol a^j and $A_h = A_h^0 + A_h^1 hD_y$, with the symbol $a = a^0 + a^1\eta$. Assume further that
$A_h^* = A_h, A_h^{1*} = A_h^1, A_h^{0*} = A_h^0 - [hD_y, A_h^1]$.
We then have the integrating by part formula

$$\frac{2}{h}\Im(P_h u|A_h u)_\Omega = (A_h^1 hD_y u|hD_y u)_{\partial\Omega} + \Re\sum_{j=0}^{2}(C_j(hD_y)^j u|u)_\Omega + O(h),$$

where C_j has principal symbol $c_j(y,x,\xi)$ and $\sum_{j=0}^{2} c_j(y,x,\xi)\eta^j = \{p,a\} + 2a\Im p_s$, where p_s is the sub-principal symbol of p, namely $2\Im p_s$ is the principal symbol of the operator $\frac{1}{ih}(R^* - R)$.

Diffractive case

Take a small neighborhood Γ_0 of ρ_0 such that $\Gamma_0 \cap \operatorname{supp}\mu = \varnothing$. Take a small neighborhood $W_0 \subset \overline{\Omega} \times \mathbb{R}^{d-1}$ such that $\frac{\partial r}{\partial y}(y,x,\xi) \geq \delta_0 > 0$. By shrinking W_0 if necessary, we assume that each point $(y,x,\xi) \in W_0$ with $r(y,x,\xi) \geq 0$ can be connected by a (possibly broken) ray issued from Γ_0 with at most one reflection or tangency at $\partial\Omega$. What we want to prove is the following statement:

For any $\chi \in C_c^\infty(\overline{\Omega} \times \mathbb{R}^{d-1})$ with $\operatorname{supp}\chi \subset W_0$, small enough, we have

$$\chi(y,x,hD_x)u = o_{L^2}(1), h \to 0.$$

Diffractive case

We make a special choice of a_0, a_1 with the following properties:

$$a(y,x,\eta,\xi) = a_0(y,x,\xi) + a_1(y,x,\xi)\eta, a_j \in C_c^\infty(W_0)$$

with the following property:

- $a_1(0,x,\xi) = -t(x,\xi)^2$, for some $t \in C_c^\infty(T^*\partial\Omega)$,
- For some $M \geq 0$, when $p = \eta^2 - r(y,x,\xi) = 0$, we have $\{p,a\} + aM|\xi| = -\psi(y,x,\eta,\xi)^2 + \varphi(y,x,\xi)(\eta - r^{1/2}(y,x,\xi))$, $a = s^2$, where $s \in C^\infty(\overline{\Omega} \times (\mathbb{R}^d \setminus \{0\}))$,$\psi \in C_c^\infty(\overline{\Omega} \times \mathbb{R}^d \setminus \{0\})$ and $\varphi \in C_c^\infty(W_0)$. Moreover, $r|_{\mathrm{supp}\ \varphi} > 0$.

Diffractive case

Now we take $\chi \in C_c^\infty(W_0)$ with $\chi \equiv 1$, in a neighborhood of supp$a_1 \cup$ supp a_2. Let $v = \chi(y, x, hD_x)u$ and use the equation:

$$\begin{aligned}(P_h v|A_h v)_\Omega =& (\mathrm{Op}_h(\chi)f|A_h v)_\Omega - (\mathrm{Op}_h(\chi)h\nabla q|A_h v)_\Omega \\ &+([P_h, \mathrm{Op}_h(\chi)]u|A_h \mathrm{Op}_h(\chi)u)_\Omega.\end{aligned}$$

Notice that $\{p, \chi\} = 0$ on suppa_j and $f = o_{L^2}(h), h\mathrm{div} u = 0$, thus

$$\begin{aligned}\frac{2}{h}\Im(P_h v|A_h v)_\Omega =& o(1) + \frac{2}{h}\Im([\mathrm{Op}_h(\chi), h\nabla]q|A_h \mathrm{Op}_h(\chi)u)_\Omega \\ &- \frac{2}{h}\Im(\mathrm{Op}_h(\chi)q|[h\mathrm{div}\ , A_h\mathrm{Op}_h(\chi)]u)_\Omega\end{aligned} \qquad (13)$$

Diffractive case

On the other hand, using integrating by part formula we have

$$\begin{aligned}\sum_{j=0}^{2}(C_j(hD_y)^j v|v) =& o(1) - (A_h^1 hD_y v|hD_y v)_{\partial\Omega} \\ &+\frac{2}{h}\Im([\mathrm{Op}_h(\chi), h\nabla]q|A_h\mathrm{Op}_h(\chi)u)_\Omega \\ &-\frac{2}{h}\Im(\mathrm{Op}_h(\chi)q|[h\mathrm{div}\ , A_h\mathrm{Op}_h(\chi)]u)_\Omega.\end{aligned} \tag{14}$$

Diffractive case

The relation

$$\sum_{j=0}^{2} c_j \eta^j + a(M|\xi| - 2\Im p_s) - \varphi(\eta - r^{1/2}) = -\psi^2$$

and Gårding inequality implies that

$$\begin{aligned}
&-\Re(\phi(y,x,hD_x)(hD_y - \widetilde{Q}_+)v|\varphi(y,x,hD_x)v)_\Omega \\
&+\Re((M|D_x| - 2\Im p_s(y,x,hD_x))v|A_h v)_\Omega + (G(y,x,hD_x)P_h v|v)_\Omega \\
&+\|\Psi_0 v + \Psi_1 hD_y v\|^2_{L^2(\Omega)} + \Re\left(\sum_{j=0}^{2} C_j (hD_y)^j u|u\right)_\Omega \\
&\leq o(1) + Ch\|v\|^2_{L^2(\Omega)},
\end{aligned} \tag{15}$$

Diffractive case

where $\phi \in C_c^\infty(W_0)$ and $r|_{\mathrm{supp}\ \phi} > 0$, $\phi = 1$ in a neighborhood of suppφ. $\widetilde{Q}_+$ be the operator constructed in the hyperbolic region with principal symbol $r^{1/2}$. This is possible since from the construction of test functions, $r \geq \delta^2|\xi|^2$ on the support of φ. Taking $M > 0$ so that $M|\xi| - 2\Im p_s > 0$ and Combining (15),(14), we can control the positive terms $-(A_h^1 hD_y v|hD_y v)_{\partial\Omega}$ and $\|\Psi_0 v + \Psi_1 hD_y v\|^2_{L^2(\Omega)}$ by $|(G(y,x,hD_x)P_h v|v)_\Omega|$,
$\left|\Re(\phi(y,x,hD_x)(hD_y - \widetilde{Q}_+)v|\varphi(y,x,hD_x)v)_\Omega\right|$,
$\left|\frac{2}{h}\Im([\mathrm{Op}_h(\chi), h\nabla]q|A_h\mathrm{Op}_h(\chi)u)_\Omega\right|$, and
$\left|\frac{2}{h}\Im(\mathrm{Op}_h(\chi)q|[h\mathrm{div}\ , A_h\mathrm{Op}_h(\chi)]u)_\Omega\right|$

Diffractive case

Namely, we have

$$
\begin{aligned}
&-(A_h^1 hD_y v|hD_y v)_{\partial\Omega}+\|\Psi_0 v+\Psi_1 hD_y v\|^2_{L^2(\Omega)}\\
\leq& o(1)+Ch\|v\|^2_{L^2(\Omega)}+C|(G(y,x,hD_x)P_h v|v)_\Omega|\\
+&\left|\Re(\phi(y,x,hD_x)(hD_y-\widetilde{Q}_+)v|\varphi(y,x,hD_x)v)_\Omega\right|\\
+&\left|\frac{2}{h}\Im([\mathrm{Op}_h(\chi),h\nabla]q|A_h\mathrm{Op}_h(\chi)u)_\Omega\right|\\
+&\left|\frac{2}{h}\Im(\mathrm{Op}_h(\chi)q|[h\mathrm{div}\ ,A_h\mathrm{Op}_h(\chi)]u)_\Omega\right|
\end{aligned}
\tag{16}
$$

Diffractive case

The last two commutator terms on the right hand side are the only new terms compared to the analysis of the Laplacian quasi-mode problem $(-h^2\Delta - 1)u = o_{L^2}(h)$. Therefore, we only need concentrated on the commutator terms. Pick a bump function $\varphi \in C_c^\infty((-2,2))$ which equals 1 on $(-1,1)$. We take $\chi_\epsilon(y,x,hD_x)$ with principal symbol $\chi_\epsilon(y,x,\xi) = \chi(y,x,\xi)\varphi\left(\frac{r(y,x,\xi)}{\epsilon}\right)$, for any $\epsilon > 0$ small enough and be fixed for the moment. We write

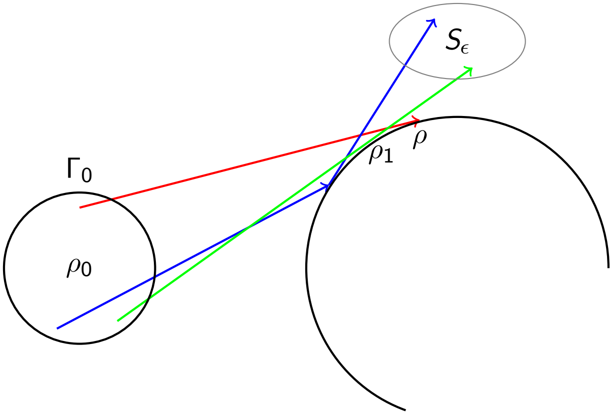
$$\begin{aligned}\frac{1}{h}([\mathrm{Op}_h(\chi), h\nabla]q|A_h v)_\Omega =&\frac{1}{h}([\mathrm{Op}_h(\chi), h\nabla]q|A_h\mathrm{Op}_h(\chi_\epsilon)u)_\Omega\\&+\frac{1}{h}([\mathrm{Op}_h(\chi), h\nabla]q|A_h\mathrm{Op}_h(\chi-\chi_\epsilon)u)_\Omega\\=&:I_{h,\epsilon} + II_{h,\epsilon}\end{aligned} \tag{17}$$

Diffractive case

For the term $II_{h,\epsilon}$, we observe that on the support of $\chi - \chi_\epsilon$, $|r(y,x,\xi)| \geq \epsilon$. Assume that there exists another $\rho_1 \in \mathcal{G}^{2,+}$ and a bicharactersitic interval issued from ρ_1, lies over W_0 and with end point $(y, x, \pm\sqrt{r}, \xi), r \geq \epsilon$, we claim that $y \geq c\epsilon$. Indeed, after the tangency at ρ_1, we rewrite the equation of bicharacterstic curve as (with notation $\rho_1 = (y(0) = 0, x(0); \eta(0) = 0, \xi(0)))$

$$\frac{dy}{dt} = 2\eta, \frac{d\eta}{dt} = \frac{\partial r}{\partial y} \geq \delta_0,$$

with $y(0) = \eta(0) = 0, \eta(T) \geq \sqrt{\epsilon}$. This implies that $T \geq \sqrt{\epsilon}/C$ and $y(T) \geq \frac{1}{2}T^2\delta_0 \geq \frac{\epsilon\delta_0}{2C}$.


S_ϵ
ρ
ρ_1
Γ_0
ρ_0

Diffractive case

Therefore, all the rays issued from Γ_0 which will enter the set $S_\epsilon = \{r \geq \epsilon, y \leq c\epsilon\} \cap \mathrm{supp}\,(\chi - \chi_\epsilon)$ can be only broken bicharacteristics before entering it. Therefore, if we further decompose $II_{h,\epsilon}$ according to S_ϵ, this part will be killed when we let $h \to 0$ first. The rest $y > c\epsilon$ can be handled in the same way by using concentration property of the pressure. $\epsilon > 0$ can be chosen arbitrarily small, therefore

$$\limsup_{h \to 0} \left| \frac{1}{h}([\mathrm{Op}_h(\chi), h\nabla]q | A_h v)_\Omega \right| = 0.$$

For the other commutator term $\left|\frac{2}{h}\Im(\mathrm{Op}_h(\chi)q|[h\mathrm{div}\,, A_h\mathrm{Op}_h(\chi)]u)_\Omega\right|$, the argument is similar.

Outline

Gliding case

Following the argument of Melrose-Sjöstrand, the propagation theorem can be deduce from induction and the following proeprty:

Proposition

For any $k \geq 2$, the following statement is true:
For any $\rho_k \in \mathcal{G}^k$, there exists a neighborhood V_k of ρ_k in $T^*\partial\Omega$, and $\delta_k > 0, \sigma_k > 0, \sigma_k \ll \delta_k$, depending on V_k, such that for any

$$\rho_0 \in \left(\mathcal{G}^{2,-} \cup \bigcup_{3 \leq j \leq k} \mathcal{G}^j \right) \cap V_k, \text{ if near } \rho_0$$

$\{(y, x; z, \zeta) : 0 \leq y \leq \delta^2, (z, \zeta) \in L^-(\delta, \delta^2)\} \cap \ supp\ (\mu) = \emptyset$ *for some* $0 < \delta < \delta_k$, *then* $\exp(sH_{-r_0})(\rho_0) \notin supp(\mu)$ *for any* $s \in [0, \sigma_k \delta)$.

Gliding case

Pick $\rho_0 \in \mathcal{G} \subset T^*\partial\Omega \setminus \{0\}$, and a small neighborhood $U \subset T^*\partial\Omega \setminus \{0\}$ of ρ_0. Let $L \subset U$ be a co-dimension 1 hypersurface containing ρ_0 in $T^*\partial\Omega$ and transversal to the vector field H_{-r_0}. For small positive numbers $\delta, \tau > 0$, define

$$L^{\pm}(\delta,\tau) := \{\exp(tH_{-r_0})(\rho) \in U : \rho \in L, \mathrm{dist}\,(\rho,\rho_0) \leq \delta^2, 0 \leq \pm t \leq \tau\},$$

and

$$F^{\pm}(\delta,\tau) := \{(y,x,\xi) : 0 \leq y \leq \delta^2, (x,\xi) \in L^{\pm}(\delta,\tau)\},$$

$$F(\delta,\tau) = F^{+}(\delta,\tau) \cup F^{-}(\delta,\tau).$$

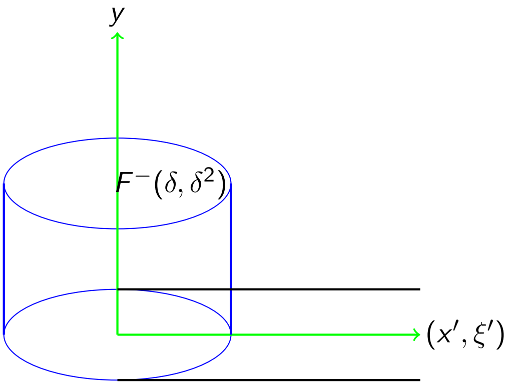
Near $\mathcal{G}^{2,-}$

We only present the proof of Proposition 13 for $k = 2$. Assume that in the neighborhood of $\rho_0 \in \mathcal{G}^{2,-}$ where we are working $-C_0 < \partial_y r \leq -c_0 < 0$. Notice that there exists $C_1 > 0$ such that for $\delta_0 > 0, \tau_0 > 0$ small enough, and for any $\delta < \delta_0, \tau < \tau_0$, we have $|r(y,x,\xi)| \leq \frac{1}{2} C_1^2 \delta^2 |\xi|^2$ for all $(y,x,\xi) \in F(\delta,\tau)$.

Now we set up the tool box due to Melrose-Sjöstrand: with the same constant C_1, we define the sets

$$
\begin{aligned}
V^{\pm}(\delta,\tau) :=& \{(y,x,\eta,\xi) : 0 \leq y \leq \delta^2/2, (x,\xi) \in L^{\pm}(\delta,\tau)\} \\
& \cup \{(y,x,\eta,\xi) : \delta^2/2 \leq y \leq \delta^2, (x,\xi) \in L^{\pm}(\delta,\tau), |\eta| \leq C_1 \delta |\xi|\}, \\
W^{\pm}(\delta,\tau) :=& \{(y,x,\eta,\xi) : 0 \leq y \leq \delta^2/2, (x,\xi) \in L^{\pm}(\delta,\tau)\} \\
& \cup \{(y,x,\eta,\xi) : \delta^2/2 \leq y \leq \delta^2, (x,\xi) \in L^{\pm}(\delta,\tau), |\eta| \leq 2C_1 \delta |\xi|\}
\end{aligned}
$$

$$
V(\delta,\tau) := V^{+}(\delta,\tau) \cup V^{-}(\delta,\tau), W(\delta,\tau) = W^{+}(\delta,\tau) \cup W^{-}(\delta,\tau).
$$


y
$F^{-}(\delta, \delta^2)$
(x', ξ')

Near $\mathcal{G}^{2,-}$

Lemma

Given $\rho_0 \in \mathcal{G}$, there exists $\delta_0 > 0, \tau_0 > 0$ such that if $dist(\rho, \rho_0) \leq \delta^2$ for some $\delta > 0$, then $dist(\gamma(s, \rho), \gamma(s, \rho_0) \leq C\delta^2$ for $|s| \leq \sigma_0 \delta$. In particular, $\gamma(s, \rho) \in W(\delta, \tau_0)$ for all $|s| \leq \tau_0 \delta$.

Proof.

For any two curves $\gamma_1(s) = (y(s), \eta(s), x(s), \xi(s))$ and $\gamma_2(s) = (\tilde{y}(s), \tilde{\eta}(s), \tilde{x}(s), \tilde{\xi}(s))$. $\dot{y} = 2\eta, \dot{\eta} = O(1)$, we have $y(s) \leq Cs^2$. Let $d(s) = |x(s) - \tilde{x}(s)|^2 + |\xi(s) - \tilde{\xi}(s)|^2$, $\dot{d}(s) \leq Cd(s) + C|y(s) - \tilde{y}(s)|\sqrt{d}$. These implies $d(s) \leq C_1\delta^2$ for all $|s| \leq \sigma_0\delta$.

□

Near $\mathcal{G}^{2,-}$

First we record several property of test functions:
Denote $I = [0, \epsilon_0)$. There exist $\sigma > 0, \delta_0 > 0, \tau_0 > 0$ small enough with $\delta \ll \sigma$ and smooth functions $a_\delta \in C_c^\infty(I \times U), g_\delta, h_\delta \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{d-1} \setminus \{0\})$ for any $0 < \delta \leq \delta_0$ with the following properties:

- $a_\delta \geq 0$, $\text{supp} a_\delta \subset F^+(\delta, \sigma\delta) \cup F^-(\delta, \delta^2)$.
- $a_\delta(0, \exp(tH_{-r_0}(\rho_0))) \neq 0, \forall 0 \leq t < \delta\sigma$.
- $a_\delta > 0$ on $\text{supp} a_{\delta'}$ if $0 < \delta' < \delta$ and independent of y for $0 \leq y < \delta^2/2$.
- $g_\delta + h_\delta = -(H_p + k)a_\delta$.
- in $W(\delta, \tau)$, $g_\delta \geq 0$ and $g_\delta > 0$ when $a_\delta \neq 0$.
- $\text{supp} h_\delta \subset I \times L^-(\delta, \delta^2) \times \mathbb{R}_\eta$, and $\text{supp} g_\delta \cup \text{supp} h_\delta \subset \text{supp} a_\delta$, g_δ, h_δ are independent of η for $0 \leq y \leq \delta^2/2$.

Near $\mathcal{G}^{2,-}$

Denote $b_\delta := g_\delta^{1/2} \in C^\infty(W(\delta,\tau))$. Note that by no means that b_δ is smooth and with compact support. We need split it into two parts as following:

Let $\phi_1 \in C^\infty(\mathbb{R})$ such that $\phi_1 \equiv 1$ if $0 \leq y \leq \frac{\delta^2}{4}$ and $\phi_1 \equiv 0$ if $y > \frac{3\delta^2}{8}$.

Let $\phi_2 \in C^\infty(\Omega \times \mathbb{R}^d \setminus \{0\})$ with compact support in x, ξ, η variables, such that $\phi_2 \geq 0$ and $\phi_2 \equiv 0$ whenever $y \leq \frac{\delta^2}{4}$ or $|\eta| > 2C_1\delta|\xi|$.

Indeed, we can choose $\Psi(x,\eta,\xi)$, non-negative, smooth and with compact support, such that $\Psi \equiv 0$ when $|\eta| > 2C_1\delta|\xi|$ and $\Psi \equiv 1$ when $|\eta| \leq \frac{3}{2}C_1\delta|\xi|$.

Near $\mathcal{G}^{2,-}$

Now let $\phi_2(y, x, \eta, \xi)^2 = (1 - \phi_1(y)^2)\Psi(x, \eta, \xi)$. We observe that

$$W^{\pm}(\delta, \tau) \cap \mathrm{supp}\ (1 - \phi_1^2 - \phi_2^2)$$
$$\subset \left\{ (y, x, \eta, \xi) : \frac{\delta^2}{4} \leq y \leq \delta^2, |\eta| > \frac{3}{2} C_1 \delta |\xi|, \right\}.$$

We finally put $b_{\delta,j} := \phi_j b_\delta, j = 1, 2$. Note that $b_{\delta,1} \in C_c^\infty(F(\delta, \tau))$ is a tangential symbol and $b_{\delta,2} \in C_c^\infty(W(\delta, \tau))$ is a usual symbol with compact support in $T^*\Omega$.

Near $\mathcal{G}^{2,-}$

Denote $\delta' < \delta$, small enough and to be chosen later, and denote the operator

$$N_{\delta'} = \frac{1}{ih}[P, A_{\delta'}] + \frac{1}{ih}(R - R^*)A_{\delta'},$$

with principal symbol

$$n_{\delta'} = -(H_p + k)a_{\delta'} = g_{\delta'} + h_{\delta'}.$$

Define the operators

$$B_{\delta',j} := \mathrm{Op}_h(b_{\delta',j}), j = 1, 2, N_{\delta,3} = \mathrm{Op}_h((1 - \phi_1^2 - \phi_2^2)n_{\delta'}).$$

Write

$$h_{\delta',j} = \phi_j^2 h_{\delta'}, H_{\delta',j} = \mathrm{Op}_h(h_{\delta',j}), j = 1, 2.$$

We remark that $h_{\delta',1}, b_{\delta',1}$ are both tangential symbols while $h_{\delta',2}, b_{\delta',2}$ are interior symbols vanishing near the boundary.

Near $\mathcal{G}^{2,-}$

Observe also that $N_{\delta',3}$ is interior pseudo-differential operator with symbol vanishing near the boundary as well as on $p^{-1}(0)$, thanks to the fact that in $W(\delta',\tau)$, $|r(y,x,\xi)| \leq \frac{1}{2}C_1^2\delta'^2|\xi|^2$.. Thus $N_{\delta',3}u = o_{L^2_{y,x}}(1)$ as $h \to 0$. Moreover, $H_{\delta',j}u = o_{L^2_{y,x}}(1)$ is clear from the assumption on the support of μ.
Now set

$$M_{\delta',j} = \phi_j^2 N_{\delta',j} - B^*_{\delta',j}B_{\delta',j} - H_{\delta',j}, j = 1, 2.$$

From symbolic calculus, we have $M_{\delta',1} = O_{L^2\to L^2}(h)$ is a tangential operator. Moreover, $M_{\delta',2} = O_{L^2\to L^2}(h)$ is an interior operator.

Near $\mathcal{G}^{2,-}$

Finally, we have obtained

$$N_{\delta'} = N_{\delta',3} + \sum_{j=1}^{2} (B^*_{\delta',j} B_{\delta',j} + H_{\delta',j}) + O_{L^2 \to L^2}(h).$$

Combining all the analysis above the integrating by part formula for self-adoint tangential operator A:

$$\frac{1}{h}(([P, A] + (R - R^*)A)u|u)_\Omega = \frac{1}{h}(Au|Pu)_\Omega - \frac{1}{h}(APu|u)_\Omega + O(h).$$

we have

$$\sum_{j=1}^{2} \|B_{\delta',j}u\|^2_{L^2} \leq o(1) + \frac{1}{h}\left|([h\mathrm{div}\ , A_{\delta'}]u|q)_\Omega\right| + \frac{1}{h}\left|([A_{\delta'}, h\nabla]q|u)_\Omega\right| \tag{18}$$

Near $\mathcal{G}^{2,-}$

To finish the proof, we need show that the right hand side of (18) is $o(1)$ as $h \to 0$.
We perform a similar decomposition: pick a$\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(s) \equiv 1$ if $0 \leq s \leq \frac{1}{2}$ and $\chi(s) \equiv 0$ is $s \geq 1$. Let $\chi_\epsilon(y,x,\xi) = \chi(\epsilon^{-1} r(y,x,\xi))$. Denote

$$I_{h,\epsilon} = \frac{1}{h} \left| ([h\mathrm{div}\ , A_\delta]\mathrm{Op}_h(\chi_\epsilon) u | q)_\Omega \right|,$$

$$II_{h,\epsilon} = \frac{1}{h} \left| ([h\mathrm{div}\ , A_\delta](1 - \mathrm{Op}_h(\chi_\epsilon)) u | q)_\Omega \right|.$$

Near $\mathcal{G}^{2,-}$

The treatment of $I_{h,\epsilon}$ is exactly the same as in the diffractive case. The treatment of $II_{h,\epsilon}$ is relied on the geometric lemma:

Lemma

If in addition, $-C_0 \leq \frac{\partial r}{\partial y}(\rho) \leq -c_0 < 0$ *for all* $\rho \in W(\delta_0, \tau_0)$*. Then for* $\delta, \tau > 0$ *small enough* $(\delta < \delta_0)$*, the following property holds: There exists a sufficiently small constant* $\delta' < \delta$ *such that for any* $\epsilon > 0, \epsilon \ll \delta, \epsilon \ll \tau$*, and for any* $\rho = (y, x, \eta, \xi) \in W(\delta', \tau)$ *with* $r(y, x, \xi) \geq \epsilon, y(\rho) \leq \epsilon$*, the real backward bicharacteristic* $\gamma(t, \rho)$ *is always contained in* $W(\delta, \tau)$ *for all* $0 \leq t \leq \tau$*. Moreover, at each time t such that* $y(\gamma(t, \rho)) = 0$*, we must have* $r \gtrsim \epsilon$ *at this point. In other words. Every backward generalized bicharactersitc issued from* ρ *is at most broken bicharactersitic.*

Near $\mathcal{G}^{2,-}$

Proof.

Assume that the initial point $\rho_0 = (y_0, x_0, \eta_0, \rho_0)$ satisfies $|\eta_0|^2 = |r(y_0, x_0, \xi_0)|^2 \geq \epsilon$.
The dynamics on the phase space is governed by the ODE $\frac{dy}{dt} = 2\eta, \frac{d\eta}{dt} = \frac{\partial r}{\partial y}$ with initial condition $y(0) = y_0 \geq 0, \eta(0) = \eta_0$.
Suppose $\eta_0 \leq -\sqrt{\epsilon} < 0$, and t_0 is the first time such that $y(t_0) = 0$. It is obvious that $\eta(t_0) < \eta(0) \leq -\sqrt{\epsilon}$. Suppose now $\eta_0 \geq \sqrt{\epsilon} > 0$. Let $t_1 > 0$ be the first time such that $\eta(t_1) = 0$, and thus $y_1 := y(t_1) > y_0$. Let $t_2 > t_1$ be the first time so that $y(t) = 0$.
First we have $-\eta_0 = \eta(t_1) - \eta(0) = \int_0^{t_1} \frac{d\eta}{dt} dt \leq -C_0 t_1$, and this implies that $t_1 \geq \frac{|\eta_0|}{C_0}$. □

Near $\mathcal{G}^{2,-}$

Proof.

Moreover,

$$y_1 - y_0 = \eta_0 t_1 + \int_0^{t_1} dt \int_0^t \frac{d^2 y}{ds^2} ds \geq \eta_0 t_1 - \frac{C_0}{2} t_1^2 \geq \frac{|\eta_0|^2}{2C_0}.$$

Now

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} dt \int_{t_1}^t \frac{d^2 y}{ds^2} ds \geq \frac{-C_0}{2}(t_2 - t_1)^2,$$

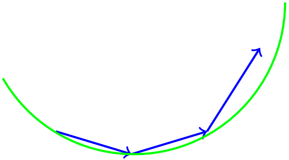
we have

$$|t_2 - t_1|^2 \geq \frac{2y_1}{C_0} \geq \frac{2(y_1 - y_0)}{C_0} \geq \frac{|\eta_0|^2}{C_0^2}.$$

Finally we have

$$\eta(t_2) = \int_{t_1}^{t_2} \frac{d\eta}{dt} dt \leq -c_0(t_2 - t_1) \leq -\frac{c_0|\eta_0|}{C_0} \leq -\frac{c_0\sqrt{\epsilon}}{C_0}.$$

□



Near $\mathcal{G}^{2,-}$

From the lemma above, the measure of $\frac{1}{h}[h\mathrm{div}\ , A_\delta](1-\mathrm{Op}_h(\chi_\epsilon))u$ vanishes since all the backward generalized rays starting from each point in $\mathrm{supp} a_\delta(1-\chi_\epsilon)$ will enter the small neighborhood of $\rho_0 \in \mathcal{G}^{2,-}$ by at most reflection at boundary. This completes the proof of Proposition 13 for $k=2$. For higher order contact, the geometric argument is a little complicated. We need a change of coordinate and make full use the fact that the Melrose-Sjöstrand flow is transverse to $\mathcal{G}^k$ for $k \geq 3$. We skip the details here.

Induction Step

We briefly explain the induction step:

Definition (k propagation property)

For $k \geq 2$, we say that $k-$propagation property holds, if along generalized ray $\gamma(s, \rho_0)$, the following statement is true: For some $\sigma_0 > 0$, if $\gamma(\cdot, \rho_0)|_{[0,\sigma_0)} \cap \text{supp } (\mu) = \emptyset$ (or $\gamma(\cdot, \rho_0)|_{(-\sigma_0, 0]} \cap \text{supp } (\mu) = \emptyset$) and $\gamma(\sigma_0, \rho_0) \in \bigcup_{2 \leq j \leq k} \mathcal{G}^j$ (or $\gamma(-\sigma_0, \rho_0) \in \bigcup_{2 \leq j \leq k} \mathcal{G}^j$), then $\gamma(\sigma_0, \rho_0) \notin \text{supp } (\mu)$ (or $\gamma(-\sigma_0, \rho_0) \notin \text{supp } (\mu)$).

The proof is complete once we have proved the k-propagation property for all $k \geq 3$.

Induction step

We also introduce the following defition

Definition (k-pre-propagation property)

For $k \geq 2$, we say that k-pre-propagation property holds, if the following statement is true:
For any $\rho_k \in \mathcal{G}^k$, there exists a neighborhood V_k of ρ_k in $T^*\partial\Omega$, and $\delta_k > 0, \sigma_k > 0, \sigma_k \ll \delta_k$, depending on V_k, such that for any $\rho_0 \in \left(\mathcal{G}^{2,-} \cup \bigcup_{3 \leq j \leq k} \mathcal{G}^j \right) \cap V_k$, if

$$\{(y, x; z, \zeta) : 0 \leq y \leq \delta^2, (z, \zeta) \in L^-(\delta, \delta^2; \rho_0)\} \cap \operatorname{supp}(\mu) = \emptyset$$

for some $0 < \delta < \delta_k$, then $\exp(sH_{-r_0})(\rho_0) \notin \operatorname{supp}(\mu)$ for any $s \in [0, \sigma_k \delta)$.

Induction step

Now the induction runs as follows:

- Deduce k-pre-propagation property by assuming the (k-1)-propagation property. The argument is similar as we have done in the case $\mathcal{G}^3$.
- Deduce k-propagation property by assuming (k-1)-propagation property and k-pre-propagation property. This step is standard.

Higher order contact

We first change to a new coordinate system near ρ_k in $\widetilde{W}_k$

$$(y,\eta,x,\xi)\mapsto(y,\eta,z,\zeta), z=(z_1,z'), \zeta=(\zeta_1,\zeta')$$

with

$$p=\eta^2-r, r=\zeta_1+yr_1(z,\zeta)+O(y^2),$$
$$\zeta_1=r_0,$$

where

$$r_0=r|_{y=0}, r_1=\partial_y r|_{y=0}.$$

This is possible since

$$d_{x,\xi}r_0\neq 0, \text{ if } \xi\neq 0.$$

Along the generalized bicharacteristic flow $\rho(s)$, (z,ζ) satisfies

$$\frac{dz}{ds}=-\frac{\partial r}{\partial\zeta}(y(s),z(s),\zeta(s)), \frac{d\zeta}{ds}=\frac{\partial r}{\partial z}(y(s),z(s),\zeta(s)).$$

Higher order contact

We deduce from this that near ρ_k,

$$-\frac{dz_1}{ds} \sim 1 > 0, \text{ as } y \to 0,$$

and thus $s \mapsto z_1(s)$ is strictly decreasing , and

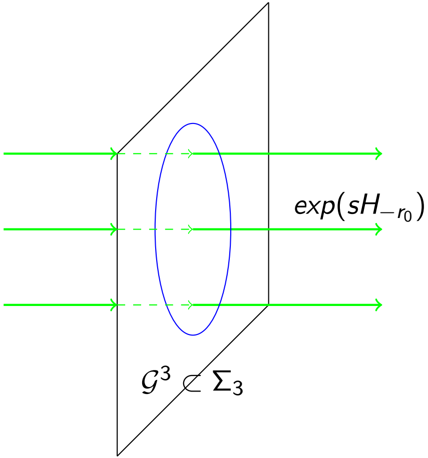
$$\frac{d\zeta_1}{ds} \sim y\frac{\partial r_1}{\partial z_1}, y \to 0.$$

Suppose now $k \geq 3$, we have locally that

$$\mathcal{G}^k := \{(z,\zeta) : \zeta_1 = 0, \partial_{z_1}^l r_1(z,\zeta) = 0, \forall l \leq k-3, \partial_{z_1}^{k-2} r_1(z,\zeta) \neq 0\}.$$

Define

$$\Sigma_k := \{(z,\zeta) : \partial_{z_1}^{k-3} r_1(z,\zeta) = 0, \partial_{z_1}^{k-2} r_1(z,\zeta) \neq 0\}.$$


$exp(sH_{-r_0})$
$\mathcal{G}^3 \subset \Sigma_3$

Higher order contact

The key fact we will use is the asymptotic behaviour of the flow $\phi_s^{H_{-r_0}} = exp(sH_{-r_0})$ when $s \to 0$, we have for any given $(z_1 = \Theta_k(z_0', \zeta_0), z_0', \zeta_0) \in \mathcal{G}^k$,

$$r_1 \circ \phi_s^{H_{-r_0}}(z_0', \zeta_0) = a_3(z_0', \zeta_0)s^{k-2} + O(s^{k-1})$$

where $a_k \neq 0$ can be viewed as a function of points in $\mathcal{G}^k$.

Thank you for your attention!