

Sufficient Conditions for the Boundary Controllability of Wave Equations with Transmission Conditions

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Outline

1 Introduction

- Presentation of the problematic and motivation
- Definition of the generalized bicharacteristics
- Statement of the main results

2 Propagation results

3 Geometrical proof

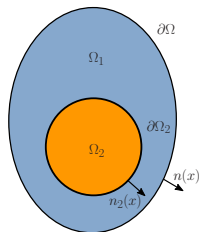
4 Conclusion

Wave equations with transmission condition

Let $T > 0$. Let $\Omega \subset \mathbb{R}^2$ an open, **strictly convex**, simply connected domain of smooth boundary $\partial\Omega$. Let $\Omega_2 \subsetneq \Omega$ an open, **strictly convex**, simply connected domain of smooth boundary $\partial\Omega_2$. Define $\Omega_1 := \Omega \setminus \Omega_2$.

(WT)

$$\begin{cases} (\partial_t^2 - c_i^2 \Delta) u^i(x, t) = 0, & (x, t) \in \Omega_i \times (0, T), \\ u^1(x, t) = u^2(x, t), & (x, t) \in \partial\Omega_2 \times (0, T), \\ c_1^2 \partial_{n_2} u^1(x, t) = c_2^2 \partial_{n_2} u^2(x, t), & (x, t) \in \partial\Omega_2 \times (0, T), \\ u^1(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u^i(0, x) = u_0^i(x), u_t^i(0, x) = u_1^i(x), & x \in \Omega_i \end{cases}$$



where $c_2 > c_1 > 0$ and $(u_0^1 \mathbf{1}_{\Omega_1} + u_0^2 \mathbf{1}_{\Omega_2}, u_1^1 \mathbf{1}_{\Omega_1} + u_1^2 \mathbf{1}_{\Omega_2}) \in H_0^1(\Omega) \times L^2(\Omega)$.

$$E(u^1, u^2)(t) := \sum_{i=1}^2 \int_0^T \int_{\Omega_i} |u_t^i|^2 + c_i^2 |\nabla u^i|^2 \, dx dt \equiv E(u^1, u^2)(0)$$

For $\Gamma \subset \partial\Omega$ and $T > 0$, we consider the observability problem : $\exists c_T > 0$ such that

$$\text{(obs)} \quad E(u^1, u^2)(t) \leq c_T \int_0^T \int_{\Gamma} |\partial_n u^1(x, t)|^2 \, dx dt.$$

Motivation : the wave equation

Let $T > 0$ and let Ω be defined as in the previous slide.

$$(W) \quad \begin{cases} (\partial_t^2 - c^2 \Delta)u(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(0, x) = u_0(x), u_t(0, x) = u_1(x), & x \in \Omega. \end{cases}$$

Conserved energy :

$$E(u)(t) := \int_0^T \int_{\Omega} |u_t|^2 + c^2 |\nabla u|^2 \, dx dt \equiv E(u)(0).$$

Observability problem : for given $T > 0$ and $\Gamma \subset \partial\Omega$, does $\exists c_T > 0$ such that

$$E(u)(t) \leq c_T \int_0^T \int_{\Gamma} |\partial_n u(x, t)|^2 \, dx dt.$$

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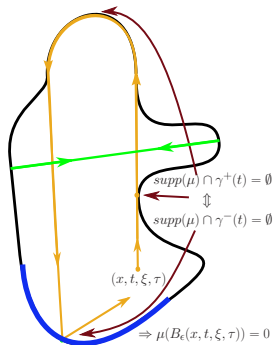
$$E(u)(t) \leq c_T \int_0^T \int_{\Gamma} |\partial_n u(x, t)|^2 \, dx dt.$$

Proof by contradiction : let $\{u_k\}_{k \in \mathbb{N}}$ be a renormalized sequence ($E(u_k) = 1$) such that

$$\int_0^T \int_{\Gamma} |\partial_n u_k(x, t)|^2 \, dx dt \rightarrow 0.$$

We associate a defect measure μ acting on $(x, t, \xi, \tau) \in T_b^*(\Omega \times]0, T[)$.

Motivation : the wave equation



Bardos, Lebeau, Rauch, '92 :

- 1 $\mu = 0$ on $\Gamma \times]0, T[$
- 2 if $\mu = 0$ on $\Omega \times]0, T[$, then $u_k \rightarrow 0$

Definition (Geometric Control Condition (GCC))

We say that Γ and $T > 0$ satisfies the geometric control condition if and only if every ray of the optic geometry of Ω intersects Γ at a non-diffractive point in time T .

Propagation of the bicharacteristics in Ω_i

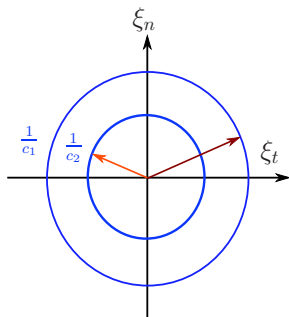
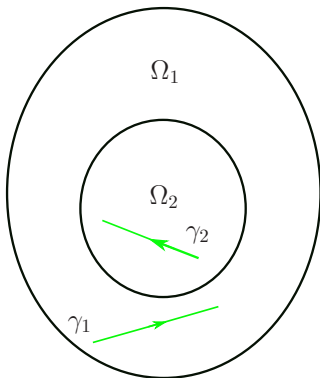
Let $T^*(\Omega_i \times \mathbb{R}_t) = \{(t, x, \tau_i, \xi_i) \in \mathbb{R}^6 \mid (t, x) \in \mathbb{R}_t \times \Omega_i\}$. The principal symbol of $P_i = \partial_t^2 - c_i^2 \Delta$ is given by

$$p_i(t, x; \tau_i, \xi) = \tau_i^2 - c_i^2 |\xi_i|^2.$$

The bicharacteristics γ_i propagates in

$$\text{Char } P_i := \{(t, x, \tau_i, \xi_i) \in T^*(\Omega_i \times \mathbb{R}_t) \mid p_i(t, x; \tau_i, \xi) = 0\}$$

in straight line and at constant speed



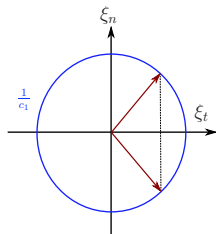
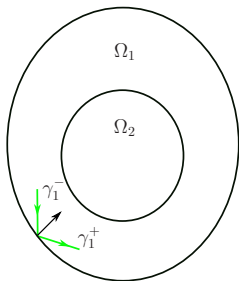
Near $\partial\Omega$

Let $(x_n, x') \in \partial\Omega$, then locally $\Delta = \partial_{x_n}^2 + R(x_n, x', D_{x'})$ where $R(x_n, x', D_{x'})$ is a second order tangential elliptic operator of real principal symbol $r(x_n, x', \xi_1')$. We have

$$\rho_1(t, x, \tau_1, \xi_1', \xi_1^n)(\sigma_0) = \tau_1^2 - c_1^2(|\xi_1^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi_1')^2) = 0$$

We decompose $T^*(\partial\Omega \times \mathbb{R}_t) = \mathcal{H} \cup \mathcal{G}^+ \cup \mathcal{E}$ where

$$\begin{aligned}\mathcal{H} &:= \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \mid r(x_n, x', \xi_1) > |\tau_1|/c_1\}, \\ \mathcal{G}^+ &:= \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \mid r(x_n, x', \xi_1) = |\tau_1|/c_1\}, \\ \mathcal{E} &:= \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \mid r(x_n, x', \xi_1) < |\tau_1|/c_1\}.\end{aligned}$$



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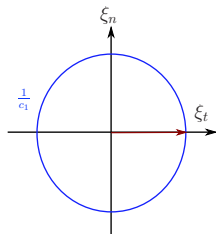
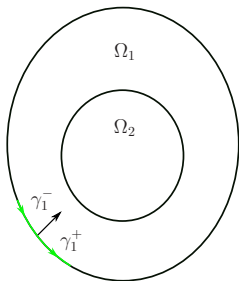
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Near $\partial\Omega_2$

$$\rho_i(t, x, \tau_i, \xi'_i, \xi_i^n)(\sigma_0) = \tau_i^2 - c_i^2(|\xi_i^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi'_i)^2) = 0,$$

$$T^*(\partial\Omega_2 \times \mathbb{R}_t) = (\mathcal{H}_1 \times \mathcal{H}^2) \cup (\mathcal{H}^1 \times \mathcal{G}^{2,+}) \cup (\mathcal{H}^1 \times \mathcal{E}^2) \cup (\mathcal{G}^{1,-} \times \mathcal{E}^2) \cup (\mathcal{E}^1 \times \mathcal{E}^2),$$

where

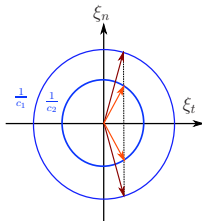
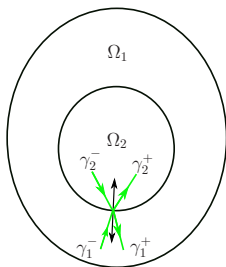
$$\mathcal{H}^1 \times \mathcal{H}^2 = \{(\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0, T)) \mid |\xi'_i| < |\tau_i|/c_i, c_1^2 \xi_1^n = -c_2^2 \xi_2^n\},$$

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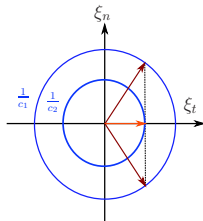
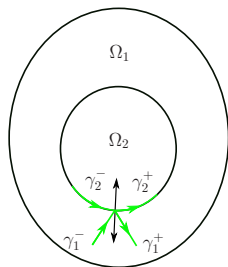
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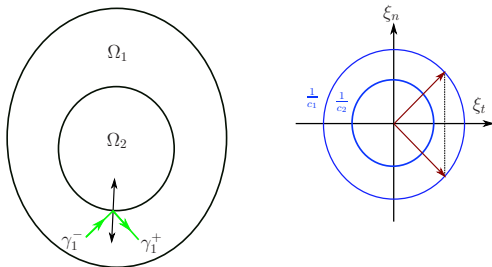
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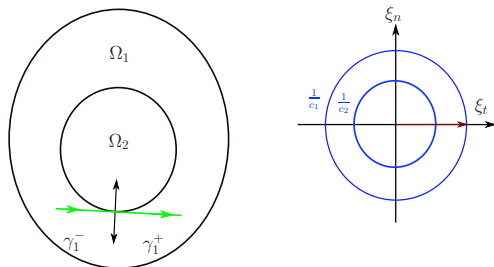
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Propagation of the measures

Let μ_1 and μ_2 be the defect measures attached to u_1 and u_2 . Then

Theorem (Bardos, Lebeau, Rauch, '92)

Let $\rho \in \mathcal{H} \cup \mathcal{G}^{2,+}$. Then,

$$((\gamma_1^-) \cap \text{supp}(\mu_1)) = \emptyset \Leftrightarrow ((\gamma_1^+) \cap \text{supp}(\mu_1)) = \emptyset.$$

An adaptation of the work of Burq and Lebeau '01 on Lamé system gives

Corollary (G., '17)

For $\rho_2 \in \mathcal{H}^2 \times \mathcal{G}^{2,+}$, we have the following equivalence

$$\begin{aligned} ((\gamma_1^-) \cap \text{supp}(\mu_1)) \cup ((\gamma_2^-) \cap \text{supp}(\mu_2)) &= \emptyset \\ \Updownarrow \\ ((\gamma_1^+) \cap \text{supp}(\mu_1)) \cup ((\gamma_2^+) \cap \text{supp}(\mu_2)) &= \emptyset \end{aligned}$$

This propagation results are sufficient for the observability in the case $\Gamma = \partial\Omega$ (noticed by Lebeau, Le Rousseau, Terpolilli and Trélat, '14).

Geometrical construction

For $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$

$$\Gamma(x_0) := \{x \in \partial\Omega \mid \langle (x - x_0), n(x) \rangle > 0\}.$$

Step 1 : Propagation of $\Gamma(x_0)$

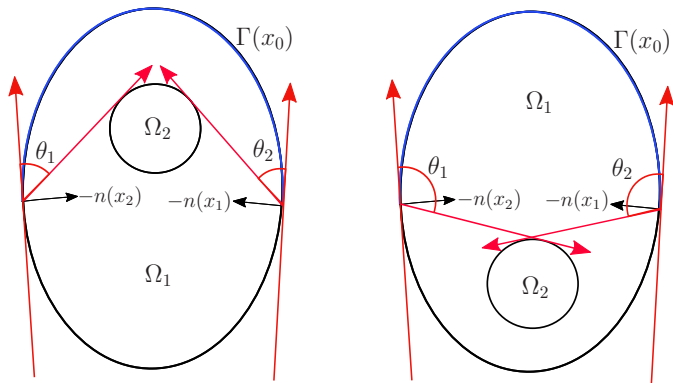


Figure: Left : case where Ω_2 is included in $\Gamma(x_0)$ - right : case where Ω_2 is not included in $\Gamma(x_0)$

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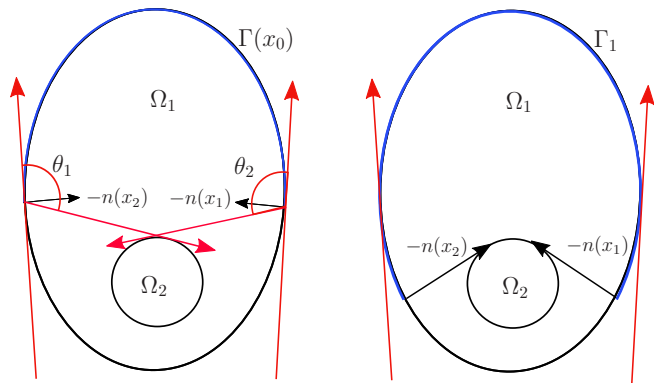


Figure: Left : case where Ω_2 is not included in $\Gamma(x_0)$ - right : Propagation of Γ

Geometrical construction

Step 2 : Construction of $\Gamma_2 \subset \partial\Omega_2$

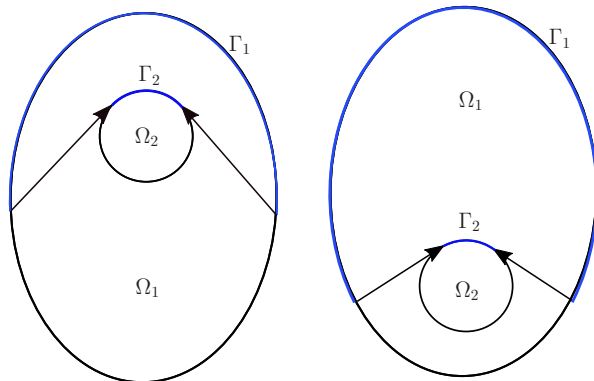


Figure: Construction of $\Gamma_2 \subset \partial\Omega_2$

Geometrical construction

Step 3 : Extension of Γ_2

As long as $\Pi_x(\mathcal{F}(x, n_2(x))) \in \Gamma_1$

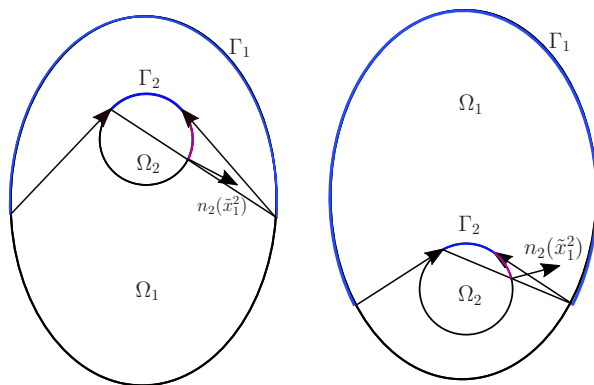


Figure: Extension of Γ_2

Geometrical construction

Step 4 : Extension of Γ_1

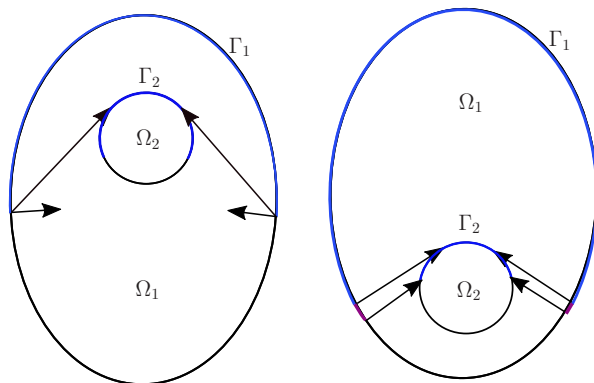


Figure: Extension of Γ_1

Geometrical construction

Step 5 : Iterate step 3 and 4

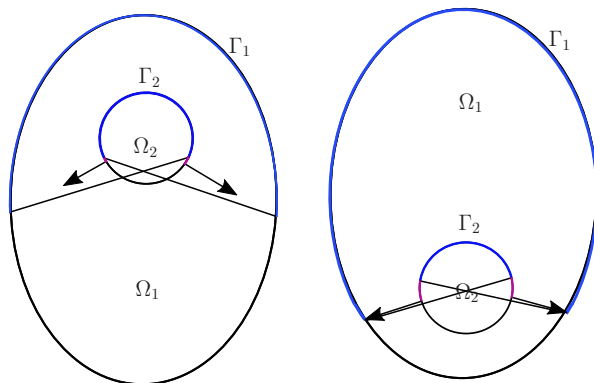


Figure: Extension of Γ_2

Geometrical construction

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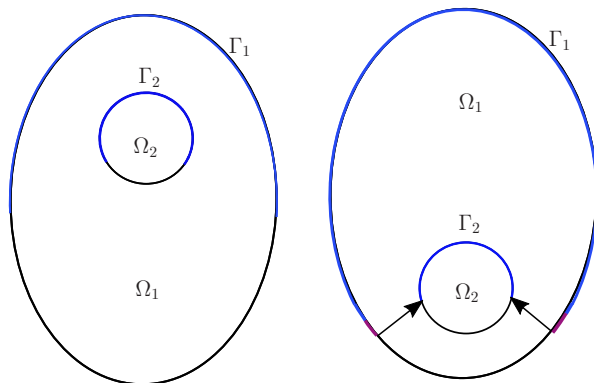


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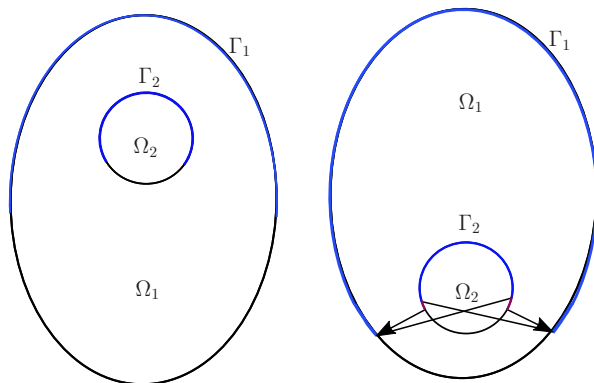


Figure: Extension of Γ_2

Geometrical construction

Step 5 : Iterate step 3 and 4

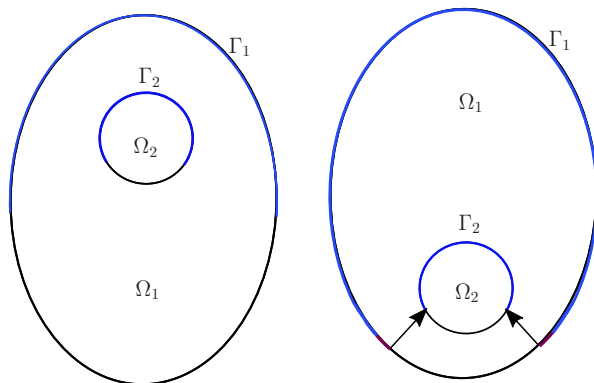


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Geometrical construction

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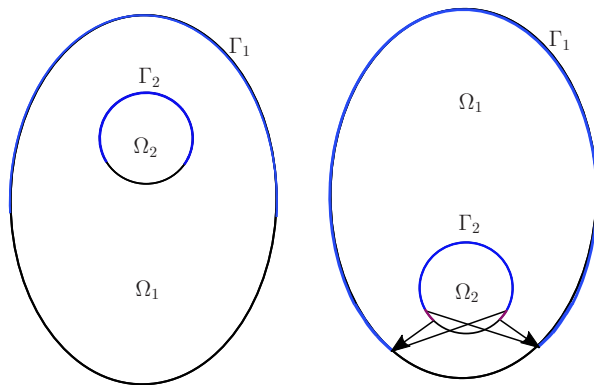


Figure: Extension of Γ_2

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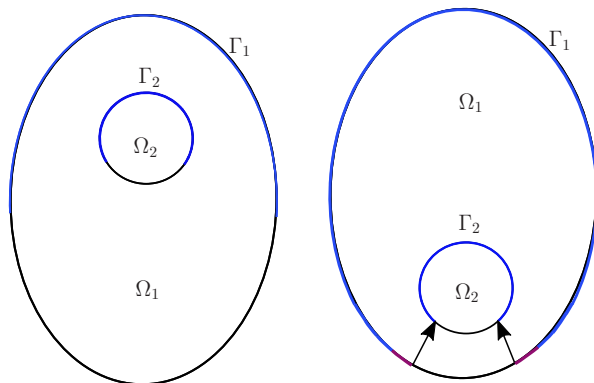


Figure: Extension of Γ_1

Geometrical construction

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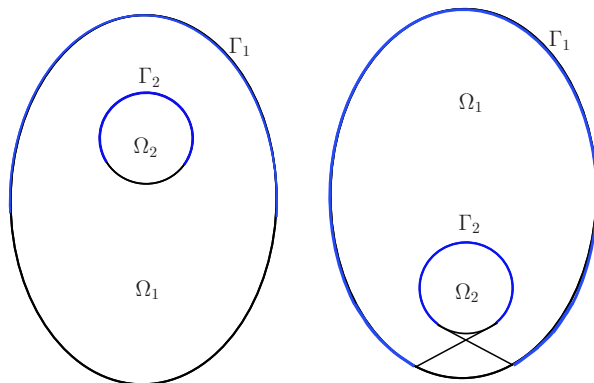


Figure: Extension of Γ_2

Geometrical construction

Final step : definition of Ω_1^f

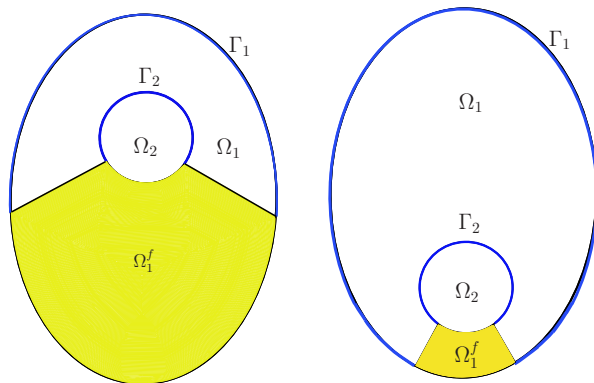


Figure: Definition of Ω_1^f

Statement of the main results

Collision map :

$$\begin{aligned}\mathcal{F} : (\partial\Omega \cup \partial\Omega_2) \times \mathbb{R}^2 &\longrightarrow (\partial\Omega \cup \partial\Omega_2) \times \mathbb{R}^2, \\ (x, \xi) &\longmapsto (x^1, \xi^1).\end{aligned}$$

Projections :

$$\Pi_x(x, \xi) = x, \quad \Pi_\xi(x, \xi) = \xi.$$

Parametrizations :

$$\partial\Omega : \delta(s), s \in [0, 1), \quad \partial\Omega_2 : \delta_2(s), s \in [0, 1),$$

Definition (Uniformly escaping geometry)

We say that Ω_1^f is a uniformly escaping geometry if the application

$$\begin{aligned}\mathcal{M} : (\partial\Omega_2 \setminus \Gamma_2) \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, \xi) &\longmapsto \langle \xi, n(\Pi_x(\mathcal{F}(x, \xi)))^\perp \rangle\end{aligned}$$

is nondecreasing for $s \mapsto \mathcal{M}(\delta_2(s), n_2(\delta_2(s))), \delta_2(s) \in \partial\Omega_2 \setminus \Gamma_2$.

Statement of the main results

Theorem (G. , '17)

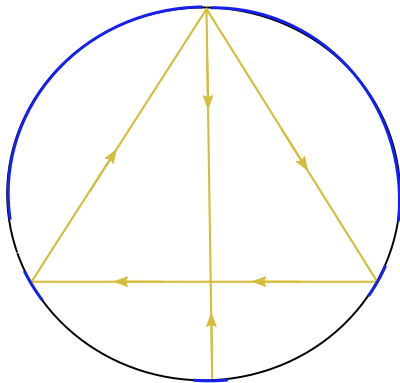
Let $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $\Gamma(x_0)$. Suppose Ω_1^f is a uniformly escaping geometry, then for every $c_2 > c_1$, there exists a time $T > 0$ such that (WT) is observable.

Theorem (G. , '17)

Let $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $\Gamma(x_0)$. Suppose Ω_1^f is not a uniformly escaping geometry but if Ω_1^f and $c_2 > c_1$ are such that for every $x \in \partial\Omega \setminus \Gamma_1$ and $\xi \in \mathbb{R}^2$ such that $\Pi_x(\mathcal{F}(x, \xi)) \in \partial\Omega_2 \setminus \Gamma_2$ and $\Pi_x(\mathcal{F}^2(x, \xi)) \in \partial\Omega \setminus \Gamma_1$, the transmitted rays in Ω_2 are such that $\Pi_x(\mathcal{F}_2^2(x, \xi^\pm)) \in \Gamma_2$, then (WT) is observable in some time $T > 0$.

Remarks on the $\Gamma(x_0)$ condition

If Ω is convex, then $\Gamma(x_0)$ is connected. An example where GCC holds but is not given by $\Gamma(x_0)$ was given by BLR, '92.



Later on, it was proved

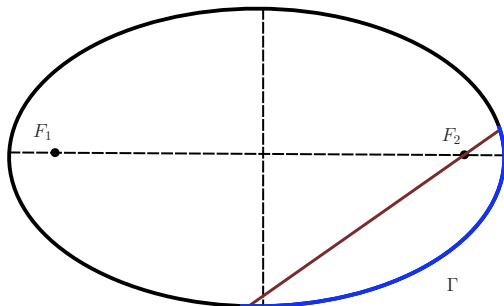
Theorem (Miller, '02)

The $\Gamma(x_0)$ condition implies GCC.

GCC is more general, even if Γ is connected and Ω convex

Lemma (G., '17)

There exists strictly convex domains $\Omega \subset \mathbb{R}^2$ and $\Gamma \subset \partial\Omega$ connected such that GCC holds but $\Gamma \not\subsetneq \Gamma(x_0)$ for any $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$.



Corollary (G., '17)

There exists strictly convex domains $\Omega \subset \mathbb{R}^2$ and $\Gamma \subset \partial\Omega$ such that Γ and $\partial\Omega \setminus \Gamma$ are observable regions.

Proof

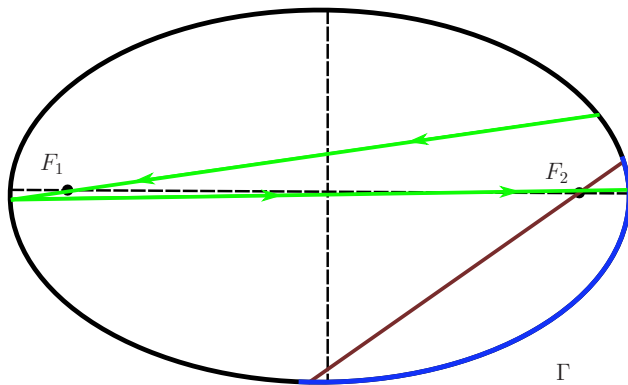


Figure: Convergence to the main axis

Proof

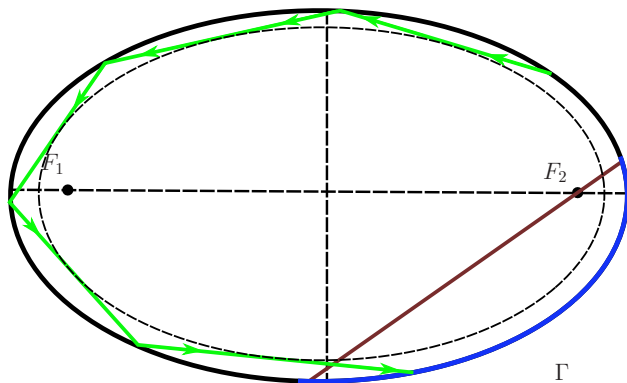


Figure: Elliptic caustic

Proof

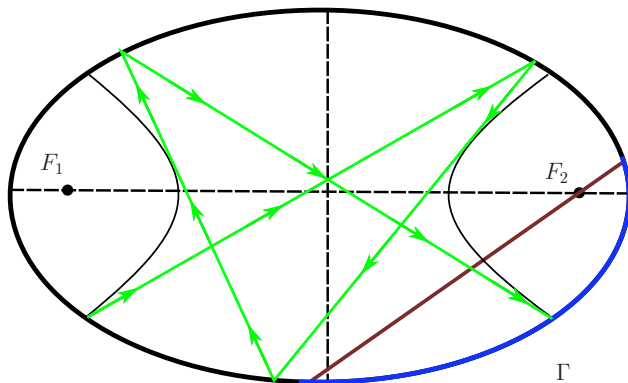


Figure: Hyperbolic caustic

Proof of the propagation of the support of the measures

Let $(\rho_1, \rho_2) \in \mathcal{H}^1 \times (\mathcal{H}^2 \cup \mathcal{G}^{2,+})$. Since $\rho_1 \in \mathcal{H}^1$, we write

$$P_i = -c_i^2 (D_x - \Lambda^+(x, y, D_y))(D_x - \Lambda^-(x, y, D_y)) + T(x, y, D_y)$$

$$P_i = -c_i^2 (D_x - \tilde{\Lambda}^-(x, y, D_y))(D_x - \tilde{\Lambda}^+(x, y, D_y)) + \tilde{T}(x, y, D_y)$$

Λ^\pm and $\tilde{\Lambda}^\pm$ are tangential pseudodifferential operators of order 1 and such that $\sigma_1(\Lambda^\pm) = \pm\sqrt{r}$ and $\sigma_1(\tilde{\Lambda}^\pm) = \pm\sqrt{r}$ and where T and \tilde{T} are tangential pseudodifferential operators of order $-\infty$. Suppose $\gamma_1^+ \cup \text{supp}(\mu_1) = \emptyset$,

$$c_1^2 Q_0(D_x u_1^k|_{x=0} - \Lambda^- u_1^k|_{x=0}) \rightarrow 0 \text{ in } L^2(Y).$$

Using the transmission conditions,

$$c_1^2 Q_0 \left(\left. \frac{D_x u_2^k}{c_2^2} \right|_{x=0} - \Lambda^- u_2^k|_{x=0} \right) \rightarrow 0 \text{ in } L^2(Y).$$

Suppose $\gamma_2^- \cup \text{supp}(\mu_2) = \emptyset$.

1 If $\rho_2 \in H^2$, we get the additional relation

$$Q_0 (D_x u_2^k|_{x=0} - c_2^2 \Lambda^- u_2^k|_{x=0}) \rightarrow 0 \text{ in } L^2(Y).$$

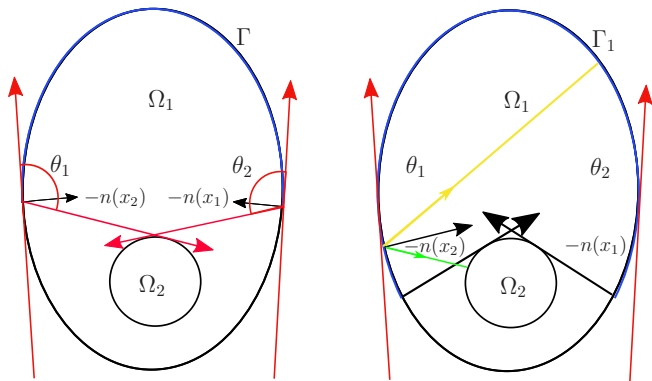
which allow us to conclude.

2 If $\rho_2 \in \mathcal{G}^{2,+}$, then we use

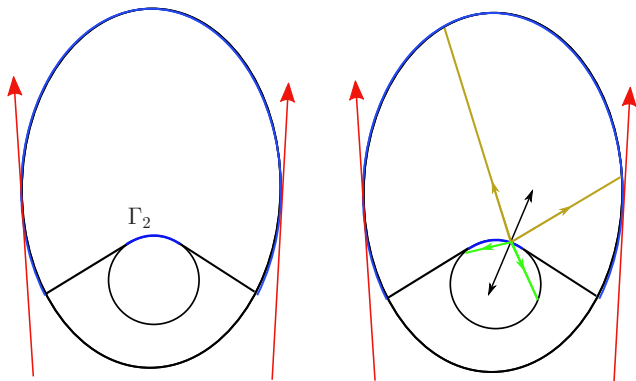
Lemma

$$\rho_2 \in \text{supp}(\mu_2) \Leftrightarrow \rho_2 \in \text{supp}(\text{meas}(u_2|_{x=0})) \cup \text{supp}(\text{meas}(D_{n_2} u_2|_{x=0})).$$

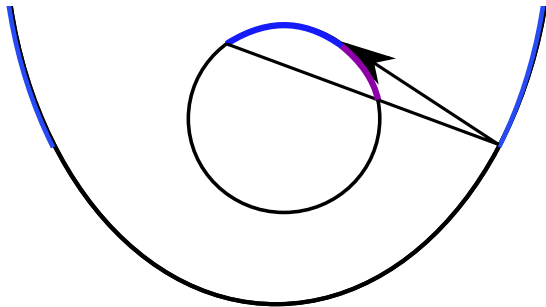
Step 1 : propagation of $\Gamma(x_0)$



Step 2 : creation of Γ_2



Step 3 : extension of Γ_2



Step 3 : extension of Γ_2

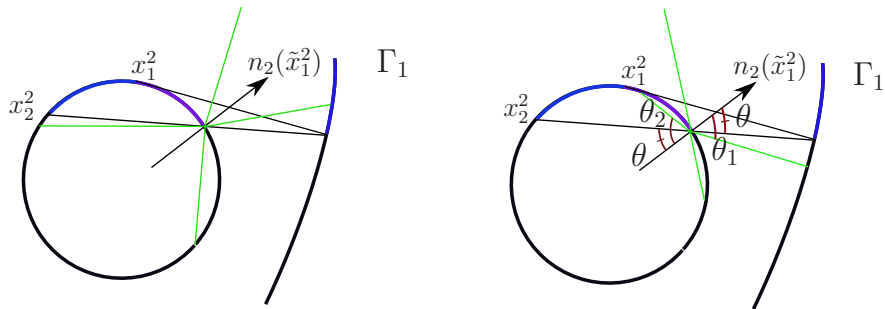


Figure: Left : γ_1^- and γ_1^+ intersects Γ_1 . Right : γ_1^- does not intersect Γ_1 but γ_2^+ hits the extension of Γ_2 .

Step 3 : extension of Γ_2

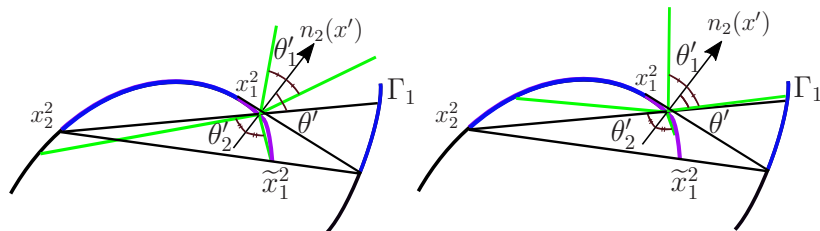
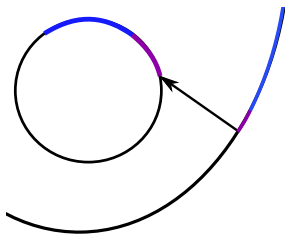


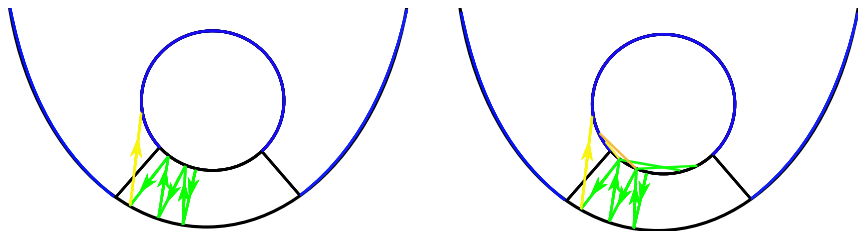
Figure: Left : γ_1^- and γ_1^+ intersects Γ_1 . Right : γ_1^- does not intersect Γ_1 but γ_2^+ hits the extension of Γ_2 .

In the second case, it may happen that γ_2^+ intersects the purple region once again. One iterates the same argument until two rays are observed.

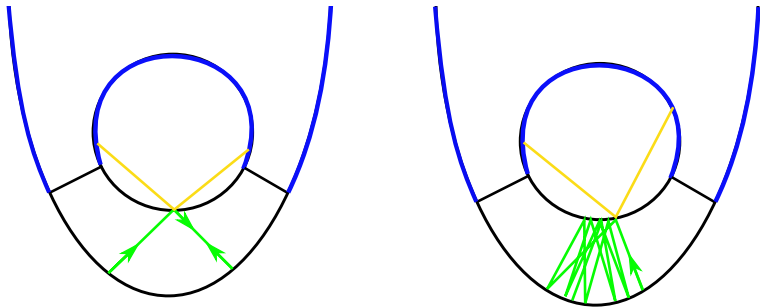
Step 3 1/2 : extension of Γ_1



Step 4 : Uniformly escaping geometry



Step 4 : non uniformly escaping geometry



Conclusion

- ① Higher dimension
- ② Ellipse case
- ③ Case where Γ is not connected

Commercial announcement

Conference in Madrid, end of February, Workshop on Microlocal analysis, kinetic equations and numerical analysis.

Scientific committee :

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Jean-Michel Coron

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