ANOMALOUS DIFFUSION FOR KINETIC EQUATIONS

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Goal : Diffusion approximation for

- The Linear Boltzmann equation
- The Fokker Planck equation

when the equilibria profile are power tail functions.

The linear Boltzmann equation

We consider a balance between an advection phenomenum and collisions between particles

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f).$$

where $f \geq 0, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^d, \quad v \in \mathbf{R}^d$
and the collision operator is given by

$$\begin{aligned} Q(f) &= \int_V \sigma(v,v')[M(v)f(t,x,v') - M(v')f(t,x,v)]dv' \\ &= \nu(v)f + K(f) \end{aligned}$$

Equilibrium: we define M by

$$Q(M) = 0 \quad \text{and} \quad \int_V M(x,v)dv = 1 \quad \forall x \in \mathbf{R}^d$$

Mass conservation: 1 belongs to the kernel of Q^*.

Diffusion approximation:

We define two small parameters

$$\begin{aligned} \varepsilon \quad &= \text{mean free path} \\ &= \text{average distance between two collisions.} \end{aligned}$$

$$1/\theta(\varepsilon) = \text{time scale.}$$

In the rescaled variable, the Boltzmann equation becomes

$$\theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon) \tag{1}$$

The classical diffusion approximation corresponds to the scaling

$$\theta(\varepsilon) = \varepsilon^2$$

Classical diffusion approximation

The solution to (1) is approximated by

$$f^0 = n(t,x)M(v)$$

where M is the equilibrium, or Mawellian profile
and the density satisfies a diffusion equation

$$\partial_t n - \nabla_x D \nabla_x n = 0 \quad \text{with} \quad D \sim \int \frac{v \otimes v}{\nu(v)} M dv$$

where ν is the collision frequency

$$\nu(v) = \int_V \sigma(v, v') dv'$$

Classical diffusion approximation: Hilbert expansion I

We write the expansion

$$f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + r^\varepsilon$$

with the different terms defined by the series of equation

$$\begin{aligned} Q(f^0) &= 0 \quad \Rightarrow f^0 = n(t,x)M(v) \\ Q(f^1) &= v \cdot \nabla_x f^0 \\ Q(f^2) &= v \cdot \nabla_x f^1 + \partial_t f^0 \end{aligned}$$

Anomalous cases

1- When the equilibrium is a power tail function, i.e.

$$M \sim \frac{1}{|v|^{d+\alpha}} \quad \text{with} \quad \alpha < 2$$

2- When the cross section is degenerated, $\nu \sim_0 |v|^{d+2+\beta}$

the diffusion coefficient obtained with $\theta = \varepsilon^2$ is no more finite.

ε^2 is not the right scaling.

Ad hoc scaling

We need to estimate the different term involved
- in the collision operator Q
- in the advection term $v \cdot \nabla_x f$.

For Boltzmann, if $M \sim \frac{1}{|v|^{d+\alpha}}$.

- For $\alpha > 2$, $\theta(\varepsilon) = \varepsilon^2$ classical diffusion
- For $\alpha = 2$, $\theta(\varepsilon) = \varepsilon^2 \log(\varepsilon)$ diffusion with a different scaling
- For $\alpha < 2$, $\theta(\varepsilon) = \varepsilon^\alpha$ fractional diffusion

Boltzmann : Hilbert method fractionnal diffusion I

Write the equation in Fourier variable in space

$$\partial_t \hat{f}(t,\xi,v) + iv\cdot\xi\hat{f}(t,\xi,v) = Q(\hat{f}).$$

then we reorganize the Hilbert expansion

$$\begin{aligned}
Q(\hat{f}^0) &= 0 \quad \Rightarrow \hat{f}^0 = n(t,x)M(v)\\
\nu\hat{f}^1 + i\varepsilon v\cdot\xi\hat{f}^1 &= \varepsilon v\cdot\nabla_x \hat{f}^0\\
Q(f^2) &= K(\hat{f}^1) + \varepsilon^\alpha \partial_t \hat{f}^0
\end{aligned}$$

Boltzmann : Hilbert method fractionnal diffusion II

By integrating the last equation with respect to velocity, we get

$$\begin{aligned}
\partial_t \hat{n} &= \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} - \int K(\hat{f}^1) dv = \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} \int \nu \hat{f}^1 dv \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} \int \nu \frac{i\varepsilon v \cdot \xi}{\nu + i\varepsilon v \cdot \xi} M dv \hat{n} \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} \int \nu \frac{i\varepsilon \frac{v}{\nu} \cdot \xi}{1 + i\varepsilon \frac{v}{\nu} \cdot \xi} \frac{1}{(1 + |v|^2)^{\frac{d+\alpha}{2}}} dv \hat{n}
\end{aligned}$$

which after a change of variable gives

$$\partial_t \hat{n} = |\xi|^\alpha \int \frac{\tilde{\nu}}{1 + i \frac{w}{\tilde{\nu}} \cdot \frac{\xi}{|\xi|}} \frac{1}{|w|^{d+\alpha}} dw \hat{n}$$

Which gives fractional diffusion.
[Mellet Mischler Mouhot], [Mellet] [Ben Abdallah Mellet P.]

Fundamental property : Poincaré inequality

The main properties that the equilibrium and the collision operator must satisfy are

- the profile M spans the kernel of Q.
- Drift condition $\int vMdv = 0$
- Convergence of f^ε thanks to Poincaré inequality
  $\Rightarrow$ spectral gap:

$$c||f^\varepsilon - nM||^2 \leq -\int_V Q(f^\varepsilon)\frac{f^\varepsilon}{M}$$

with an appropriate norm.

Fokker Planck equation

We consider the same kinetic equation

$$\partial_t f(t,x,v) + v \cdot \nabla f(t,x,v) = Q(f).$$

where $f \geq 0, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^d, \quad v \in R^d$
with the Fokker Planck collision operator

$$Q(f) = \nabla_v \cdot (M(\nabla_v \frac{f}{M})).$$

and we focus on the anomalous case when

$$M = \frac{1}{(1+|v|^2)^{\frac{d+\alpha}{2}}}$$

Particularity of Fokker Planck equation

- The Poisson equation $QH = v$ can be solved explicitly
- definition of the diffusion coefficient $\beta > d + 4$
- No Poincare Inequality.

The EDP method to get the classical diffusion depends on the power α and degenerates for the critical power.
[Nasreddine P.]

Fractionnal diffusion for Fokker Planck : spectral method

When $\alpha < 4$, work in progress in collaboration with G. Lebeau.
Consider the whole operator in dimension 1 and compute the first eigenvalue and eigenvector.

$\Longrightarrow$ the first eigenvalue will give the right time scaling
$\Longrightarrow$ the time scaling gives the right power of the fractionnal diffusion operator.

Construction of the first eigenvector, computation of the first eigenvalue : 1dimension

First, change of unknown $g\varepsilon = \frac{f^\varepsilon}{\sqrt{M}}$, the equation becomes

$$\partial_t g^\varepsilon + i\varepsilon v\xi g^\varepsilon - \partial_v^2 g^\varepsilon + W(v)g^\varepsilon = 0$$

Then, for any λ, we construct a solution in $L^2[0,\infty)$ to

$$(i\varepsilon v + W(v) - \partial_v^2)F^\varepsilon = \lambda F^\varepsilon$$

with $W(v) = \frac{\gamma}{(1+v^2)^2}[v^2(\gamma+1)-1] \sim_{v\sim\infty} \frac{\gamma(\gamma+1)}{v^2}$

Computation of the eigenvalue

We need to proceed to a reconnection at $v = 0$.
For that purpose, compute the derivative at $v = 0$ of the solution coming from the L^2 Airy like profile.
Then use the symmetries of the equation, parity of the equilibrium, to get a compatibility equation.
There exists a unique value of λ such that the compatibility equation is fulfilled.
This gives the eigenvalue.

Construction of the eigenvector : the different profiles

Lemma

The $L^2 51, \infty)$ *solution is given by*

$$H_\lambda(s) = s^{-\gamma} F_{+,\lambda}(s) + d(\lambda) F_{-,\lambda}(s) s^{\gamma+1}$$

where $d(\lambda)$ *is an holomorphic function for* $|\lambda| \leq \lambda_0$ *and* $d(0) \neq 0$*.*

When v large, the potential can be approximated by its asymptotic

$$(-\partial_v^2 + i\varepsilon v\xi - \lambda + \frac{\gamma(\gamma+1)}{v^2})g^\varepsilon = 0$$

When $v >> \varepsilon^{-\frac{1}{3}}$, then a good approximation of the equation is

$$(-\partial_v^2 + i\varepsilon v\xi - \lambda)g^\varepsilon = 0$$

then the solution is close to an Airy function.

Convergence toward limiting equation

We obtain the eigenvalue

$$\lambda(\varepsilon) = \kappa\varepsilon^{\frac{\alpha+1}{3}}\xi^{\frac{\alpha+1}{3}}$$

which gives a fractional diffusion equation for the density obtained formally by integrating the equation.
Momentum method with the eigenvector as a test function.
Need weak Poincare inequality.

Other models

Linear Boltzmann with degenerate cross sections for large velocities

Linearized Boltzmann equation or "linearized Fokker Planck"

Linearized Boltzmann on a bounded domain.

Stochastic differential equation

The characteristic equations corresponding to the kinetic equation is the following stochastic system

$$
\begin{aligned}
dv_t &= \sqrt{2}\,dB_t - \frac{\nabla\omega}{\omega}(v_t)\,dt \\
dx_t &= v_t\,dt\,,
\end{aligned}
$$

where $\omega = (1+|v|^2)^{\frac{d+\alpha}{2}}$.
We want to characterize the asymptotic behavior of the

$$
\left(\varepsilon S_t = \varepsilon \int_0^{\frac{t'}{\theta(\varepsilon)}} v_s ds, v_{\frac{t'}{\theta(\varepsilon)}}\right).
$$

Scaling correspondig to Fokker Planck : critical case

Theorem ([Cattiaux, Nasreddine, P.])

If $\alpha = 4$, $\exists \kappa_{\alpha,d} > 0$ *such that*

$$S_t / t \log t \to_{t \to \infty} \mathcal{N}(\kappa_{\alpha,d}\, Id)$$

a centered gaussian random vector with covariance matrix $\kappa_{\alpha,d}\, Id$.

- Variance breacking.
- Anomalous time scaling.
- Diffusion equation for the corresponding law.