S. Margherita di Pula, Sardinia,

October 2, 2017

Random data nonlinear wave equations I

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The Hamiltonian structure

- Consider the cubic defocusing wave equation

$$\partial_t^2 u - \Delta u + u^3 = 0, \tag{1}$$

where $u : \mathbb{R} \times \mathbb{T}^d \longrightarrow \mathbb{R}$.

- We can rewrite (1) as the first order system

$$\partial_t u = v, \quad \partial_t v = \Delta u - u^3. \tag{2}$$

- One can rewrite (2) as

$$\partial_t u = \frac{\delta E}{\delta v}, \quad \partial_t v = -\frac{\delta E}{\delta u},$$

where

$$E(u,v) = \frac{1}{2}\int_{\mathbb{T}^d} \left(|\nabla u|^2 + v^2\right) + \frac{1}{4}\int_{\mathbb{T}^d} u^4 \,. \tag{3}$$

- As a consequence $E(u,v)$ is a first integral for (2).

The classical global well-posedness

• In view of the Hamiltonian structure and the properties of the linear wave equation, a natural phase space for

$$\partial_t u = v, \quad \partial_t v = \Delta u - u^3 \tag{4}$$

is $\mathcal{H}^s(\mathbb{T}^d) \equiv H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$, where $H^s(\mathbb{T}^d)$ is the classical Sobolev space of order $s \in \mathbb{R}$. One has

$$\|(u,v)\|_{\mathcal{H}^1(\mathbb{T}^d)} \approx \int_{\mathbb{T}^d} \left(|\nabla u|^2 + u^2 + v^2 \right).$$

Theorem 1 (classical)
Let $d \leq 3$. Then (4) is globally well-posed in $\mathcal{H}^s(\mathbb{T}^d)$, $s \geq 1$. More precisely, for every $(u_0, v_0) \in \mathcal{H}^s(\mathbb{T}^d)$ there is a unique solution of (4) with initial data $(u,v)|_{t=0} = (u_0, v_0)$ in $C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^d))$.

• The result still holds true for $d = 4$ thanks to the concentration compactness method. For $d > 5$ the problem has some similarities with the 3d Navier-Stokes global regularity problem ($\frac{d-2}{2} > 1$).

On the proof of the classical global well-posedness

• Using the variation of the constants method, we obtain that the solutions of non homogeneous problem

$$(\partial_t^2 - \Delta)u = F(t,x), \quad u(0,x) = 0, \quad \partial_t u(0,x) = 0 \tag{5}$$

are given by

$$u(t) = \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}((F(\tau))d\tau\,.$$

• As a consequence, we obtain that the solution of the non homogeneous problem (5) is one derivative smoother than the source term F. Namely, for $T > 0$ we have the "wave-regularity" bound

$$\|u\|_{L^\infty([0,T];H^{s+1}(\mathbb{T}^d))} \leq C\|F\|_{L^1([0,T];H^s(\mathbb{T}^d))}\,. \tag{6}$$

On the proof of the classical global well-posedness (sequel)

- Let $d = 3$. Using the "wave-regularity" bound in the context of

$$(\partial_t^2 - \Delta)u = -u^3, \quad u(0,x) = u_0, \quad \partial_t u(0,x) = u_1, \tag{7}$$

we obtain that a "solution" of (7) satisfies

$$\|u\|_{L^\infty([0,T];H^1(\mathbb{T}^3))} \leq C(\|u_0\|_{H^1(\mathbb{T}^3)} + \|u_1\|_{L^2(\mathbb{T}^3)}) + C\|u^3\|_{L^1([0,T];L^2(\mathbb{T}^3))}.$$

- The key point is to use the $3d$ Sobolev embedding and to write

$$\|u^3\|_{L^2(\mathbb{T}^3)} = \|u\|^3_{L^6(\mathbb{T}^3)} \leq C\|u\|^3_{H^1(\mathbb{T}^3)}$$

and thus

$$\|u^3\|_{L^1([0,T];L^2(\mathbb{T}^3))} \leq CT\|u\|^3_{L^\infty([0,T];H^1(\mathbb{T}^3))}.$$

- Therefore, we "closed the circle" which yields the local well-posedness.
- The global well-posedness results for the energy conservation law.

Low regularity well-posedness

• Consider again

$$\partial_t u = v, \quad \partial_t v = \Delta u - u^3. \tag{8}$$

• Thanks to the work of Ginibre-Velo, Lindblad-Sogge, Kenig-Ponce-Vega, Gallagher-Planchon, we have the following result.

Theorem 2

Let $d = 3$. Then (8) is locally well-posed in $\mathcal{H}^s(\mathbb{T}^d)$, $s > 1/2$ and globally well-posed in $\mathcal{H}^s(\mathbb{T}^d)$, $s > 3/4$.

• The regularity restriction $s > 1/2$ is related to the scaling symmetry: if (u, v) is a solution of (8) then so is

$$u_\lambda(t,x) = \lambda u(\lambda t, \lambda x),\ v_\lambda(t,x) = \lambda^2 v(\lambda t, \lambda x), \quad \lambda > 0. \tag{9}$$

In $3d$, the $\dot{\mathcal{H}}^{1/2}$ norm is invariant under (9).

• A result of similar spirit holds for $d = 2$. One should also take into account another invariance ...

• **From now on in the today lecture, we fix $d = 3$.**

On the proof of the low regularity well-posedness

• The new point with respect to the classical well-posedness is the use of **the Strichartz estimates**. A typical Strichartz estimates is

$$\left\| e^{\pm it\sqrt{-\Delta}}(f) \right\|_{L^2([0,1];L^\infty(\mathbb{T}^3))} \leq C\|f\|_{H^s(\mathbb{T}^3)}, \quad s > 1. \tag{10}$$

Therefore, we obtain that for $f \in H^s(\mathbb{T}^3)$, $s > 1$, the function $e^{it\sqrt{-\Delta}}(f)$ which is a priori defined as an element of $C([0,1]; H^s(\mathbb{T}^3))$ has the remarkable property that

$$e^{it\sqrt{-\Delta}}(f) \in L^\infty(\mathbb{T}^3)$$

for almost every $t \in [0,1]$.

• Recall that the Sobolev embedding requires the condition $s > 3/2$ in order to ensure that an $H^s(\mathbb{T}^3)$ function is in $L^\infty(\mathbb{T}^3)$.

• Therefore, one may wish to see (10) as an almost sure in t improvement (with 1/2 derivative) of the Sobolev embedding

$$H^\sigma(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3), \quad \sigma > 3/2$$

under the evolution of the linear wave equation.

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$

• Thanks to the work by Lebeau, Christ-Colliander-Tao, Burq-Gérard-Tz., Xia, we have the following result.

Theorem 3

Let us fix $s \in (0, 1/2)$ *and* $(u_0, v_0) \in \mathcal{H}^s(\mathbb{T}^3)$. *Then there exists a sequence* $(u_n(t,x), v_n(t,x))_{n=1}^{\infty}$ *of* $C(\mathbb{R}; C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3))$ *functions such that*

$$\partial_t u_n = v_n, \quad \partial_t v_n = \Delta u_n - u_n^3$$

with

$$\lim_{n\to\infty} \|(u_n(0) - u_0, v_n(0) - v_0)\|_{\mathcal{H}^s(\mathbb{T}^3)} = 0$$

but for every $T > 0$,

$$\lim_{n\to\infty} \|(u_n(t), v_n(t))\|_{L^\infty([0,T];\mathcal{H}^s(\mathbb{T}^3))} = \infty.$$

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$ (sequel)

- It suffices to prove the statement for $(u_0, u_1) \in C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$.
- Consider

$$(\partial_t^2 - \Delta)u + u^3 = 0 \tag{11}$$

subject to initial conditions

$$(u_0(x) + \kappa_n n^{\frac{3}{2}-s}\varphi(nx), u_1(x)), \qquad n \gg 1\,, \tag{12}$$

where φ is a nontrivial bump function on $\mathbb{R}^3$ and

$$\kappa_n \equiv [\log(n)]^{-\delta_1},$$

with $\delta_1 > 0$, small, to be fixed.

- (11)-(12) has a unique global smooth solution which we denote by u_n.
- Moreover $u_n \in C(\mathbb{R}; C^\infty(\mathbb{T}^3))$ thanks the propagation of the higher Sobolev regularity and the Sobolev embeddings.

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$ (sequel)

• Consider the ODE

$$V'' + V^3 = 0, \quad V(0) = 1,\ V'(0) = 0. \tag{13}$$

• The Cauchy problem (13) has a global smooth (non constant) solution $V(t)$ which is periodic in time.

• Denote by v_n the solution of

$$\partial_t^2 v_n + v_n^3 = 0, \quad (v_n(0), \partial_t v_n(0)) = (\kappa_n n^{\frac{3}{2}-s}\varphi(nx), 0).$$

Clearly

$$v_n(t,x) = \kappa_n n^{\frac{3}{2}-s}\varphi(nx) V\Big(t\kappa_n n^{\frac{3}{2}-s}\varphi(nx)\Big).$$

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$ (sequel)

Let

$$t_n \equiv [\log(n)]^{\delta_2} n^{-(\frac{3}{2}-s)}, \quad \delta_2 > \delta_1$$

Then, we have the following bounds for $t \in [0, t_n]$,

$$\|v_n(t,\cdot)\|_{H^\sigma(\mathbb{T}^3)} \le C\kappa_n (t_n \kappa_n n^{\frac{3}{2}-s})^\sigma n^{\sigma - s}, \ \sigma \ge 0 \tag{14}$$

and

$$\|v_n(t,\cdot)\|_{L^\infty(\mathbb{T}^3)} \le C n^{\frac{3}{2}-s}. \tag{15}$$

Most importantly, there exists $n_0 \gg 1$ such that for $n \ge n_0$,

$$\|v_n(t_n,\cdot)\|_{H^s(\mathbb{T}^3)} \ge C\kappa_n (t_n \kappa_n n^{\frac{3}{2}-s})^s = C[\log(n)]^{-(s+1)\delta_1 + s\delta_2}. \tag{16}$$

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$ (sequel)

• Consider the semi-classical energy

$$E_n(u) \equiv n^{-(1-s)}\Big(\|\partial_t u\|^2_{L^2(\mathbb{T}^3)} + \|\nabla u\|^2_{L^2(\mathbb{T}^3)}\Big)^{\frac{1}{2}} + n^{-(2-s)}\Big(\|\partial_t u\|^2_{H^1(\mathbb{T}^3)} + \|\nabla u\|^2_{H^1(\mathbb{T}^3)}\Big)^{\frac{1}{2}}.$$

• The key point is that for very small times u_n and v_n are close with respect to E_n. But these small times are long enough to get the needed amplification of the H^s norm ! This amplification is a phenomenon only related to the solution of the dispersionless model.

Lemma 4

There exist $\varepsilon > 0$, $\delta_2 > 0$ and $C > 0$ such that for $\delta_1 < \delta_2$, if we set

$$t_n \equiv [\log(n)]^{\delta_2} n^{-(\frac{3}{2}-s)}$$

then for every $n \gg 1$, every $t \in [0, t_n]$, $E_n(u_n(t) - v_n(t)) \leq Cn^{-\varepsilon}$. As a consequence,

$$\|u_n(t) - v_n(t)\|_{H^s(\mathbb{T}^3)} \leq Cn^{-\varepsilon}.$$

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$ (sequel)

• Set $w_n = u_n - v_n$ and

$$e_n(w_n(t)) \equiv \sup_{0\leq\tau\leq t} E_n(w_n(\tau))\,.$$

Then w_n solves the equation

$$(\partial_t^2 - \Delta)w_n = \Delta v_n - 3v_n^2 w_n - 3v_n w_n^2 - w_n^3,$$

with initial data

$$(w_n(0,\cdot), \partial_t w_n(0,\cdot)) = (u_0, u_1)\,.$$

• Using the wave-regularity bounds for the equation satisfied by w_n together with the bounds on the explicit object v_n, we get that for $t \in [0, t_n]$,

$$\frac{d}{dt}(e_n(w_n(t))) \leq C[\log(n)]^{3\delta_2} n + C[\log(n)]^{2\delta_2} n^{\frac{3}{2}-s} e_n(w_n(t))$$

and consequently

$$\frac{d}{dt}\Big(e^{-Ct[\log(n)]^{2\delta_2} n^{\frac{3}{2}-s}} e_n(w_n(t))\Big) \leq C[\log(n)]^{3\delta_2}\, n\, e^{-Ct[\log(n)]^{2\delta_2} n^{\frac{3}{2}-s}}\,.$$

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$ (sequel)

• An integration of the last estimate gives that for $t \in [0, t_n]$,

$$
\begin{aligned}
e_n(w_n(t)) &\leq C\Big(n^{-(1-s)} + [\log(n)]^{\delta_2} n^{s-\frac{1}{2}}\Big) e^{Ct[\log(n)]^{2\delta_2} n^{\frac{3}{2}-s}} \\
&\leq C\Big(n^{-(1-s)} + [\log(n)]^{\delta_2} n^{s-\frac{1}{2}}\Big) e^{C[\log(n)]^{3\delta_2}} .
\end{aligned}
$$

(one should see δ_2 as $3\delta_2 - 2\delta_2$ and $s - 1/2$ as $1 - (3/2 - s)$).
• Since $s < 1/2$, by taking $\delta_2 > 0$ small enough, we obtain that there exists $\varepsilon > 0$ such that for $t \in [0, t_n]$,

$$
E_n(w_n(t)) \leq C n^{-\varepsilon} .
$$

Ill-posedness in $\mathcal{H}^s$ for $s < 1/2$ (sequel)

- The estimate

$$E_n(w_n(t)) \leq C n^{-\varepsilon}$$

in particular implies that one has for $t \in [0, t_n]$,

$$\|\partial_t w_n(t,\cdot)\|_{L^2(\mathbb{T}^3)} + \|\nabla w_n(t,\cdot)\|_{L^2(\mathbb{T}^3)} \leq C n^{1-s-\varepsilon}\,. \tag{17}$$

- We next estimate $\|w_n(t,\cdot)\|_{L^2}$. We may write for $t \in [0, t_n]$,

$$\|w_n(t,\cdot)\|_{L^2(\mathbb{T}^3)} = \|\int_0^t \partial_t w_n(\tau,\cdot)d\tau\|_{L^2(\mathbb{T}^3)} \leq c t_n \sup_{0\leq\tau\leq t} \|\partial_t w_n(\tau,\cdot)\|_{L^2(\mathbb{T}^3)}\,.$$

Thanks to (17) and the definition of t_n, we get

$$\|w_n(t,\cdot)\|_{L^2(\mathbb{T}^3))} \leq C[\log(n)]^{\delta_2} n^{-(\frac{3}{2}-s)} n^{1-s} n^{-\varepsilon}\,.$$

Therefore, since $s < 1/2$,

$$\|w_n(t,\cdot)\|_{L^2(\mathbb{T}^3)} \leq C n^{-s-\varepsilon}\,. \tag{18}$$

- An interpolation between (17) and (18) yields

$$\|u_n(t) - v_n(t)\|_{H^s(\mathbb{T}^3)} \leq C n^{-\varepsilon}\,.$$

- This yields the claimed ill-posedness result.

Probabilistic well-posedness in $\mathcal{H}^s$ for $s \in (0,1/2)$

• Consider the Cauchy problem for the cubic defocusing wave equation

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad (u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)\,. \qquad (19)$$

• We have seen that (19) is well-posed (at least locally in time) in $\mathcal{H}^s(\mathbb{T}^3)$, $s > 1/2$. Moreover, for $s \in (0,1/2)$ the Cauchy problem (19) is ill-posed in $\mathcal{H}^s(\mathbb{T}^3)$.

• One may however ask whether some sort of well-posedness for (19) survives for $s < 1/2$ (a priori even the existence is not known). We will show that this is indeed the case, if we accept to "randomise" the initial data. This means that we will endow $\mathcal{H}^s(\mathbb{T}^3)$, $s \in (0,1/2)$ with suitable non degenerate probability measures and we will show that the Cauchy problem (19) is well-posed in a suitable sense for initial data (u_0, u_1) on a set of a full measure.

Description of the measures describing the initial data

Starting from $(u_0, u_1) \in \mathcal{H}^s$ given by their Fourier series

$$u_j(x) = a_j + \sum_{n \in \mathbb{Z}^3_\star} \Big(b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x) \Big), \quad j = 0, 1,$$

we define u_j^ω by

$$u_j^\omega(x) = \alpha_j(\omega) a_j + \sum_{n \in \mathbb{Z}^3_\star} \Big(\beta_{n,j}(\omega) b_{n,j} \cos(n \cdot x) + \gamma_{n,j}(\omega) c_{n,j} \sin(n \cdot x) \Big),$$

where $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$, $n \in \mathbb{Z}^3_\star$, $j = 0, 1$ is a sequence of real random variables on a probability space $(\Omega, p, \mathcal{F})$. We assume that the random variables $(\alpha_j, \beta_{n,j}, \gamma_{n,j})_{n \in \mathbb{Z}^3_\star, j=0,1}$ are independent identically distributed real random variables with a joint distribution θ satisfying

$$\exists\, c > 0, \quad \forall\, \gamma \in \mathbb{R}, \quad \int_{-\infty}^{\infty} e^{\gamma x} d\theta(x) \leq e^{c\gamma^2}. \tag{20}$$

Typical examples satisfying (20) are the standard gaussians and the Bernoulli measures.

Description of the measures (sequel)

• For fixed $(u_0, u_1) \in \mathcal{H}^s$, the map

$$\omega \in \Omega \longmapsto (u_0^\omega, u_1^\omega) \in \mathcal{H}^s \tag{21}$$

is a measurable map from $(\Omega, \mathcal{F})$ to $\mathcal{H}^s$ endowed with the Borel sigma algebra since the partial sums from a Cauchy sequence in $L^2(\Omega; \mathcal{H}^s)$.
• Thus the map (21) endows the space $\mathcal{H}^s(\mathbb{T}^3)$ with a probability measure. Let us denote this measure by $\mu_{(u_0,u_1)}$. Then

$$\forall\, A \subset \mathcal{H}^s,\ \mu_{(u_0,u_1)}(A) = p(\omega \in \Omega\,:\,(u_0^\omega, u_1^\omega) \in A).$$

• Denote by $\mathcal{M}^s$ the set of measures obtained following this construction :

$$\mathcal{M}^s = \bigcup_{(u_0,u_1)\in\mathcal{H}^s} \{\mu_{(u_0,u_1)}\}\,.$$

Properties of the measures

• For any $s' > s$, if $(u_0, u_1) \notin \mathcal{H}^{s'}$, then

$$\mu_{(u_0,u_1)}(\mathcal{H}^{s'}) = 0\,.$$

In other words, the randomisation (21) does not regularise in the scale of the L^2-based Sobolev spaces (this fact is obvious for the Bernoulli randomisation).

• If (u_0, u_1) have all their Fourier coefficients different from zero and if the measure θ charges all open sets of $\mathbb{R}$ then the support of $\mu_{(u_0,u_1)}$ is $\mathcal{H}^s$. In other words, under these assumptions, for any $(w_0, w_1) \in \mathcal{H}^s$ and any $\varepsilon > 0$,

$$\mu_{(u_0,u_1)}(\{(v_0, v_1) \in \mathcal{H}^s \,:\, \|(w_0, w_1) - (v_0, v_1)\|_{\mathcal{H}^s} < \varepsilon\}) > 0. \qquad (22)$$

Yet in other words, any set of full $\mu_{(u_0,u_1)}$ measure is dense in $\mathcal{H}^s$.

Statement of the results (joint work with Nicolas Burq)

• Let $S(t)(v_0, v_1)$ be the solution of the linear wave equation with data (v_0, v_1).

Theorem 5 (existence and uniqueness)

Let us fix $s \in [0, 1)$ and $\mu \in \mathcal{M}^s$. Then, there exists a full μ measure set $\Sigma \subset \mathcal{H}^s(\mathbb{T}^3)$ such that for every $(v_0, v_1) \in \Sigma$, there exists a unique global solution v of of the non linear wave equation

$$(\partial_t^2 - \Delta)v + v^3 = 0, \quad (v(0), \partial_t v(0)) = (v_0, v_1) \tag{23}$$

satisfying

$$(v(t), \partial_t v(t)) \in \Big(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)\Big) + C(\mathbb{R}_t; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)).$$

Furthermore, if we denote by

$$\Phi(t)(v_0, v_1) \equiv (v(t), \partial_t v(t))$$

the flow thus defined, the set Σ is invariant by the map $\Phi(t)$, namely

$$\Phi(t)(\Sigma) = \Sigma, \qquad \forall\, t \in \mathbb{R}.$$

Remark. The result also (trivially) holds for $s \geq 1$.

Theorem 6 (quantitative bounds)
Let us fix $s \in (0,1)$ and $\mu \in \mathcal{M}^s$. Let Σ be the set constructed in the previous theorem. Then for every $\varepsilon > 0$ there exist $C, \delta > 0$ such that for every $(v_0, v_1) \in \Sigma$, there exists $M > 0$ such that the global solution of the nonlinear wave equation, constructed in the previous theorem satisfies

$$v(t) = S(t)\Pi_0^{\perp}(v_0, v_1) + w(t),$$

with

$$\|(w(t), \partial_t w(t))\|_{\mathcal{H}^1(\mathbb{T}^3)} \leq C(M + |t|)^{\frac{1-s}{s}+\varepsilon}$$

and

$$\mu((v_0, v_1) \, : \, M > \lambda) \leq Ce^{-\lambda^{\delta}}.$$

- $\Pi_0^{\perp}$ is the projection on the $\neq 0$ Fourier modes.

Statement of the results (sequel)

Theorem 7 (unique limit of smooth solutions for $s > 1/2$)
Let $s \in (1/2, 1)$. With the notations of the statement of the previous theorem, let us fix an initial datum $(v_0, v_1) \in \Sigma$ with a corresponding global solution $v(t)$. Let $(v_{0,n}, v_{1,n})_{n=1}^{\infty}$ be a sequence of $\mathcal{H}^1(\mathbb{T}^3)$ such that

$$\lim_{n\to\infty} \|(v_{0,n} - v_0, v_{1,n} - v_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} = 0 .$$

Denote by $v_n(t)$ the solution of the cubic defocusing wave equation with data $(v_{0,n}, v_{1,n})$ defined in the classical global well-posedness result. Then for every $T > 0$,

$$\lim_{n\to\infty} \|(v_n(t) - v(t), \partial_t v_n(t) - \partial_t v(t))\|_{L^\infty([0,T];\mathcal{H}^s(\mathbb{T}^3))} = 0 .$$

Remark. Thanks to the ill-posedness result, we know that for $s \in (0, 1/2)$ the result of Theorem 7 cannot hold true ! We only have a partial statement.

Statement of the results (sequel)

Theorem 8 (unique limit of particular smooth solutions for $s < 1/2$) *Let $s \in (0, 1/2)$. With the notations of the statement of the global existence result, let us fix an initial datum $(v_0, v_1) \in \Sigma$ with a corresponding global solution $v(t)$. Let $(v_{0,n}, v_{1,n})_{n=1}^{\infty}$ be the sequence of $C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$ defined by the usual regularisation by convolution, i.e.*

$$v_{0,n} = v_0 \star \rho_n, \quad v_{1,n} = v_1 \star \rho_n\,,$$

where $(\rho_n)_{n=1}^{\infty}$ is an approximate identity. Denote by $v_n(t)$ the solution of the cubic defocusing wave equation with data $(v_{0,n}, v_{1,n})$ defined by the classical global well-posedness result. Then for every $T > 0$,

$$\lim_{n\to\infty} \|(v_n(t) - v(t), \partial_t v_n(t) - \partial_t v(t))\|_{L^\infty([0,T];\mathcal{H}^s(\mathbb{T}^3))} = 0\,.$$

Remark

• The choice of the particular regularisation of the initial data in the previous theorem is of key importance.

• It would be interesting to classify the ”admissible type of regularisations” allowing to get a statement such as the previous theorem.

Statement of the results (sequel)

Theorem 9 (conditioned continuous dependence)
Let us fix $s \in (0,1)$, *let* $A > 0$ *and let* $B_A \equiv (V \in \mathcal{H}^s : \|V\|_{\mathcal{H}^s} \leq A)$ *be the closed ball of radius* A *centered at the origin of* $\mathcal{H}^s$ *and let* $T > 0$. *Let* $\mu \in \mathcal{M}^s$ *and suppose that* θ *(the law of our random variables) is symmetric. Let* $\Phi(t)$ *be the flow of the cubic wave equations defined* μ *almost everywhere. Then for* $\varepsilon, \eta > 0$, *we have the bound*

$$\mu \otimes \mu\Big((V, V') \in \mathcal{H}^s \times \mathcal{H}^s : \|\Phi(t)(V) - \Phi(t)(V')\|_{X_T} > \varepsilon \;\Big|\; \|V - V'\|_{\mathcal{H}^s} < \eta \text{ and } (V, V') \in B_A \times B_A \Big) \leq g(\varepsilon, \eta),$$

where $X_T \equiv (C([0,T]; \mathcal{H}^s) \cap L^4([0,T] \times \mathbb{T}^3)) \times C([0,T]; H^{s-1})$ *and* $g(\varepsilon, \eta)$ *is such that*

$$\lim_{\eta \to 0} g(\varepsilon, \eta) = 0, \qquad \forall \varepsilon > 0.$$

Moreover, if for $s \in (0, 1/2)$ we assume in addition that the support of μ is the whole $\mathcal{H}^s$ (which is true iff in the definition of the measure μ, we have $a_i, b_{n,j}, c_{n,j} \neq 0, \forall n \in \mathbb{Z}^d$ and the support of the distribution function of the random variables is $\mathbb{R}$), then there exists $\varepsilon > 0$ such that for every $\eta > 0$

$$\mu \otimes \mu\Big((V, V') \in \mathcal{H}^s \times \mathcal{H}^s \, : \|\Phi(t)(V) - \Phi(t)(V')\|_{X_T} > \varepsilon \;\Big|\; \|V - V'\|_{\mathcal{H}^s} < \eta \text{ and } (V, V') \in B_A \times B_A \Big) > 0.$$

Extension to more general nonlinearities

• In the remarkable subsequent work by Oh-Pocovnicu it is shown that the results can be extended to the energy critical equation

$$(\partial_t^2 - \Delta)v + v^5 = 0$$

with data being a typical element with respect to $\mu \in \mathcal{M}^s$, $s > 1/2$.

• This equation is H^1 critical and the proof relies on a much more complicated deterministic analysis (such as the concentration compactness ideas) and also on a significant extension of the probabilistic energy bound used in the proof of our result.

Probabilistic Strichartz estimates, soft version

Theorem 10

Let us fix $s \in (0,1)$ *and let* $\mu \in \mathcal{M}^s$. *Then for every* $T > 0$ *and* $p_1 \in [1, \infty)$, $p_2 \in [2, \infty]$,

$$\|S(t)(u_0, u_1)\|_{L^{p_1}([0,T];L^{p_2}(\mathbb{T}^3))} < \infty, \quad \mu - \text{almost surely.}$$

• Using the Picard iteration scheme, for small times depending on (u_0, u_1), we can hope to represent the solution of the nonlinear wave equation as

$$u = \sum_{j=1}^{\infty} Q_j(u_0, u_1),$$

where Q_j is homogeneous of order j in (u_0, u_1).

The first non trivial Picard iteration

We have that

$$\begin{aligned} Q_1(u_0,u_1) &= S(t)(u_0,u_1), \\ Q_2(u_0,u_1) &= 0, \\ Q_3(u_0,u_1) &= -\int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}\Big(S(\tau)(u_0,u_1)\Big)^3 d\tau, \end{aligned}$$

etc.

• We have that μ a.s. $Q_1 \notin H^\sigma$ for $\sigma > s$. However, using the probabilistic Strichartz estimates, we have that for $T > 0$,

$$\|Q_3(u_0,u_1)\|_{L^\infty_T H^1(\mathbb{T}^3)} \lesssim \|S(t)(u_0,u_1)\|^3_{L^3_T L^6(\mathbb{T}^3)} < \infty, \quad \mu-\text{almost surely.}$$

Therefore the second non trivial term in the formal expansion defining the solution is μ a.s. more regular than the initial data !

The strategy

• The strategy will therefore be to write the solution of the nonlinear wave equation as

$$u = Q_1(u_0, u_1) + v,$$

where $v \in H^1$ and solve the equation for v by a deterministic method.

• In the case of the cubic nonlinearity the deterministic analysis used to solve the equation for v is particularly simple, it is in fact very close to the analysis in the proof of the classical $\mathcal{H}^1$ well-posedness result. For more complicated problems the analysis of the equation for v could involve more advanced deterministic arguments.

Remarks

• In principle the argument should not be particularly restricted to Q_3. One can imagine situations when for some $m > 3$, Q_m is the first element in the expansion whose regularity fits well in a deterministic analysis. Then we can equally well look for the solutions under the form

$$u = \sum_{j=1}^{m-1} Q_j(u_0, u_1) + v,$$

and treat v by a deterministic analysis. This is precisely done in the work of Hairer and others on singular stochastic PDE. In these works m is finite.

• In a very recent work by Oh-Tz.-Wang, on the invariance of the white noise for 4th order NLS, we have an exemple where an infinite expansion is needed.

Probabilistic Strichartz estimates, complete version.

Theorem 11

Let us fix $s \in (0,1)$ *and let* $\mu \in \mathcal{M}^s$ *be induced via the map*

$$\omega \in \Omega \longmapsto (u_0^\omega, u_1^\omega) \in \mathcal{H}^s$$

from the couple $(u_0, u_1) \in \mathcal{H}^s$*. Let us also fix* $\sigma \in (0, s]$, $2 \leq p_1 < +\infty$, $2 \leq p_2 \leq +\infty$ *and* $\delta > 1 + \frac{1}{p_1}$*. Then there exists a positive constant* C *such that for every* $p \geq 2$,

$$\Big\| \|\langle t \rangle^{-\delta} S(t)(v_0, v_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} \Big\|_{L^p(\mu)} \leq C\sqrt{p} \|(u_0, u_1)\|_{\mathcal{H}^\sigma(\mathbb{T}^3)}$$

As a consequence for every $T > 0$ *and* $p_1 \in [1, \infty)$, $p_2 \in [2, \infty]$,

$$\|S(t)(v_0, v_1)\|_{L^{p_1}([0,T]; L^{p_2}(\mathbb{T}^3))} < \infty, \quad \mu-\text{almost surely.}$$

Moreover, there exist two positive constants C *and* c *such that for every* $\lambda > 0$,

$$\mu\Big((v_0, v_1) \in \mathcal{H}^s : \|\langle t \rangle^{-\delta} S(t)(v_0, v_1)\|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} > \lambda \Big) \leq$$

$$C \exp\Big(- \frac{c\lambda^2}{\|(u_0, u_1)\|^2_{\mathcal{H}^\sigma(\mathbb{T}^3)}} \Big).$$

Comments

• The probabilistic Strichartz estimate, applied for $p_2 = \infty$ displays an improvement of 3/2 derivatives with respect to the Sobolev embedding which is stronger than the improvement obtained by the (deterministic) Strichartz estimates, we already discussed.
• The proof of the probabilistic Strichartz estimate exploits the random oscillations of the initial data while the proof of the deterministic Strichartz estimates exploits in a crucial manner the time oscillations of $S(t)$.
• In the proof of the probabilistic Strichartz estimates, we simply neglect these times oscillations.
• In some other situations, as for instance the work of Oh-Pocovnicu, the work on invariant measures for Benjamin-Ono or the derivative nonlinear Schrödinger equations etc. one exploits the regularisation effects coming both from the random and the time oscillations.
• The "art" is to understand how to make "collaborate" in the best way the random oscillations of the data and the time oscillations induced by the free evolution.

Lemma 12 (Khinchin inequality)
Let $(l_n(\omega))_{n=1}^{\infty}$ be a sequence of real, independent random variables with associated sequence of distributions $(\theta_n)_{n=1}^{\infty}$. Assume that θ_n satisfy the property

$$\exists\, c > 0 \, : \, \forall\, \gamma \in \mathbb{R}, \forall\, n \geq 1, \Big| \int_{-\infty}^{\infty} e^{\gamma x} d\theta_n(x) \Big| \leq e^{c\gamma^2} .$$

Then there exists $\alpha > 0$ such that for every $\lambda > 0$, every sequence $(c_n)_{n=1}^{\infty} \in l^2$ of real numbers,

$$p\Big(\omega \, : \, \Big| \sum_{n=1}^{\infty} c_n l_n(\omega) \Big| > \lambda \Big) \leq 2e^{-\frac{\alpha\lambda^2}{\sum_n c_n^2}} .$$

As a consequence there exists $C > 0$ such that for every $p \geq 2$, every $(c_n)_{n=1}^{\infty} \in l^2$,

$$\Big\| \sum_{n=1}^{\infty} c_n l_n(\omega) \Big\|_{L^p(\Omega)} \leq C\sqrt{p} \Big(\sum_{n=1}^{\infty} c_n^2 \Big)^{1/2} .$$

Proof of the probabilistic Strichartz estimates (sequel)

• We shall use the Sobolev spaces $W^{\sigma,q}(\mathbb{T}^3)$, $\sigma \geq 0$, $q \in (1,\infty)$, defined via the norm

$$\|u\|_{W^{\sigma,q}(\mathbb{T}^3)} = \|(1-\Delta)^{\sigma/2}u\|_{L^q(\mathbb{T}^3)}\,.$$

• We have that

$$\Big\| \|\langle t\rangle^{-\delta}\Pi_0 S(t)(v_0,v_1)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))}\Big\|_{L^p(\mu)}$$

equals

$$\Big\| \|\langle t\rangle^{-\delta}(\alpha_0(\omega)a_0 + t\alpha_1(\omega)a_1)\|_{L^{p_1}(\mathbb{R}_t;L^{p_2}(\mathbb{T}^3))}\Big\|_{L^p_\omega}\,. \qquad (24)$$

A trivial application of the Khinchin inequality implies that

$$\|\alpha_j(\omega)\|_{L^p_\omega} \leq C\sqrt{p}, \quad j=0,1.$$

Therefore, using that $\delta > 1+1/p_1$ the expression (24) can be bounded by

$$(2\pi)^{\frac{3}{p_2}}\Big\| \|\langle t\rangle^{-\delta}(\alpha_0(\omega)a_0 + t\alpha_1(\omega)a_1)\|_{L^{p_1}(\mathbb{R}_t)}\Big\|_{L^p_\omega} \leq C\sqrt{p}(|a_0|+|a_1|)\,.$$

Proof of the probabilistic Strichartz estimates (sequel)

• Therefore, it remains to estimate

$$\Big\| \| \langle t \rangle^{-\delta} \Pi_0^{\perp} S(t)(v_0, v_1) \|_{L^{p_1}(\mathbb{R}_t; L^{p_2}(\mathbb{T}^3))} \Big\|_{L^p(\mu)}$$

• By a use of the Hölder inequality on $\mathbb{T}^3$, we observe that it suffices to estimate

$$\Big\| \| \langle t \rangle^{-\delta} \Pi_0^{\perp} S(t)(v_0, v_1) \|_{L^{p_1}(\mathbb{R}_t; L^{\infty}(\mathbb{T}^3))} \Big\|_{L^p(\mu)} .$$

Let $q < \infty$ be such that $\sigma > 3/q$. Then by the Sobolev embedding

$$W^{\sigma,q}(\mathbb{T}^3) \subset C^0(\mathbb{T}^3) ,$$

we have

$$\| \Pi_0^{\perp} S(t)(v_0, v_1) \|_{L^{\infty}(\mathbb{T}^3)} \leq C \| (1 - \Delta)^{\sigma/2} \Pi_0^{\perp} S(t)(v_0, v_1) \|_{L^q(\mathbb{T}^3)} .$$

Proof of the probabilistic Strichartz estimates (sequel)

- Therefore, we need to estimate

$$\Big\| \| \langle t \rangle^{-\delta} (1-\Delta)^{\sigma/2} \Pi_0^{\perp} S(t)(v_0, v_1) \|_{L^{p_1}(\mathbb{R}_t; L^q(\mathbb{T}^3))} \Big\|_{L^p(\mu)}$$

which equals

$$\Big\| \| \langle t \rangle^{-\delta} (1-\Delta)^{\sigma/2} \Pi_0^{\perp} S(t)(u_0^{\omega}, u_1^{\omega}) \|_{L^{p_1}(\mathbb{R}_t; L^q(\mathbb{T}^3))} \Big\|_{L^p_{\omega}}. \tag{25}$$

By using the Hölder inequality in ω, we observe that it suffices to evaluate the last quantity only for $p > \max(p_1, q)$. For such values of p, using the Minkowski inequality, we can estimate (25) by

$$\Big\| \Big\| \langle t \rangle^{-\delta} (1-\Delta)^{\sigma/2} \Pi_0^{\perp} S(t)(u_0^{\omega}, u_1^{\omega}) \Big\|_{L^p_{\omega}} \Big\|_{L^{p_1}(\mathbb{R}_t; L^q(\mathbb{T}^3))}. \tag{26}$$

Proof of the probabilistic Strichartz estimates (sequel)

• Now, we can write $(1-\Delta)^{\sigma/2}\Pi_0^{\perp} S(t)(u_0^\omega, u_1^\omega)$ as

$$\sum_{n\in\mathbb{Z}^3_\star} \langle n\rangle^{\sigma}\Big(\Big(\beta_{n,0}(\omega) b_{n,0}\cos(t|n|) + \beta_{n,1}(\omega) b_{n,1}\frac{\sin(t|n|)}{|n|}\Big)\cos(n\cdot x)$$

$$+\Big(\gamma_{n,0}(\omega) c_{n,0}\cos(t|n|) + \gamma_{n,1}(\omega) c_{n,1}\frac{\sin(t|n|)}{|n|}\Big)\sin(n\cdot x)\Big),$$

with

$$\sum_{n\in\mathbb{Z}^3_\star} \langle n\rangle^{2\sigma}\Big(|b_{n,0}|^2+|c_{n,0}|^2+|n|^{-2}(|b_{n,1}|^2+|c_{n,1}|^2)\Big) \leq C\|(u_0,u_1)\|^2_{\mathcal{H}^\sigma(\mathbb{T}^3)}\,.$$

• Using the Khinchin inequality and the boundedness of sin and cos functions, we obtain that

$$\Big\|\Big\|\langle t\rangle^{-\delta}(1-\Delta)^{\sigma/2}\Pi_0^{\perp} S(t)(u_0^\omega, u_1^\omega)\Big\|_{L^p_\omega}\Big\|_{L^{p_1}(\mathbb{R}_t;L^q(\mathbb{T}^3))}$$

can be bounded by

$$C\Big\|\langle t\rangle^{-\delta} C\sqrt{p}\|(u_0,u_1)\|_{\mathcal{H}^\sigma(\mathbb{T}^3)}\Big\|_{L^{p_1}(\mathbb{R}_t;L^q(\mathbb{T}^3))}\,. \qquad (27)$$

Since $\delta > 1 + 1/p_1$, we can estimate

$$\left\| \langle t \rangle^{-\delta} C\sqrt{p} \|(u_0, u_1)\|_{\mathcal{H}^\sigma(\mathbb{T}^3)} \right\|_{L^{p_1}(\mathbb{R}_t; L^q(\mathbb{T}^3))}$$

by

$$C\sqrt{p}\|(u_0, u_1)\|_{\mathcal{H}^\sigma(\mathbb{T}^3)} \, .$$

This completes the proof.

• **Remark.** There are probabilistic Strichartz estimates if $\mathbb{T}^3$ is replaced by a three dimensional riemannian manifold or a domain of $\mathbb{R}^3$ (with suitable boundary conditions). One however needs to impose a suitable conditions on the "Fourier coefficients" related to the fact that the analogues of

$$\cos(n \cdot x), \quad \sin(n \cdot x)$$

on a manifold can grow in L^p, $p > 2$.

Theorem 13 (existence and uniqueness)
Let us fix $s \in [0,1)$ and $\mu \in \mathcal{M}^s$. Then, there exists a full μ measure set $\Sigma \subset \mathcal{H}^s(\mathbb{T}^3)$ such that for every $(v_0, v_1) \in \Sigma$, there exists a unique global solution v of of the non linear wave equation

$$(\partial_t^2 - \Delta)v + v^3 = 0, \quad (v(0), \partial_t v(0)) = (v_0, v_1)$$

satisfying

$$(v(t), \partial_t v(t)) \in \Big(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)\Big) + C(\mathbb{R}_t; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)).$$

Furthermore, if we denote by

$$\Phi(t)(v_0, v_1) \equiv (v(t), \partial_t v(t))$$

the flow thus defined, the set Σ is invariant by the map $\Phi(t)$, namely

$$\Phi(t)(\Sigma) = \Sigma, \qquad \forall\, t \in \mathbb{R}.$$

• **Remark.** Thanks to Cameron-Martin, we can also sate global existence results for datum which is a low regularity random perturbation (noise ?) of a regular object.

Proof of the existence and uniqueness result

Lemma 14

Consider the problem

$$(\partial_t^2 - \Delta)v + (f + v)^3 = 0\,. \tag{28}$$

There exists a constant C such that for every time interval $I = [a, b]$ of size 1*, every* $\Lambda \geq 1$*, every*

$$(v_0, v_1, f) \in H^1 \times L^2 \times L^3(I, L^6)$$

satisfying

$$\|v_0\|_{H^1} + \|v_1\|_{L^2} + \|f\|^3_{L^3(I,L^6)} \leq \Lambda$$

there exists a unique solution on the time interval $[a, a + C^{-1}\Lambda^{-2}]$ of (28) *with initial data*

$$v(a, x) = v_0(x), \quad \partial_t v(a, x) = v_1(x)\,.$$

Proof of the existence and uniqueness result (sequel)

- The solution, we obtained in the previous statement satisfies

$$\|(v, \partial_t v)\|_{L^\infty([a,a+C\Lambda^{-2}],H^1\times L^2)} \leq C\Lambda,$$

- $(v, \partial_t v)$ is unique in the class

$$L^\infty([a, a + C\Lambda^{-2}], H^1 \times L^2)$$

and the dependence in time is continuous.

- The proof is very similar to the proof of the classical local well-posedness result.

Proof of the existence and uniqueness result (sequel)

• Let $s > 0$. We search the solution v under the form

$$v(t) = S(t)(v_0, v_1) + w(t)$$

• Then w solves

$$(\partial_t^2 - \Delta)w + (S(t)(v_0, v_1) + w)^3 = 0, \quad w\,|_{t=0} = 0, \quad \partial_t w\,|_{t=0} = 0. \tag{29}$$

• Thanks to the probabilistic Strichartz estimates, we have that μ-almost surely,

$$\begin{aligned} g(t) &= \|S(t)(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t), \\ f(t) &= \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t). \end{aligned} \tag{30}$$

• The local existence for (29) follows from local existence result of the previous slide and the first estimate in (30).

• We also deduce from the local existence result that as long as the $H^1 \times L^2$ norm of $(w, \partial_t w)$ remains bounded, the solution w of (29) exists.

Proof of the existence and uniqueness result (sequel)

- Set

$$\mathcal{E}(w(t)) = \frac{1}{2}\int_{\mathbb{T}^3}\Big((\partial_t w)^2 + |\nabla_x w|^2 + \frac{1}{2}w^4\Big)dx\,.$$

Using the equation solved by w, we now compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(w(t)) &= \int_{\mathbb{T}^3}\Big(\partial_t w\partial_t^2 w + \nabla_x\partial_t w\cdot\nabla_x w + \partial_t w\, w^3\Big)dx \\
&= \int_{\mathbb{T}^3}\partial_t w\Big(\partial_t^2 w - \Delta w + w^3\Big)dx \\
&= \int_{\mathbb{T}^3}\partial_t w\Big(w^3 - (S(t)(v_0,v_1)+w)^3\Big)dx.
\end{aligned}$$

Proof of the existence and uniqueness result (sequel)

- Now, using the Cauchy-Schwarz inequality, we write

$$\frac{d}{dt}\mathcal{E}(w(t)) \leq C\Big(\mathcal{E}(w(t))\Big)^{1/2}\|w^3 - (S(t)(v_0, v_1) + w)^3\|_{L^2(\mathbb{T}^3)}\,.$$

- Using the Hölder inequality, we can estimate right hand-side by

$$C\Big(\mathcal{E}(w(t))\Big)^{1/2}\Big(\|S(t)(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)} + \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)}\|w^2\|_{L^2(\mathbb{T}^3)}\Big)$$

- But

$$\|w^2\|_{L^2(\mathbb{T}^3)} \leq C\Big(\mathcal{E}(w(t))\Big)^{1/2}$$

and consequently, we get the key bound

$$\frac{d}{dt}\mathcal{E}(w(t)) \leq C\Big(\mathcal{E}(w(t))\Big)^{1/2}\Big(g(t) + f(t)\Big(\mathcal{E}(w(t))\Big)^{1/2}\Big).$$

Proof of the existence and uniqueness result (sequel)

- Therefore, according to the Gronwall inequality and

$$g(t) = \|S(t)(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t), \quad \mu - \text{ a.s.}$$

$$f(t) = \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t), \quad \mu - \text{ a.s.}$$

we deduce that w exists globally in time.

- This completes the proof of the existence and uniqueness.

The invariant set

- Define the sets

$$\Theta \equiv \Big\{ (v_0, v_1) \in \mathcal{H}^s \,:\, \|S(t)(v_0,v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t), \\ \|S(t)(v_0,v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t) \Big\}$$

and

$$\Sigma \equiv \Theta + \mathcal{H}^1.$$

- Then Σ is of full μ measure for every $\mu \in \mathcal{H}^s$, since so is Θ.
- The set Σ is invariant under the dynamics.

Remarks

• Following Bourgain, in some **very specific situations** we can also globalise using invariant measures.

• For singular stochastic PDE similar schemes are used to pass from local to global solutions.