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October 6, 2017

Random data nonlinear wave equations II

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Probabilistic well-posedness in $\mathcal{H}^s$ for $s \in (0, 1/2)$

• Consider the Cauchy problem for the cubic defocusing wave equation

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad (u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)\,. \qquad (1)$$

• We have seen that (1) is well-posed (at least locally in time) in $\mathcal{H}^s(\mathbb{T}^3)$, $s > 1/2$. Moreover, for $s \in (0, 1/2)$ the Cauchy problem (1) is ill-posed in $\mathcal{H}^s(\mathbb{T}^3)$.

• One may however ask whether some sort of well-posedness for (1) survives for $s < 1/2$ (a priori even the existence is not known). It turns out that this is indeed the case, if we accept to "randomise" the initial data. This means that we will endow $\mathcal{H}^s(\mathbb{T}^3)$, $s \in (0, 1/2)$ with suitable non degenerate probability measures and we will show that the Cauchy problem (1) is well-posed in a suitable sense for initial data (u_0, u_1) on a set of a full measure.

Description of the measures describing the initial data

Starting from $(u_0, u_1) \in \mathcal{H}^s$ given by their Fourier series

$$u_j(x) = a_j + \sum_{n \in \mathbb{Z}^3_\star} \Big(b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x) \Big), \quad j = 0, 1,$$

we define u_j^ω by

$$u_j^\omega(x) = \alpha_j(\omega) a_j + \sum_{n \in \mathbb{Z}^3_\star} \Big(\beta_{n,j}(\omega) b_{n,j} \cos(n \cdot x) + \gamma_{n,j}(\omega) c_{n,j} \sin(n \cdot x) \Big),$$

where $(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega))$, $n \in \mathbb{Z}^3_\star$, $j = 0, 1$ is a sequence of real random variables on a probability space $(\Omega, p, \mathcal{F})$. We assume that the random variables $(\alpha_j, \beta_{n,j}, \gamma_{n,j})_{n \in \mathbb{Z}^3_\star, j=0,1}$ are independent identically distributed real random variables with a joint distribution θ satisfying

$$\exists\, c > 0, \quad \forall\, \gamma \in \mathbb{R}, \quad \int_{-\infty}^{\infty} e^{\gamma x} d\theta(x) \leq e^{c\gamma^2}. \tag{2}$$

Typical examples satisfying (2) are the standard gaussians and the Bernoulli measures.

Description of the measures (sequel)

• For fixed $(u_0, u_1) \in \mathcal{H}^s$, the map

$$\omega \in \Omega \longmapsto (u_0^\omega, u_1^\omega) \in \mathcal{H}^s \tag{3}$$

is a measurable map from $(\Omega, \mathcal{F})$ to $\mathcal{H}^s$ endowed with the Borel sigma algebra since the partial sums from a Cauchy sequence in $L^2(\Omega; \mathcal{H}^s)$.
• Thus the map (3) endows the space $\mathcal{H}^s(\mathbb{T}^3)$ with a probability measure. Let us denote this measure by $\mu_{(u_0,u_1)}$. Then

$$\forall\, A \subset \mathcal{H}^s,\ \mu_{(u_0,u_1)}(A) = p(\omega \in \Omega\ :\ (u_0^\omega, u_1^\omega) \in A).$$

• Denote by $\mathcal{M}^s$ the set of measures obtained following this construction :

$$\mathcal{M}^s = \bigcup_{(u_0,u_1)\in\mathcal{H}^s} \{\mu_{(u_0,u_1)}\}\,.$$

Probabilistic Strichartz estimates

Theorem 1

Let us fix $s \in (0,1)$ *and let* $\mu \in \mathcal{M}^s$. *Then for every* $T > 0$ *and* $p_1 \in [1,\infty)$, $p_2 \in [2,\infty]$,

$$\|S(t)(v_0, v_1)\|_{L^{p_1}([0,T];L^{p_2}(\mathbb{T}^3))} < \infty, \quad \mu - \text{almost surely.}$$

• Recall that $S(t)(v_0, v_1)$ denotes the solution of the linear wave equation with data (v_0, v_1).

Comments

• The probabilistic Strichartz estimate, applied for $p_2 = \infty$ displays an improvement of 3/2 derivatives with respect to the Sobolev embedding which is stronger than the improvement obtained by the (deterministic) Strichartz estimates, we already discussed.

• The proof of the probabilistic Strichartz estimate exploits the random oscillations of the initial data while the proof of the deterministic Strichartz estimates exploits in a crucial manner the time oscillations of $S(t)$.

• In the proof of the probabilistic Strichartz estimates, we simply neglect these times oscillations.

• In some other situations, as for instance the work on invariant measures for Benjamin-Ono or the derivative nonlinear Schrödinger equations etc. one exploits the regularisation effects coming both from the random and the time oscillations.

• The "art" is to understand how to make "collaborate" in the best way the random oscillations of the data and the time oscillations induced by the free evolution.

Random or time oscillations ?

- Let $\Phi(t)$ a flow and a measure of type

$$e^{-F(u)}du,$$

where du is invariant under $\Phi(t)$.

- For A a set, we typically face with the estimate of

$$\int_A e^{-F(u)}du - \int_{\Phi(t)(A)} e^{-F(u)}du\,.$$

- Method 1. Write

$$\frac{d}{dt}\int_{\Phi(t)(A)} e^{-F(u)}du\Big|_{t=t_0} = \frac{d}{dt}\int_{\Phi(t)(\Phi(t_0)(A))} e^{-F(u)}du\Big|_{t=0}$$

and exploit the random oscillations of the initial data.

- Method 2. Write

$$\int_A e^{-F(u)}du - \int_{\Phi(t)(A)} e^{-F(u)}du = \int_A\int_0^t \partial_t F(\Phi(\tau)(u))e^{-F(\Phi(\tau)(u))}d\tau du$$

and exploit the time oscillations of the solution (Strichartz, $X^{s,b}$ and so on ...).

Theorem 2 (existence and uniqueness)
Let us fix $s \in [0,1)$ *and* $\mu \in \mathcal{M}^s$. *Then, there exists a full* μ *measure set* $\Sigma \subset \mathcal{H}^s(\mathbb{T}^3)$ *such that for every* $(v_0, v_1) \in \Sigma$, *there exists a unique global solution* v *of of the non linear wave equation*

$$(\partial_t^2 - \Delta)v + v^3 = 0, \quad (v(0), \partial_t v(0)) = (v_0, v_1)$$

satisfying

$$(v(t), \partial_t v(t)) \in \Big(S(t)(v_0, v_1), \partial_t S(t)(v_0, v_1)\Big) + C(\mathbb{R}_t; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)).$$

Furthermore, if we denote by

$$\Phi(t)(v_0, v_1) \equiv (v(t), \partial_t v(t))$$

the flow thus defined, the set Σ *is invariant by the map* $\Phi(t)$, *namely*

$$\Phi(t)(\Sigma) = \Sigma, \qquad \forall\, t \in \mathbb{R}.$$

• **Remark.** Thanks to Cameron-Martin, we can also sate global existence results for datum which is a low regularity random perturbation (noise ?) of a regular object.

Lemma 3

Consider the problem

$$(\partial_t^2 - \Delta)v + (f+v)^3 = 0\,. \qquad (4)$$

There exists a constant C *such that for every time interval* $I = [a,b]$ *of size* 1*, every* $\Lambda \geq 1$*, every*

$$(v_0, v_1, f) \in H^1 \times L^2 \times L^3(I, L^6)$$

satisfying

$$\|v_0\|_{H^1} + \|v_1\|_{L^2} + \|f\|^3_{L^3(I,L^6)} \leq \Lambda$$

there exists a unique solution on the time interval $[a, a + C^{-1}\Lambda^{-2}]$ *of* (4) *with initial data*

$$v(a,x) = v_0(x), \quad \partial_t v(a,x) = v_1(x)\,.$$

• The proof is very similar to the proof of the classical local well-posedness result.

Proof of the existence and uniqueness result (sequel)

• Let $s > 0$. We search the solution v under the form

$$v(t) = S(t)(v_0, v_1) + w(t)$$

• Then w solves

$$(\partial_t^2 - \Delta)w + (S(t)(v_0, v_1) + w)^3 = 0, \quad w\,|_{t=0} = 0, \quad \partial_t w\,|_{t=0} = 0. \tag{5}$$

• Thanks to the probabilistic Strichartz estimates, we have that μ-almost surely,

$$\begin{aligned} g(t) &= \|S(t)(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t), \\ f(t) &= \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t). \end{aligned} \tag{6}$$

• The local existence for (5) follows from local existence result of the previous slide and the first estimate in (6).

• We also deduce from the local existence result that as long as the $H^1 \times L^2$ norm of $(w, \partial_t w)$ remains bounded, the solution w of (5) exists.

Proof of the existence and uniqueness result (sequel)

- Set

$$\mathcal{E}(w(t)) = \frac{1}{2}\int_{\mathbb{T}^3}\Big((\partial_t w)^2 + |\nabla_x w|^2 + \frac{1}{2}w^4\Big)dx\,.$$

Using the equation solved by w, we now compute

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(w(t)) &= \int_{\mathbb{T}^3}\Big(\partial_t w\partial_t^2 w + \nabla_x\partial_t w\cdot\nabla_x w + \partial_t w\, w^3\Big)dx \\ &= \int_{\mathbb{T}^3}\partial_t w\Big(\partial_t^2 w - \Delta w + w^3\Big)dx \\ &= \int_{\mathbb{T}^3}\partial_t w\Big(w^3 - (S(t)(v_0,v_1)+w)^3\Big)dx. \end{aligned}$$

Proof of the existence and uniqueness result (sequel)

- Now, using the Cauchy-Schwarz inequality, we write

$$\frac{d}{dt}\mathcal{E}(w(t)) \leq C\Big(\mathcal{E}(w(t))\Big)^{1/2}\|w^3 - (S(t)(v_0, v_1) + w)^3\|_{L^2(\mathbb{T}^3)}\,.$$

- Using the Hölder inequality, we can estimate right hand-side by

$$C\Big(\mathcal{E}(w(t))\Big)^{1/2}\Big(\|S(t)(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)} + \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)}\|w^2\|_{L^2(\mathbb{T}^3)}\Big)$$

- But

$$\|w^2\|_{L^2(\mathbb{T}^3)} \leq C\Big(\mathcal{E}(w(t))\Big)^{1/2}$$

and consequently, we get the key bound

$$\frac{d}{dt}\mathcal{E}(w(t)) \leq C\Big(\mathcal{E}(w(t))\Big)^{1/2}\Big(g(t) + f(t)\Big(\mathcal{E}(w(t))\Big)^{1/2}\Big).$$

Proof of the existence and uniqueness result (sequel)

- Therefore, according to the Gronwall inequality and

$$g(t) = \|S(t)(v_0, v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t), \quad \mu- \text{ a.s.}$$
$$f(t) = \|S(t)(v_0, v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t), \quad \mu- \text{ a.s.}$$

we deduce that w exists globally in time.

- This completes the proof of the existence and uniqueness.

The invariant set

- Define the sets

$$\Theta \equiv \Big\{ (v_0, v_1) \in \mathcal{H}^s \,:\, \|S(t)(v_0,v_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t), \\ \|S(t)(v_0,v_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t) \Big\}$$

and

$$\Sigma \equiv \Theta + \mathcal{H}^1.$$

- Then Σ is of full μ measure for every $\mu \in \mathcal{H}^s$, since so is Θ.
- The set Σ is invariant under the dynamics.

Remarks

• Following Bourgain, in some **very specific situations** we can also globalise using invariant measures.

• For singular stochastic PDE similar schemes are used to pass from local to global solutions.

Quasi-invariant measures

joint work with Tadahiro Oh

the starting point being ideas from previous works with
Nicolas Burq and Nicolas Visciglia

The linear equation

- Consider the linear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \tag{7}$$

where u_0 and u_1 are real valued and $u : \mathbb{R} \times \mathbb{T}^d \longrightarrow \mathbb{R}$, $d \geq 1$. The solutions of (7) are generated by the map $S(t)$, defined as follows

$$S(t)(u_0, u_1) \equiv \cos(t\sqrt{1-\Delta})(u_0) + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}(u_1),$$

where

$$\cos(t\sqrt{1-\Delta})(u_0) \equiv \sum_{n \in \mathbb{Z}^d} \cos(t\langle n \rangle)\widehat{u_0}(n)\, e^{in\cdot x}\,,$$

$$\frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}(u_1) \equiv \sum_{n \in \mathbb{Z}^d} \frac{\sin(t\langle n \rangle)}{\langle n \rangle}\widehat{u_1}(n)\, e^{in\cdot x}\,,$$

where $\langle n \rangle = (1 + |n|^2)^{1/2} = (1 + n_1^2 + \cdots + n_d^2)^{1/2}$.

The linear equation in Sobolev spaces

- For a function f on $\mathbb{T}^d$ given by its Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x},$$

we define the Sobolev norm $H^s(\mathbb{T}^d)$ of f as

$$\|f\|_{H^s}^2 = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s}\, |\widehat{f}(n)|^2\,.$$

- It follows directly from the definition that the operator

$$\bar{S}(t) \equiv (S(t), \partial_t S(t)),$$

where

$$\partial_t S(t)(u_0, u_1) \equiv -\sqrt{1-\Delta}\,\sin(t\sqrt{1-\Delta})(u_0) + \cos(t\sqrt{1-\Delta})(u_1)$$

is bounded on $H^s \times H^{s-1}$, $\bar{S}(0) = \mathrm{Id}$ and $\bar{S}(t+\tau) = \bar{S}(t)\circ\bar{S}(\tau)$.

Remark. In the proof of the boundedness on $H^s \times H^{s-1}$, we only use the boundedness of $\cos(t|n|)$ and $\sin(t|n|)$. One may use the oscillations of $\cos(t|n|)$ and $\sin(t|n|)$ for $|n| \gg 1$ in order to get more involved L^p, $p > 2$ properties of the map $S(t)$ (Strichartz estimates).

Invariant spaces of the linear evolution

Set

$$l_1 = \begin{pmatrix} \cos(n \cdot x) \\ 0 \end{pmatrix}, \; l_2 = \begin{pmatrix} 0 \\ \cos(n \cdot x) \end{pmatrix},$$

Then by definition for real number α, β, we can write

$$\begin{aligned} \bar{S}(t)(\alpha l_1 + \beta l_2) = (\alpha \cos(t\langle n\rangle) + \beta \langle n\rangle^{-1} \sin(t\langle n\rangle))l_1 \\ + (-\alpha\langle n\rangle \sin(t\langle n\rangle) + \beta \cos(t\langle n\rangle))l_2 \end{aligned}$$

Hence in the plane spanned by l_1, l_2, the map $\bar{S}(t)$ is represented by

$$A = \begin{pmatrix} \cos(t\langle n\rangle) & \langle n\rangle^{-1} \sin(t\langle n\rangle) \\ -\langle n\rangle \sin(t\langle n\rangle) & \cos(t\langle n\rangle) \end{pmatrix}.$$

We have that $\det(A) = 1$ and that for every s, the quadratic form

$$Q(X, Y) = \langle n\rangle^{2s+2} X^2 + \langle n\rangle^{2s} Y^2$$

is preserved by $\bar{S}(t)$.

Invariant spaces of the linear evolution (sequel)

Let us equip the line spanned by l_1 with the gaussian measure

$$\frac{\langle n\rangle^{s+1}}{\sqrt{2\pi}} e^{-\frac{\langle n\rangle^{2s+2}x^2}{2}} dx,$$

the line spanned by l_2 with the gaussian measure

$$\frac{\langle n\rangle^{s}}{\sqrt{2\pi}} e^{-\frac{\langle n\rangle^{2s}x^2}{2}} dx.$$

Denote by ρ the natural product measure in the plane spanned by l_1 and l_2. Then thanks to the previous discussion, we have :

Proposition 4

The measure ρ is invariant under $\bar{S}(t)$.

A similar analysis holds concerning the plane spanned by

$$\begin{pmatrix} \sin(n\cdot x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sin(n\cdot x) \end{pmatrix} .$$

The nonlinear equation

- Consider next the nonlinear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u + u^3 = 0, \tag{8}$$

where $u : \mathbb{R} \times \mathbb{T}^d \longrightarrow \mathbb{R}$, $d \geq 1$.

- We rewrite (8) as the first order system

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - u^3. \tag{9}$$

- One can rewrite (9) as a Hamiltonian system

$$\partial_t u = \frac{\delta E}{\delta v}, \quad \partial_t v = -\frac{\delta E}{\delta u},$$

where

$$E(u,v) = \frac{1}{2}\int_{\mathbb{T}^d}\left(u^2 + |\nabla u|^2 + v^2\right) + \frac{1}{4}\int_{\mathbb{T}^d} u^4 .$$

- Therefore $E(u,v)$ is a first integral for (9).

The global well-posedness for $d \leq 3$

• In view of the Hamiltonian structure and the properties of the linear equation, a natural phase space for

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - u^3. \tag{10}$$

is $\mathcal{H}^s(\mathbb{T}^d) \equiv H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$.

Theorem 5 (classical)
Let $d \leq 3$. Then (10) is globally well-posed in $\mathcal{H}^s(\mathbb{T}^d)$, $s \geq 1$. More precisely, for every $(u_0, v_0) \in \mathcal{H}^s(\mathbb{T}^d)$ there is a unique solution of (10) in $C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^d))$.

• Denote by $\Phi(t) : \mathcal{H}^s(\mathbb{T}^d) \to \mathcal{H}^s(\mathbb{T}^d)$ the resulting flow in Theorem 5.
• **In this lecture, we are interested in the statistical description of the flow $\Phi(t)$.**

Formal definition of the gaussian measures

• Let $\mu_{s,d}$ be the measure **formally** defined as

$$d\mu_{s,d} = Z_{s,d}^{-1} e^{-\frac{1}{2}\|(u,v)\|^2_{\mathcal{H}^{s+1}}} du dv$$

or

$$\prod_{n\in\mathbb{Z}^d} Z_{s,d,n}^{-1} \; e^{-\frac{1}{2}\langle n\rangle^{2(s+1)}|\widehat{u}_n|^2} e^{-\frac{1}{2}\langle n\rangle^{2s}|\widehat{v}_n|^2} d\widehat{u}_n d\widehat{v}_n \, ,$$

where $\widehat{u}_n$ and $\widehat{v}_n$ denote the Fourier transforms of u and v respectively.

• Recall that : $\langle n\rangle = (1+|n|^2)^{\frac{1}{2}}$.

Rigorous definition of the gaussian measures

• $\mu_{s,d}$ is the induced probability measure under the map

$$\omega \longmapsto (u^\omega(x), v^\omega(x))$$

with

$$u^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^{s+1}} e^{in\cdot x}, \quad v^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{h_n(\omega)}{\langle n\rangle^{s}} e^{in\cdot x}\,. \tag{11}$$

• In (11), $(g_n)_{n\in\mathbb{Z}^d}$, $(h_n)_{n\in\mathbb{Z}^d}$ are two sequences of "standard" complex gaussian random variables, such that $g_n = \overline{g_{-n}}$, $h_n = \overline{h_{-n}}$ and such that $\{g_n, h_n\}$ are independent, modulo the central symmetry.

• The partial sums of the series in (11) are a Cauchy sequence in $L^2(\Omega; \mathcal{H}^\sigma(\mathbb{T}^d))$ for every $\sigma < s + 1 - \frac{d}{2}$ and therefore one can see $\mu_{s,d}$ as a probability measure on $\mathcal{H}^\sigma$ for a fixed $\sigma < s + 1 - \frac{d}{2}$.

Remark. For the same range of σ, the triplet $\left(\mathcal{H}^{s+1}(\mathbb{T}^d), \mathcal{H}^\sigma(\mathbb{T}^d), \mu_{s,d}\right)$ forms an abstract Wiener space.

Invariance under the free evolution

We (essentially) have already established that :

Proposition 6
The measures $\mu_{s,d}$ *is invariant under* $\bar{S}(t)$.

Question : How much this property survives for the nonlinear flow ?

Statement of the results

- By the global well-posedness result, we also have that the flow $\Phi(t)$ is defined $\mu_{s,2}$ almost surely, provided $s > 1$.

Theorem 7
Let $s \geq 2$ be an even integer. Then $\mu_{s,2}$ is quasi-invariant under $\Phi(t)$.

- In $1d$ one can show the quasi-invariance of $\mu_{s,1}$ for $s \geq 0$ with a much simpler proof.
- The result of Theorem 7 holds also for the cubic wave equation

$$\partial_t^2 u - \Delta u + u^3 = 0\,.$$

Comments

• The proof crucially exploits the "dispersion" for any s. More precisely, it is likely that $\mu_{s,2}$ is not quasi-invariant under the flow of

$$\partial_t u = v, \quad \partial_t v = -u - u^3.$$

• We expect that the same result should hold for any $s > 0$ (for $s \in (0, 1]$ one should use a probabilistic global well-posedness in the sense of Burq-Tz.).

• We have some hope to extend the results to $d = 3$ ...

• Main remaining issue : What can be said about the resulting densities ?

From now on we only consider $d = 2$ and we denote $\mu_{s,2}$ simply by μ_s.

Related results 1 (Cameron-Martin 1944)

Theorem 8 (CM in the context of the measure μ_s)
For a fixed $(h_1, h_2) \in \mathcal{H}^\sigma$, $\sigma < s$, the transport of μ_s under the shift

$$(u_1, u_2) \longmapsto (u_1, u_2) + (h_1, h_2),$$

is absolutely continuous with respect to μ_s if and only if

$$(h_1, h_2) \in \mathcal{H}^{s+1}\,.$$

Proof. Denote by $(\cdot, \cdot)$ the scalar product in $\mathcal{H}^{s+1}(\mathbb{T}^2)$.
Let first $h = (h_1, h_2) \in \mathcal{H}^{s+1}$. If $u = (u_1, u_2) \in \mathcal{H}^{s+1}(\mathbb{T}^2)$ then

$$\|u + h\|^2_{\mathcal{H}^{s+1}} = \|u\|^2_{\mathcal{H}^{s+1}} + \|h\|^2_{\mathcal{H}^{s+1}} + 2(u, h).$$

Therefore, at least "formally" the transported measure is

$$e^{-\frac{1}{2}\|h\|^2_{\mathcal{H}^{s+1}}}\, e^{-(u,h)} d\mu_s(u)\,.$$

Proof of the Cameron-Martin theorem

• We are therefore reduced to check that μ_s almost surely the scalar product (u, h) is finite.
• Let us now observe that the proof of the fact that the scalar product (u, h) is μ_s almost surely finite reduces to check that for $(c_n) \in l^2(\mathbb{Z}^2)$,

$$\sum_{n\in\mathbb{Z}^2} c_n g_n(\omega) < \infty \quad \text{a.s} \quad \text{in} \quad \omega$$

which holds thanks to an orthogonality property resulting from the independence.
• Let now $h \notin \mathcal{H}^{s+1}$. Then there exists $g \in \mathcal{H}^{s+1}$ such that $(g, h) = \infty$.
• Define $A := \{w : (w, g) < \infty\}$. Then as above $\mu_s(A) = 1$.
• Denote by ρ_s the image measure of μ_s under the map $u \mapsto u + h$. Then

$$\rho_s(A) = \mu_s(B), \quad B = \{w - h, w \in A\}.$$

Hence for every $u \in B$, $(u, g) = \infty$. Therefore $B \subset A^c$ and $\mu_s(B) = 0$. This in turn implies that $\rho_s(A) = 0$.
This completes the proof.

"Comparing" our result Cameron-Martin's theorem

• For $(u,v) \in \mathcal{H}^\sigma$, we classically have

$$\Phi(t)(u,v) = \bar{S}(t)\Big((u,v) + (h_1,h_2)\Big),$$

where $(h_1,h_2) = (h_1(u,v), h_2(u,v)) \in \mathcal{H}^{\sigma+1}$ (one smoothing and not more).

• If $\sigma < s$ then $\sigma + 1 < s + 1$ and therefore our result displays a remarkable property of the vector field generating $\Phi(t)$.

Related results 2 (Ramer 1974)

• For $\sigma < s$, let us consider a diffeo Φ on $\mathcal{H}^\sigma(\mathbb{T}^2)$ of the form

$$\Phi(u,v) = (u,v) + F(u,v),$$

where $F : \mathcal{H}^\sigma(\mathbb{T}^2) \to \mathcal{H}^{s+1}(\mathbb{T}^2)$. Suppose that

$$DF(u,v) : \mathcal{H}^{s+1}(\mathbb{T}^2) \to \mathcal{H}^{s+1}(\mathbb{T}^2)$$

is Hilbert-Schmidt.

• Ramer (1974) : under the above assumption μ_s is quasi-invariant under Φ.

• Typical example :

$$F(u,v) = \varepsilon(1-\Delta)^{-1-\delta}(u^2, v^2), \quad \delta > 0,\ |\varepsilon| \ll 1,$$

i.e. 2-smoothing is needed.

• The Ramer's result would apply in the context of

$$\partial_t^2 u + (-\Delta)^\alpha u + u + u^3 = 0, \quad \alpha > 2.$$

• Therefore our result *seems* to go much beyond Ramer's framework because for the wave equation there is only 1-smoothing.

Related results 3. (A.B. Cruzeiro 1983)

• In her work Ana Bela Cruzeiro considers a general equation of the form

$$\partial_t u = X(u),$$

where X is a vector field on $\mathcal{H}^\sigma$, $\sigma < s$.

• A.B. Cruzeiro 1983 : the resulting flow has μ_s as a quasi-invariant measure provided that several assumptions are satisfied, the most important being

$$\int_{\mathcal{H}^\sigma} e^{\mathrm{div}(X(u))} d\mu_s(u) < \infty. \tag{12}$$

• Very roughly speaking, our work consists in verifying in practice a condition of type (12).

The approximated model

• We consider the approximated models

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - \pi_N((\pi_N u)^3), \quad N \to \infty, \tag{13}$$

where π_N denotes the Dirichlet projector on Fourier modes $\leq N$, i.e.

$$(\pi_N u)(x) = \sum_{|n| \leq N} \widehat{u}(n)\, e^{in\cdot x}\,.$$

• The quantity

$$E_N(u,v) = \frac{1}{2}\int_{\mathbb{T}^2}\left(u^2 + |\nabla u|^2 + v^2\right) + \frac{1}{4}\int_{\mathbb{T}^2}(\pi_N u)^4$$

is conserved under the flow of (13).

• We can therefore obtain that as is the case of the non-truncated model, the Cauchy problem for (13) is still globally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$, $s \geq 1$.

• For shortness, in the sequel we denote $\pi_N u$ by u_N and $\pi_N v$ by v_N.

The generalised energies

• Taking into account the definition of the gaussian measure μ_s, it is natural to study the expression

$$\frac{1}{2}\frac{d}{dt}\|(u_N(t), v_N(t))\|^2_{\mathcal{H}^{s+1}},$$

where (u,v) is a solution of

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - \pi_N((\pi_N u)^3)\,. \tag{14}$$

• For that purpose, we observe that if (u,v) is a solution of (14) then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(u_N(t), v_N(t))\|^2_{\mathcal{H}^{s+1}} &= \partial_t\Big[\frac{1}{2}\int_{\mathbb{T}^2}(J^s v_N)^2 + \frac{1}{2}\int_{\mathbb{T}^2}(J^{s+1}u_N)^2\Big] \\ &= \int_{\mathbb{T}^2}(J^{2s}v_N)(-u_N^3)\,,\end{aligned}$$

where $J \equiv \sqrt{1-\Delta}$. Of course, if $s=0$, the term in the r.h.s is

$$-\frac{1}{4}\partial_t\Big[\int_{\mathbb{T}^2} u_N^4\Big]$$

and we recover the conservation of $E(u_N, v_N)$.

The generalised energies (sequel)

- For $s \geq 2$, an even integer, using the Leibniz rule, we can write

$$\int_{\mathbb{T}^2} (J^{2s} v_N)(-u_N^3) = -3 \int_{\mathbb{T}^2} J^s v_N \, J^s u_N \, u_N^2 + \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ \max(|\alpha|,|\beta|,|\gamma|)<s}} c_{\alpha,\beta,\gamma} \int_{\mathbb{T}^2} J^s v_N \, \partial^\alpha u_N \partial^\beta u_N \partial^\gamma u_N,$$

where $c_{\alpha,\beta,\gamma}$ are unessential constants.

- Let us analyse the quantity

$$-3 \int_{\mathbb{T}^2} J^s v_N \, J^s u_N \, u_N^2 \, .$$

A need of a renormalisation

- Recalling that $\partial_t u_N = v_N$, we can write

$$
\begin{aligned}
-3\int_{\mathbb{T}^2} J^s v_N\, J^s u_N\, u_N^2 &= -\frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}(J^s u_N)^2 u_N^2\Big] + 3\int_{\mathbb{T}^2}(J^s u_N)^2\, v_N\, u_N \\
&= -\frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}\mathbf{P}_{\neq 0}[(J^s u_N)^2]\,\mathbf{P}_{\neq 0}[u_N^2]\Big] \\
&\quad + 3\int_{\mathbb{T}^2}\mathbf{P}_{\neq 0}[(J^s u_N)^2]\,\mathbf{P}_{\neq 0}[v_N\, u_N] \\
&\quad - \frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}(J^s u_N)^2\int_{\mathbb{T}^2} u_N^2\Big] + 3\int_{\mathbb{T}^2}(J^s u_N)^2\int_{\mathbb{T}^2} v_N\, u_N,
\end{aligned}
$$

where $\mathbf{P}_{\neq 0}$ is the projection on the non zero frequencies.

- The last two terms on the right-hand side are problematic because

$$
\lim_{N\to\infty}\mathbb{E}_{\mu_s}\Big[\int_{\mathbb{T}^2}(J^s\pi_N u)^2\Big] = \infty\,.
$$

Therefore, we need to use a suitable renormalisation !

The renormalised energies

- Define σ_N by

$$\sigma_N = \mathbb{E}_{\mu_s}\Big[\int_{\mathbb{T}^2}(J^s\pi_N u)^2\Big] = \sum_{\substack{n\in\mathbb{Z}^2\\|n|\leq N}} \frac{1}{1+|n|^2} \sim \log N\,.$$

- Then, we have

$$-\frac{3}{2}\partial_t\Big[\int_{\mathbb{T}^2}(J^s u_N)^2\int u_N^2\Big] + 3\int_{\mathbb{T}^2}(J^s u_N)^2\int v_N\,u_N$$
$$= -\frac{3}{2}\partial_t\Big[\Big(\int_{\mathbb{T}^2}(J^s u_N)^2 - \sigma_N\Big)\int_{\mathbb{T}^2} u_N^2\Big] + 3\Big(\int_{\mathbb{T}^2}(J^s u_N)^2 - \sigma_N\Big)\int v_N\,u_N.$$

- The term

$$\int_{\mathbb{T}^2}(J^s u_N)^2 - \sigma_N$$

is now a "good" term because there is C such that for every $p \geq 2$ and every $N \geq 1$

$$\Big\|\int_{\mathbb{T}^2}(J^s\pi_N u)^2 - \sigma_N\Big\|_{L^p(d\mu_s(u,v))} \leq C\sqrt{p}\,.$$

The renormalised energies (sequel)

• In view of the above discussion, it is now natural to define the modified energy $E_{s,N}(u,v)$ by

$$E_{s,N}(u,v)=\frac{1}{2}\int(J^s v)^2+\frac{1}{2}\int(J^{s+1}u)^2+\frac{3}{2}\int(J^s u)^2u^2-\frac{3}{2}\sigma_N\int u^2$$

and we have that if that if (u,v) is a solution of the truncated problem then

$$\partial_t E_{s,N}(u_N,v_N)=3\int_{\mathbb{T}^2}\mathbf{P}_{\neq 0}[(J^s u_N)^2]\,\mathbf{P}_{\neq 0}[v_N\,u_N]+$$
$$\sum_{\substack{|\alpha|+|\beta|+|\gamma|=s\\ \max(|\alpha|,|\beta|,|\gamma|)<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^2}J^s v_N\,\partial^\alpha u_N\partial^\beta u_N\partial^\gamma u_N+$$
$$3\Big(\int_{\mathbb{T}^2}(J^s u_N)^2-\sigma_N\Big)\int_{\mathbb{T}^2}v_N\,u_N. \quad (15)$$

• Now all terms in the right hand-side of (15) are suitable for a perturbative analysis.

The key estimate

Theorem 9
Let $s \geq 2$ *be an even integer and let us denote by* $\Phi_N(t)$ *the flow of*

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - \pi_N((\pi_N u)^3)\,.$$

Then for every $r > 0$ *there is a constant* C *such that for every* $p \geq 2$ *and every* $N \geq 1$,

$$\Big(\int_{E_N(u,v)\leq r} \Big| \partial_t E_{s,N}(\pi_N \Phi_N(t)(u,v))|_{t=0} \Big|^p d\mu_s(u,v) \Big)^{\frac{1}{p}} \leq Cp.$$

On the proof of the quasi-invariance

• Recall that

$$E_{s,N}(u,v) = \frac{1}{2}\int (J^s v)^2 + \frac{1}{2}\int (J^{s+1}u)^2 + \frac{3}{2}\int (J^s u)^2 u^2 - \frac{3}{2}\sigma_N \int u^2$$

• By classical arguments from QFT, we can define

$$\lim_{N\to\infty} \Big(\frac{3}{2}\int (J^s \pi_N u)^2 (\pi_N u)^2 - \frac{3}{2}\sigma_N \int (\pi_N u)^2\Big)$$

in $L^p(d\mu_s(u,v))$, $p < \infty$.

• Denote this limit by $R(u)$. Essentially speaking, once we have the key estimate, we study the quasi-invariance of

$$\mathbf{1}_{E(u,v)\leq r}\, e^{-R(u)}\, d\mu_s(u,v)$$

by soft analysis techniques.

On the proof of the quasi-invariance (sequel)

• Let us be a little more precise. Denote by $x(t)$ the measure evolution of a set having zero measure with respect to

$$\mathbf{1}_{E(u,v)\leq r}\, e^{-R(u)}\, d\mu_s(u,v)\,.$$

• Essentially speaking, using the key estimate and the Liouville argument, we obtain that $x(t)$ satisfy the estimate

$$\dot{x}(t) \leq Cp(x(t))^{1-\frac{1}{p}}, \quad x(0) = 0\,. \tag{16}$$

Integrating the last estimate leads to $x(t) \leq (Ct)^p$. Taking the limit $p \to \infty$, we infer that $x(t) = 0$ for $0 \leq t < 1/C$. Since C is an absolute constant, we can iterate the argument and show that $x(t)$ is vanishing.

• Observe that this argument would not work if in (16), we have p^α, $\alpha > 1$ instead of p.

• In order to make the previous reasoning rigorous, we need to use some more or less standard approximation arguments.

On the proof of the key proposition

- We have that

$$\partial_t E_{s,N}(\pi_N \Phi_N(t)(u,v))|_{t=0} = Q_1(u,v) + Q_2(u,v) + Q_3(u,v),$$

where

$$Q_1(u,v) = 3 \int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s \pi_N u)^2] \, \mathbf{P}_{\neq 0}[\pi_N v \, \pi_N u],$$

$$Q_2(u,v) = \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ \max(|\alpha|,|\beta|,|\gamma|)<s}} c_{\alpha,\beta,\gamma} \int_{\mathbb{T}^2} J^s \pi_N v \, \partial^\alpha \pi_N u \partial^\beta \pi_N u \partial^\gamma \pi_N u,$$

$$Q_3(u,v) = 3 \Big(\int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \Big) \int_{\mathbb{T}^2} \pi_N v \, \pi_N u.$$

Estimate of $Q_3(u,v)$

- For $r > 0$, we define $\mu_{s,r}$ as

$$d\mu_{s,r}(u,v) = \mathbf{1}_{E_N(u,v)\leq r}\, d\mu_s(u,v)\,.$$

- The goal is to show that

$$\|Q_3(u,v)\|_{L^p(d\mu_{s,r}(u,v))} \leq Cp,$$

with a constant C independent of N and p. Since

$$\Big| \int_{\mathbb{T}^2} \pi_N v\, \pi_N u \Big| \leq \|\pi_N u\|_{L^2} \|\pi_N v\|_{L^2} \leq E_N(u,v),$$

we obtain that

$$\begin{aligned} \|Q_3(u,v)\|_{L^p(d\mu_{s,r}(u,v))} &\leq C_r \Big\| \int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \Big\|_{L^p(d\mu_{s,r}(u,v))} \\ &\leq C_r \Big\| \int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \Big\|_{L^p(d\mu_s(u,v))}\,. \end{aligned}$$

Estimate of $Q_3(u,v)$ (sequel)

- On the other hand

$$\Big\| \int_{\mathbb{T}^2} (J^s \pi_N u)^2 - \sigma_N \Big\|_{L^p(d\mu_s(u,v))} = \Big\| \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{|g_n(\omega)|^2 - 1}{1+|n|^2} \Big\|_{L^p(\Omega)}$$

and by using Wiener chaos estimates, we have

$$\Big\| \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{|g_n(\omega)|^2 - 1}{1+|n|^2} \Big\|_{L^p(\Omega)} \le Cp \Big\| \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{|g_n(\omega)|^2 - 1}{1+|n|^2} \Big\|_{L^2(\Omega)} \le Cp$$

which provides the needed bound for $Q_3(u,v)$.

On the estimate of $Q_1(u,v)$

• The bound for

$$Q_1(u,v) = 3\int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]\, \mathbf{P}_{\neq 0}[\pi_N v\, \pi_N u]$$

is the most delicate part of the analysis and relies on subtle multi-linear arguments.

• Basically, we are allowed to have outputs as

$$\|J^\sigma u\|_{L^\infty(\mathbb{T}^2)}, \quad \sigma < s$$

with a loss $\sqrt{p}$ and $E(u,v)$.

• A naive Hölder inequality approach clearly fails.

• A purely probabilistic argument based on Wiener chaos estimates fails because the output power of p is too large.

• The basic strategy is to perform a multi-scale analysis redistributing properly the derivative losses by never having more then quadratic weight of the contribution of the Wiener chaos estimate.

On the estimate of $Q_1(u,v)$ (sequel)

• When analysing the 4-linear expression

$$Q_1(u,v) = 3\int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s\pi_N u)^2]\, \mathbf{P}_{\neq 0}[\pi_N v\, \pi_N u],$$

we suppose that

$$J^s\pi_N u,\ J^s\pi_N u,\ \pi_N v,\ \pi_N u$$

are localised at frequencies N_1, N_2, N_3, N_4 respectively.

• We first consider the case when $N_4 \gtrsim (\max(N_1, N_2))^{\frac{1}{100}}$. In this case we exchange some regularity of $J^s\pi_N u$ with this of $\pi_N u$ and we perform the naive linear analysis.

On the estimate of $Q_1(u,v)$ (sequel)

• Therefore, in the analysis of

$$Q_1(u,v) = 3\int_{\mathbb{T}^2} \mathbf{P}_{\neq 0}[(J^s\pi_N u)^2]\,\mathbf{P}_{\neq 0}[\pi_N v\,\pi_N u],$$

we can suppose that $N_4 \ll (\max(N_1,N_2))^{\frac{1}{100}}$. In this case, we have that

$$\max(N_1,N_2) \sim \max(N_j,\, j=1,2,3,4).$$

• By symmetry, we can suppose that $N_1 = \max(N_1,N_2)$.
• We next consider the case $N_3 \ll N_1^{1-a}$, $a = a(s)$-small. In this case, we perform a bi-linear Wiener chaos estimate and we have **some gain of regularity** in the localisation of $\mathbf{P}_{\neq 0}[(J^s\pi_N u)^2]$.

On the estimate of $Q_1(u,v)$ (sequel)

- Finally, we consider the case

$$N_1 \sim \max(N_j,\ j=1,2,3,4),\ N_4 \ll (\max(N_1,N_2))^{\frac{1}{100}},\ N_3 \gtrsim N_1^{1-a}$$

In this case, we perform a tri-linear Wiener chaos estimate and we have **enough gain of regularity** in the localisation of

$$\mathbf{P}_{\neq 0}[(J^s \pi_N u)^2]\pi_N v\,.$$

- This essentially explains the argument leading to the key estimate.