Non Null-Controllability of the Grushin Equation in Dimension 2

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S. Margherita di Pula, Sardinia, Italy

Some Controllability Results for the Grushin Equation

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Grushin equation

$$\begin{aligned} \Omega &= [-1,1] \times \mathbb{T} \\ \partial_t f - \partial_x^2 f - x^2 \partial_y^2 f &= 1_\omega u \\ f_{|\partial\Omega} &= 0 \end{aligned}$$

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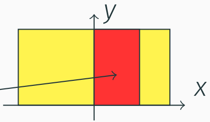
  (Beauchard, Miller & Morancey 2015)



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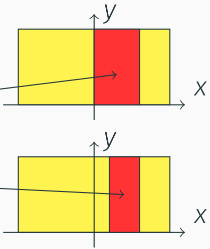

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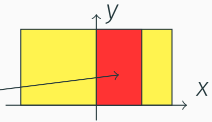

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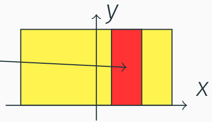

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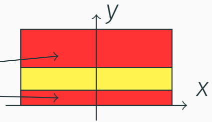
- Never null-controllable if ω



Non Null-Controllability of the Grushin Equation: Heuristics

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- Goal: disprove the observability inequality

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- Approximation of $-\partial_x^2 + (nx)^2$ on $(-1,1)$ by itself on $\mathbf{R}$: we expect $v_n \sim \left(\frac{n}{4\pi}\right)^{1/4} e^{-nx^2/2}$ et $\lambda_n \sim n$
- Observability inequality with $f(t,x,y) = \sum a_n v_n(x) e^{iny-\lambda_n t}$, heuristics $\lambda_n = n$ and $\int v_n v_m = 1$:

$$\int_{\mathrm{T}} \left|\sum a_n e^{-nT} e^{iny}\right|^2 \mathrm{d}y \leq C \int_{[0,T]\times\omega_y} \left|\sum a_n e^{-nt} e^{iny}\right|^2 \mathrm{d}t\,\mathrm{d}y$$

The toy model: $D = \sqrt{-\Delta}$

Theorem

Let $H = \{\sum_{n\geq 0} a_n e^{iny}, \sum |a_n|^2 < +\infty\}$ and $D \sum a_n e^{iny} = \sum n a_n e^{iny}$.
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Observability inequality $f(t,y) = \sum a_n e^{-nt} e^{iny}$:

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It is the approximate observability inequality of the Grushin equation!

Proof of the non null-controllability of the toy model

Observability inequality with $f(t,y) = \sum a_n e^{-nt} e^{iny}$:

$$\sum |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \left|\sum a_n e^{-nt} e^{iny}\right|^2 \,\mathrm{d}t\,\mathrm{d}y$$

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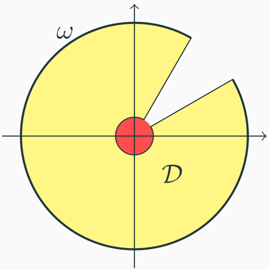
Let $z = e^{-t+iy} = \mu + i\nu$ and $f(z) = \sum_{n\geq 1} a_n z^{n-1}$

$$\begin{aligned} &\int_{z\in D(0,e^{-T})} |f(z)|^2 \,\mathrm{d}\mu\,\mathrm{d}\nu \\ &\leq \pi \sum |a_n|^2 e^{-2nT} \\ &\leq \pi C \int_{z\in\mathcal{D}} |f(z)|^2 \,\mathrm{d}\mu\,\mathrm{d}\nu \end{aligned}$$



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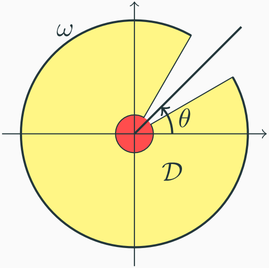

False thanks to Runge's theorem (take $f_k \longrightarrow 1/z$ uniformly on every compact subset of $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_+$) $\square$

Differences between the Grushin equation and the toy model

"Holomorphic" observability inequality (we know it is false):

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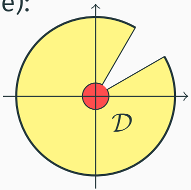

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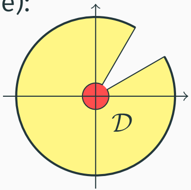

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Observability inequality of the Grushin equation

- Let $\lambda_n = n + \rho_n$ and $z = \mu + i\nu$:

$$\sum |a_n|^2 e^{-2nT} e^{-2\rho_n T} \leq C \int_{-1}^{1} \int_{z\in\mathcal{D}} \left|\sum v_n(x) a_n z^{n-1} |z|^{\rho_n}\right|^2 \,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{d}x$$

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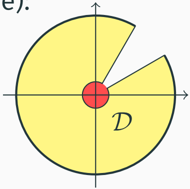

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- Solution:
  - $e^{-2\rho_n T}$ in the LHS: not a problem
  - Treat x as a parameter
  - Prove

$$\left| \sum v_n(x) |\zeta|^{\rho_n} a_n z^n \right|_{L^\infty(\mathcal{D})} \leq C \left| \sum a_n z^n \right|_{L^\infty(U)}$$

The technical inequality: the complex analysis part

Theorem (LeRoy & Lindelöf 1906)

Let $\gamma : \{\Re(z) > 0\} \to \mathbb{C}$ *be holomorphic and bounded. Then* $K(z) = \sum_{n>0} \gamma(n) z^n$ *admits an analytic extension to* $\mathbb{C} \setminus [1, +\infty)$.

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- with $f(z) = \sum a_n z^n$:

$$\sum \gamma(n) a_n z^n = \frac{1}{2i\pi} \sum \gamma(n) \overbrace{\oint_{\partial D} \frac{f(\zeta)}{\zeta^{n+1}} \,\mathrm{d}\zeta}^{a_n} z^n$$
$$= \frac{1}{2i\pi} \oint_{\partial D} \frac{1}{\zeta} K\left(\frac{z}{\zeta}\right) f(\zeta) \,\mathrm{d}\zeta$$

- change the path of integration:

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- *mumble...* quasi-mode $e^{-\alpha x^2/2}$ *mumble...* stationary phase theorem *mumble mumble*: for all $0 < \theta < \frac{\pi}{2}$:

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- for $v_\alpha(x)$: Agmon's inequality $\square$

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That's all folks!