Nonlinear Radon and Fourier Transforms

François Rouvière
Université de Nice
Laboratoire Dieudonné, UMR 7351

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Abstract

In this note we explain a generalization, due to Leon Ehrenpreis, of the classical Radon transform on hyperplanes. A function \( f \) on \( \mathbb{R}^n \) can be reconstructed from nonlinear Radon transforms, obtained by integrating \( f \) and a finite number of multiples \( \alpha^m f \) over a family of algebraic hypersurfaces of degree \( m \). This follows by solving a Cauchy problem for the nonlinear Fourier transform of \( f \). We also give an inversion formula for this Radon transform.

1 Introduction

This expository note is an attempt at explaining the pages from Ehrenpreis’ treatise [5] in which he develops the nonlinear Radon and Fourier transforms he had introduced in his previous papers [1][2][3][4]. The goal is to extend the classical hyperplane Radon transform \( R_0 f \) (integrals of a function \( f \) over all hyperplanes in \( \mathbb{R}^n \)) to a family of algebraic submanifolds defined by higher degree polynomial equations. Is the generalized transform \( R \) still injective? Can we give an inversion formula? Unfortunately it is readily seen that \( R \) is no more injective (in general): reconstructing \( f \) from Radon transforms needs more than \( Rf \) alone.

We shall explain here several results of the following type: there exists a finite number of low-degree polynomial functions \( a_k \) (with \( a_1 = 1 \)) such that \( f \) is determined by the Radon transforms \( R(a_k f) \). Besides, the restriction of the \( R(a_k f) \)'s to a certain subfamily of algebraic manifolds may even be sufficient, provided one increases the number of polynomials \( a_k \).

After a brief reminder of the classical hyperplane transform (this Section) we shall introduce Ehrenpreis’ nonlinear Radon transform and the related nonlinear Fourier transform, so as to get a projection slice theorem which plays a crucial role in this study (Section 2). The reconstruction problem boils down to a Cauchy problem for a system of partial differential equations, solved in a naive way in Section 3 then, in Section 4, by the more sophisticated tools of harmonic polynomials. In Section 5 we discuss an inversion formula for the nonlinear Radon transform.

In order to motivate the forthcoming construction, let us briefly recall a few facts about the classical Radon transform \( R_0 \). In the Euclidean space \( \mathbb{R}^n \) it is given by integration of a compactly supported smooth function \( f \in \mathcal{D}(\mathbb{R}^n) \) over the family of
all hyperplanes. A hyperplane being defined by the equation $\omega \cdot x = t$ where $\omega$ is a unit vector, $t$ a real number and $\cdot$ denotes the scalar product, we consider

$$R_0 f(t, \omega) := \int_{\omega \cdot x = t} f,$$

an integral with respect to the measure induced on the hyperplane by the Euclidean measure $dx$ of $\mathbb{R}^n$. Note that $(t, \omega)$ and $(-t, -\omega)$ define the same hyperplane, thus $R_0 f(t, \omega) = R_0 f(-t, -\omega)$. For any $\tau \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} e^{i\tau \omega \cdot x} f(x) dx = \int_{\mathbb{R}} dt \int_{\omega \cdot x = t} e^{i\tau \omega \cdot x} f(x) = \int_{\mathbb{R}} e^{i\tau t} R_0 f(t, \omega) dt.$$

This gives the projection slice theorem

$$\hat{f}(\tau \omega) = \overline{R_0 f(\tau)}$$

for $\tau \in \mathbb{R}$, $\omega \in \mathbb{R}^n$ and $||\omega|| = 1$.

Caution: on the left-hand side of (1) the hat denotes the $n$-dimensional Fourier transform on $x$ but on the right-hand side it denotes the 1-dimensional Fourier transform on $t$. Both sides are smooth functions on $\mathbb{R} \times S^{n-1}$, rapidly decreasing with respect to $\tau$.

Knowing the integrals of $f$ over all hyperplanes, i.e. $R_0 f$, the Fourier transform $\hat{f}$ is therefore known and $R_0$ is easily inverted as follows. Writing the Fourier inversion formula for $f$ in spherical coordinates we have

$$f(x) = (2\pi)^{-n} \int_{||\omega|| = 1} d\omega \int_0^\infty e^{-i\tau \omega \cdot x} \overline{R_0 f(\tau, \omega)} \tau^{n-1} d\tau$$

where $d\omega$ is the Euclidean measure on the unit sphere of $\mathbb{R}^n$. In order to use Fourier analysis in one variable we can replace $\int_0^\infty$ by $\int_{\mathbb{R}}$: indeed $\overline{R_0 f(\tau, \omega)} = \overline{R_0 f(-\tau, -\omega)}$ and, changing $\tau$ into $-\tau$ then $\omega$ into $-\omega$, we obtain

$$f(x) = C \int_{||\omega|| = 1} d\omega \int_{\mathbb{R}} e^{-i\tau \omega \cdot x} \overline{R_0 f(\tau, \omega)} |\tau|^{n-1} d\tau$$

with $C := \frac{1}{2} (2\pi)^{-n}$. Let $F(t, \omega)$ be a smooth function on $\mathbb{R} \times S^{n-1}$, rapidly decreasing with respect to $t$, and let the operator $|\partial_t|^{n-1}$ be defined by

$$(|\partial_t|^{n-1} F)(\tau, \omega) = \hat{F}(\tau, \omega) |\tau|^{n-1}.$$

Thus $|\partial_t|^{n-1} = (-1)^k \partial_t^k$ if $n = 2k + 1$ is odd; if $n$ is even $|\partial_t|^{n-1}$ is the composition of $\partial_t^{n-1}$ and a Hilbert integral operator (see Helgason [7] p. 22). We infer the following inversion formula

$$f = CR_0^* |\partial_t|^{n-1} R_0 f$$

(2)

where the dual transform $R_0^*$ is defined by

$$R_0^* F(x) := \int_{||\omega|| = 1} F(\omega \cdot x, \omega) d\omega$$

(integration over the set of all hyperplanes containing $x$).
2 A Nonlinear Radon Transform

2.1 Integration on Hypersurfaces

Let \( \varphi : \Omega \to \mathbb{R} \) be a smooth function on an open subset \( \Omega \) of the Euclidean space \( \mathbb{R}^n \). A convenient way to introduce our Radon transform is to consider first, for \( f \in \mathcal{D}(\Omega) \) (a smooth function with compact support contained in \( \Omega \)) and \( t \in \mathbb{R} \),

\[
 f_\varphi(t) := \int_{\varphi(x) < t} f(x) \, dx
\]

where \( dx \) is the Lebesgue measure of \( \mathbb{R}^n \). Let \( m \) and \( M \) denote the lower and upper bounds of \( \varphi(x) \) for \( x \in \text{supp} \, f \); then \( f_\varphi(t) = 0 \) for \( t \leq m \) and \( f_\varphi(t) = \int_{\Omega} f(x) \, dx \) for \( t \geq M \).

The example \( \Omega = \mathbb{R} \) and \( \varphi(x) = x^3 \) gives \( f_\varphi(t) = F(t^{1/3}) \) with \( F(u) = \int_{-\infty}^{u} f(x) \, dx \); thus \( f_\varphi \) is not necessarily smooth. However the following result holds true.

**Proposition 1** Assume the gradient \( \varphi' \) of \( \varphi \) never vanishes on \( \Omega \). For \( f \in \mathcal{D}(\Omega) \), \( f_\varphi \) is then a smooth function on \( \mathbb{R} \) and we may define

\[
 R_\varphi f(t) := (f_\varphi)'(t) = \partial_t \int_{\varphi(x) < t} f(x) \, dx.
\]

(i) \( R_\varphi f \) is a smooth function on \( \mathbb{R} \) and supp \( R_\varphi f \subset [m, M] \).

(ii) For any \( u \in C^\infty(\mathbb{R}) \)

\[
 \int_{\mathbb{R}^n} u(\varphi(x)) f(x) \, dx = \int_{\mathbb{R}} u(t) R_\varphi f(t) \, dt.
\]

(iii) Let \( dS_t \) be the Euclidean measure on the hypersurface \( S_t := \{ x \in \Omega | \varphi(x) = t \} \). Then

\[
 R_\varphi f(t) = \int_{S_t} f(x) \frac{1}{\|\varphi'(x)\|} \, dS_t(x).
\]

Formula (5) gives the geometrical meaning of \( R_\varphi f \) as an integral of \( f \) over the level hypersurface \( \varphi(x) = t \); we may write it for short as

\[
 R_\varphi f(t) = \int_{\varphi(x) = t} f.
\]

According to (4) it may also be viewed as \( R_\varphi f(t) = (\varphi^* \delta_t, f) \) where \( \varphi^* \delta_t \) is the pullback by \( \varphi \) of the Dirac measure \( \delta_t \) of \( \mathbb{R} \) at \( t \) (see Friedlander [6] Section 7.2 or Hörmander [8] Section 6.1).

**Proof.** (i) and (iii) Given \( a \in \Omega \) we have \( \varphi'(a) \neq 0 \) thus (for instance) \( \partial_n \varphi(a) \neq 0 \). By the inverse function theorem there exists an open neighborhood \( U \) of \( a \) such that the map \( x = (x', x_n) \mapsto y = (x', \varphi(x)) \) is a diffeomorphism of \( U \) onto \( V \times I \), where \( x' = (x_1, ..., x_{n-1}) \), \( V \) is an open neighborhood of \( (a_1, ..., a_{n-1}) \) in \( \mathbb{R}^{n-1} \) and \( I \) is an open interval containing \( \varphi(a) \). Let \( y = (y', y_n) \mapsto x = (y', \psi(y', y_n)) \) denote the inverse map. Then \( dy = |\partial_n \varphi(x)| \, dx \) and, assuming \( \text{supp} \, f \subset U \), we have

\[
 f_\varphi(t) = \int_{\varphi(x) < t} f(x) \, dx = \int_{y_n < t} f \left( y', \psi(y', y_n) \right) dy' dy_n.
\]
The $y_n$ integral actually runs over $[a, b] \cap \mathbb{R} + t$ where $[a, b]$ is compact and contained in $I$. Thus $f_\varphi$ is a smooth function of $t \in \mathbb{R}$ and

$$R_\varphi f(t) = (f_\varphi)'(t) = \int_V \frac{f}{|\partial_n \varphi|}(y', \psi(y', t))dy' \text{ for } t \in I$$

$$= 0 \text{ for } t \notin I$$

is smooth on $\mathbb{R}$.

Besides, $\varphi(y', \psi(y', t)) = t$ for $y' \in V$ and $t \in I$ therefore

$$\partial_i \varphi(y', \psi(y', t)) + \partial_n \varphi(y', \psi(y', t))\partial_i \psi(y', t) = 0$$

for $i = 1, \ldots, n - 1$. It follows that $||\varphi'|| = |\partial_n \varphi| \left(1 + \sum_{i=1}^{n-1} (\partial_i \psi)^2\right)^{1/2}$ and, for $t \in I$,

$$R_\varphi f(t) = \int_V \frac{f}{||\varphi'||}(y', \psi(y', t)) \left(1 + \sum_{i=1}^{n-1} (\partial_i \psi(y', t))^2\right)^{1/2} dy'$$

$$= \int_{S_t} \frac{f}{||\varphi'||}(x)dS_t(x),$$

the hypersurface integral being computed by means of the parameters $y'$. The latter equality also holds for $t \notin I$ (both sides vanish) and this proves (i) and (iii) for $\text{supp } f \subset U$. The general case follows by partition of unity.

(ii) Since $\text{supp } R_\varphi f \subset [m, M]$ we have

$$\int_{\mathbb{R}} u(t)R_\varphi f(t)dt = \int_{m}^{M} u(t)(f_\varphi)'(t)dt = [u(t)f_\varphi(t)]_{m}^{M} - \int_{m}^{M} u'(t)f_\varphi(t)dt$$

$$= u(M) \int_{\Omega} f(x)dx - \int_{\varphi(x) < t < M} u'(t)f(x)dtdx.$$

The latter integral is

$$\int_{\Omega} f(x)dx \int_{\varphi(x)}^{M} u'(t)dt = \int_{\Omega} f(x)(u(M) - u(\varphi(x)))dx$$

and (4) follows. □

2.2 Nonlinear Radon and Fourier Transforms

We now wish to extend the classical Radon transform of Section 1, replacing the hyperplanes $\omega \cdot x = t$ by level hypersurfaces of homogeneous polynomials of given degree $m \geq 1$ in $\mathbb{R}^n$. We write such polynomials as

$$\lambda \cdot p(x) := \sum_{|\alpha|=m} \lambda_\alpha x^\alpha$$

where $x \in \mathbb{R}^n$ and, in multi-index notation, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_1^n \alpha_i$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\lambda_\alpha \in \mathbb{R}$. 

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It is easily checked that the number of terms in \( \sum_{|\alpha|=m} \) is the binomial coefficient \( N = N(m, n) = \frac{(m+n-1)!}{m!(n-1)!} \). Indeed let us consider

\[
\prod_{j=1}^{n} \frac{1}{1 - tx_j} = \prod_{j=1}^{n} (1 + tx_j + t^2x_j^2 + \cdots).
\]

Expanding the product we see that the coefficient of \( t^m \) is \( \sum_{|\alpha|=m} x_{\alpha} \), therefore equals \( N(m, n) \) when all \( x_i \)'s are 1. Thus \( N(m, n) \) is the coefficient of \( t^m \) in the expansion of \( (1 - t)^{-n} \) and the result follows. Note that \( N > n \) for \( n \geq 2 \) and \( m \geq 2 \).

Let \( \lambda \in \mathbb{R}^n \), \( \lambda \neq 0 \), and \( \Omega := \{ x | \lambda \cdot p(x) \neq 0 \} \). By Euler’s identity for the homogeneous function \( \varphi(x) = \lambda \cdot p(x) \) on \( \mathbb{R}^n \) the gradient \( \varphi' \) does not vanish on \( \Omega \).

The level surface \( \lambda \cdot p(x) = t \) is thus a smooth hypersurface of \( \mathbb{R}^n \) for \( t \in \mathbb{R}, t \neq 0 \). The nonlinear Radon transform of a test function \( f \in \mathcal{D}(\Omega) \) is then defined, in the notation of (6), by

\[
Rf(t, \lambda) := R_{\varphi}f(t) = \int_{\lambda \cdot p(x) = t} f.
\]

For \( m = 1 \) we have \( N = n \) and \( R \) is the classical hyperplane Radon transform \( R_0 \).

**Properties of \( R \).**

(i) By Proposition 1, for \( f \in \mathcal{D}(\Omega) \) and \( \lambda \neq 0 \), \( Rf(., \lambda) \) is a compactly supported smooth function of \( t \) on \( \mathbb{R} \). By (4)

\[
\int_{\mathbb{R}^n} F(\lambda \cdot p(x), \lambda) f(x)dx = \int_{\mathbb{R}} F(t, \lambda)Rf(t, \lambda)dt
\]

for \( \lambda \neq 0 \) and any \( F \) continuous on \( \mathbb{R} \times \mathbb{R}^N \). In particular, for \( \tau \in \mathbb{R}, \)

\[
\int_{\mathbb{R}^n} e^{i\tau \lambda \cdot p(x)} f(x)dx = \int_{\mathbb{R}} e^{i\tau t}Rf(t, \lambda)dt = \widehat{Rf}(\tau, \lambda) = \widehat{f}(1, \tau, \lambda)
\]

is the one-dimensional Fourier transform of \( Rf \) with respect to the variable \( t \). This extends the projection slice theorem (1).

(ii) The left-hand side of (8) is well-defined for all \( f \in \mathcal{D}(\mathbb{R}^n) \) (without assuming \( \text{supp} f \subset \Omega \)), and extends to an entire function of \( (\tau, \lambda) \) on \( \mathbb{C} \times \mathbb{C}^N \). This suggests defining \( \widehat{Rf}(\tau, 0) = \int f \), that is \( Rf(t, 0) = (\int_{\mathbb{R}^n} f(x)dx) \delta(t) \) where \( \delta \) is the Dirac measure at the origin of \( \mathbb{R} \).

Actually, the restrictive assumptions \( \text{supp} f \subset \Omega, t \neq 0, \lambda \neq 0 \) may be left out in the sequel, as we shall work with \( \widehat{Rf} \) rather than \( Rf \).

(iii) From (8) it follows that

\[
\partial_{\lambda^\alpha} \widehat{Rf}(\tau, \lambda) = i\tau \int_{\mathbb{R}^n} e^{i\tau \lambda \cdot p(x)} x^\alpha f(x)dx = i\tau \overline{R(x^\alpha f)}(\tau, \lambda),
\]

therefore

\[
\partial_{\lambda^\alpha} Rf(t, \lambda) = - \partial t R(x^\alpha f) (t, \lambda)
\]

for \( f \in \mathcal{D}(\Omega), \lambda \neq 0 \) and \( \alpha \in \mathbb{N}^n, |\alpha| = m. \)
(iv) Note that, for $m$ even, $Rf = 0$ whenever $f$ is an odd function: $R$ is not an injective map and, in this case, $f$ cannot be reconstructed from $Rf$ alone. We shall see in the next sections how to circumvent this difficulty.

Let us introduce the **nonlinear Fourier transform of** $f$ defined, for all $f \in \mathcal{D}(\mathbb{R}^n)$, by

$$
\tilde{f}(\xi, \lambda) := \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda \cdot p(x))} f(x) dx , \xi \in \mathbb{R}^n , \lambda \in \mathbb{R}^N. 
$$

(11)

It extends to an entire function of $(\xi, \lambda) \in \mathbb{C}^n \times \mathbb{C}^N$. As a function on $\mathbb{R}^n \times \mathbb{R}^N$ it is bounded by $\int_{\mathbb{R}^n} |f(x)| dx$ and, for fixed $\lambda$, it is rapidly decreasing with respect to $\xi$.

On the one hand $\tilde{f}(\xi, 0) = \hat{f}(\xi)$ is the classical $n$-dimensional Fourier transform of $f$; on the other hand $\tilde{f}(0, \tau \lambda) = \tilde{R}f(\tau, \lambda)$ is the $1$-dimensional Fourier transform of $Rf$:

$$
\tilde{f}(\xi, 0) = \hat{f}(\xi) \quad \tilde{f}(0, \lambda) = \tilde{R}f(1, \lambda).
$$

Reconstructing $\tilde{f}(\xi, \lambda)$ from $\tilde{f}(0, \lambda)$ would therefore allow to reconstruct $f$ from $Rf$. For this we shall consider partial differential equations satisfied by $\tilde{f}$.

### 2.3 Partial Differential Equations

Taking derivatives of (11) under the integral sign we get, for $j = 1, ..., n$ and $\alpha \in \mathbb{N}^n$, $|\alpha| = m$,

$$
\partial_{\xi_j} \tilde{f}(\xi, \lambda) = i \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda \cdot p(x))} x_j f(x) dx = i(x_j \hat{f})(\xi, \lambda) 
$$

(12)

$$
\partial_{\lambda_\alpha} \tilde{f}(\xi, \lambda) = i \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda \cdot p(x))} x^\alpha f(x) dx = i(x^\alpha \hat{f})(\xi, \lambda). 
$$

(13)

Thus $\tilde{f}$ satisfies the system of $N$ linear partial differential equations on $\mathbb{R}^n \times \mathbb{R}^N$

$$
im^{-1} \partial_{\lambda_\alpha} \tilde{f} = \partial^2_{\xi_\alpha} \tilde{f} \quad \text{for } \alpha \in \mathbb{N}^n, |\alpha| = m.
$$

(14)

For any $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$ of length $m$ such that $x^\alpha x^\beta = x^\gamma x^\delta$ we infer that, as a function of $\lambda$, $\tilde{f}$ satisfies the **Plücker equations**

$$
(\partial_{\lambda_\beta} \partial_{\lambda_\delta} - \partial_{\lambda_\delta} \partial_{\lambda_\beta}) \tilde{f} = 0.
$$

(15)

Given $\alpha, \beta$, all such multi-indices $\gamma, \delta$ are obtained as $\gamma = \alpha - \varepsilon, \delta = \beta + \varepsilon$, where $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in \mathbb{Z}^n$ satisfies $-\beta_j \leq \varepsilon_j \leq \alpha_j$ for $j = 1, ..., n$ and $\sum_n \varepsilon_j = 0$.

**Example.** For $m = n = 2$ we have $\lambda \cdot p(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_1 x_2$ (here $N = 3$) and

$$
i \partial_{\lambda_1} \tilde{f} = \partial^2_{\xi_1} \tilde{f} , \quad i \partial_{\lambda_2} \tilde{f} = \partial^2_{\xi_2} \tilde{f} , \quad i \partial_{\lambda_3} \tilde{f} = \partial_{\xi_1} \partial_{\xi_2} \tilde{f}.
$$

The identity $(x_1 x_2)^2 = x_1^2 x_2^2$ leads to the hyperbolic equation $\partial^2_{\xi_3} \tilde{f} = \partial_{\xi_1} \partial_{\xi_2} \tilde{f}$. 

6
3 A Cauchy Problem

Given \( f \in \mathcal{D}(\mathbb{R}^n) \) let us now try to reconstruct \( \tilde{f}(\xi, \lambda) \) from \( \tilde{f}(0, \lambda) = \tilde{R}f(1, \lambda) \) by solving a Cauchy problem for the system (14) with data on \( \xi = 0 \). In order to achieve this goal we shall of course need more than \( \tilde{R}f(1, \lambda) \); let us recall that \( \tilde{f}(0, \lambda) = 0 \) for \( m \) even and \( f \) odd, though \( \tilde{f} \) may be not identically zero. It should be noted that \( \tilde{f}(0, \lambda) \) satisfies the Plücker equations (15), but this fact will not be taken into account here (see Remark below however).

Since \( \tilde{f} \) is an entire function we have

\[
\tilde{f}(\xi, \lambda) = \sum_{\alpha \in \mathbb{N}^n} \partial_\xi^\alpha \tilde{f}(0, \lambda) \frac{\xi^\alpha}{\alpha!},
\]

an absolutely convergent series for all \( \xi \in \mathbb{C}^n \) and \( \lambda \in \mathbb{C}^N \).

To work it out we shall only need the derivatives \( \partial_\xi^\alpha \tilde{f}(0, \lambda) \) for \( |\alpha| < m \); the higher order derivatives will be given by (14). More precisely, \( \partial_\xi^\alpha \tilde{f} = i^{\alpha} (x^\gamma f) \) for all \( \alpha \) by (12), and equals \( i^{m-1} \partial_\lambda \tilde{f} \) by (14) if \( |\alpha| = m \). For any \( \alpha \in \mathbb{N}^n \) we may write \( |\alpha| = qm + r \) with \( q, r \in \mathbb{N}, 0 \leq r < m \), and factorize \( \partial_\xi^\alpha \) as

\[
\partial_\xi^\alpha = \partial_\xi^{\beta_1} \cdots \partial_\xi^{\beta_q} \partial_\lambda^{\beta_{q+1}} \cdots \partial_\lambda^{\beta_r}
\]

with \( \beta_1, \ldots, \beta_q \in \mathbb{N}^n, |\beta_1| = \cdots = |\beta_q| = m \) and \( |\gamma| = r \); this factorization is not unique. It follows that

\[
\partial_\xi^\alpha \tilde{f} = i^{\alpha} (x^\gamma f)
\]

and

\[
\tilde{f}(\xi, \lambda) = \sum_{\alpha \in \mathbb{N}^n} i^{\alpha} \partial_\lambda^{\beta_{q+1}} \cdots \partial_\lambda^{\beta_r} (x^\gamma f)(0, \lambda) \frac{\xi^\alpha}{\alpha!}
\]

(with \( q, \beta_1, \ldots, \beta_q, \gamma \) depending on \( \alpha \) in the sum).

Remembering \( (x^\gamma f)(0, \lambda) = \tilde{R}(x^\gamma f)(1, \lambda) \) for \( \lambda \neq 0 \), we see that \( \tilde{f} \) is determined by the nonlinear Radon transforms of all functions \( x^\gamma f \) for \( \gamma \in \mathbb{N}^n \) and \( |\gamma| < m \). Their number is \( \sum_{k=0}^{m-1} N(k, n) = N(m - 1, n + 1) = \frac{m}{n} N(m, n) \) (induction on \( m \)). In particular if \( \tilde{R}(x^\gamma f) = 0 \) for all \( \gamma \) with \( |\gamma| < m \), then \( f = 0 \).

**Example.** For \( m = n = 2 \) (Section 2.3), \( \partial_{\xi_1}^2 \partial_{\xi_2}^2 \) factorizes as powers of \( \partial_{\xi_1}^2 \) and \( \partial_{\xi_2}^2 \), possibly composed with \( \partial_{\xi_1} \) or \( \partial_{\xi_2} \) or \( \partial_{\xi_1} \partial_{\xi_2} \) according to the parity of \( \alpha_1 \) and \( \alpha_2 \).

Gathering together similar terms the above result reads

\[
\tilde{f}(\xi, \lambda) = C(D_1)C(D_2)\tilde{f} + S(D_1)S(D_2)D_3\tilde{f} +
\]

\[
+i\xi_1 S(D_1)C(D_2)(x_1 f) + i\xi_2 C(D_1)S(D_2)(x_2 f)
\]

(16)

where

\[
D_1 = i\xi_1^2 \partial_{\xi_1}, \quad D_2 = i\xi_2^2 \partial_{\xi_2}, \quad D_3 = i\xi_1 \xi_2 \partial_{\lambda_3}
\]

\[
C(z) = \sum_{k=0}^{\infty} \frac{z^k}{(2k)!}, \quad S(z) = \sum_{k=0}^{\infty} \frac{z^k}{(2k+1)!}
\]
and, in the right-hand side of (16), \( \tilde{f}, (\tilde{x}_1 f), (\tilde{x}_2 f) \) are evaluated at \((0, \lambda)\). Thus the knowledge of the three Radon transforms \( Rf, R(x_1 f) \) and \( R(x_2 f) \) determines \( \tilde{f} \).

**Remark.** The Plücker equations (15), here \( \partial_{\lambda_3}^2 \tilde{f} = \partial_{\lambda_1} \partial_{\lambda_2} \tilde{f} \), haven’t been taken into account. They imply \( \partial_{\lambda_3}^{2k} \tilde{f} = (\partial_{\lambda_1} \partial_{\lambda_2})^k \tilde{f}, \partial_{\lambda_3}^{2k+1} \tilde{f} = (\partial_{\lambda_1} \partial_{\lambda_2})^k \partial_{\lambda_3} \tilde{f} \) for \( k \in \mathbb{N} \), hence the Taylor expansion

\[
\tilde{f}(0, \lambda_1, \lambda_2, \lambda_3) = \sum_{k \in \mathbb{N}} \partial_{\lambda_3}^k \tilde{f}(0, \lambda_1, \lambda_2, 0) \frac{\lambda_3^k}{k!}
\]

where \( E = \lambda_3^2 \partial_{\lambda_1} \partial_{\lambda_2} \), and similarly

\[
\partial_{\lambda_3} \tilde{f}(0, \lambda_1, \lambda_2, \lambda_3) = \lambda_3 \partial_{\lambda_1} \partial_{\lambda_2} S (E) \tilde{f}(0, \lambda_1, \lambda_2, 0) + C (E) (\partial_{\lambda_3} \tilde{f})(0, \lambda_1, \lambda_2, 0).
\]

Combining (16) (17) and (18) it follows that \( \tilde{f} \) can be reconstructed from \( \tilde{f}, \partial_{\lambda_3} \tilde{f}, (\tilde{x}_1 f), \partial_{\lambda_3}(\tilde{x}_1 f), (\tilde{x}_2 f) \) and \( \partial_{\lambda_3}(\tilde{x}_2 f) \) at \((0, \lambda_1, \lambda_2, 0)\) only.

Remembering (13) \( \partial_{\lambda_3} \tilde{f} = i(\tilde{x}_1 \tilde{x}_2 f) \), these 6 functions can be replaced by \( \tilde{f}, (\tilde{x}_1 f), (\tilde{x}_2 f), (\tilde{x}_1 \tilde{x}_2 f), (x_1 \tilde{x}_2 f) \) and \((x_1 x_2^2 f)\), that is \( \tilde{Rf}, \tilde{(x_1 f)}, \tilde{(x_2 f)} \) evaluated at \((1; \lambda_1, \lambda_2, 0)\). In other words the integrals of \( f, x_1 f, ..., x_1 x_2^2 f \) over the conics \( \lambda_1 x_1^2 + \lambda_2 x_2^2 = t \) will determine \( f \). A stronger (and more general) result is given in the next section.

## 4 Harmonic Polynomials and the Cauchy Problem

Two chapters of [5] are devoted to a general theory of harmonic polynomials which, when applied to nonlinear Radon transforms, leads to a refined version of the results of Section 3. We shall only present here a simplified approach to the harmonic polynomials relevant to our problem.

**Notation.** All polynomials considered here have complex coefficients. Let us order the \( N \) monomials \((x^n)_{|\alpha|=m} \) as \( x_1^m, ..., x_n^m \) first, then \((x^\beta)_{\beta \in B}\) where \( B \) is the set of the \( N - n \) remaining multi-indices of length \( m \). In accordance with this we replace our previous notation \( \lambda = (\lambda_\alpha)_{|\alpha|=m} \in \mathbb{R}^N \) by \( (\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^{N-n} \) with \( \lambda = (\lambda_1, ..., \lambda_n) \) and \( \mu = (\mu_\beta)_{\beta \in B} \); the former \( \sum_\alpha \lambda_\alpha x^\alpha \) is replaced by \( \sum_{j=1}^n \lambda_j x_j^m + \sum_{\beta \in B} \mu_\beta x^\beta \). Let \((x, p, q) \in \mathbb{R}^{n+N}\) denote dual variables to \((\xi, \lambda, \mu)\), with \( x = (x_1, ..., x_n) \in \mathbb{R}^n, \)

\( p = (p_1, ..., p_n) \in \mathbb{R}^n \) and \( q = (q_\beta)_{\beta \in B} \in \mathbb{R}^{N-n} \).

In this new notation the partial differential equations (14) become

\[
(-i\partial_{\xi_j})^m \tilde{f} = -i\partial_{\lambda_j} \tilde{f}, \quad (-i\partial_{\xi})^\beta \tilde{f} = -i\partial_{\mu} \tilde{f} \text{ for } j = 1, ..., n \text{ and } \mu \in B.
\]

They are dual to

\[
x_j^m F = p_j F, \quad (x^\beta - q_\beta) F = 0 \text{ for } j = 1, ..., n \text{ and } \mu \in B,
\]

where \( F \) is the tempered distribution on \( \mathbb{R}^{n+N} \) corresponding to \( \tilde{f} \) via the Fourier transform on \( \mathbb{R}^{n+N} \) (being smooth and bounded, \( \tilde{f} \) is tempered on \( \mathbb{R}^{n+N} \)).
Let us introduce the following $N$ polynomials on $\mathbb{R}^n$: 
\[ u_j(x,q) := x_j^m, \quad u_\beta(x,q) := x^\beta - q_\beta \text{ for } j = 1, \ldots, n \text{ and } \beta \in B. \quad (21) \]

The system (20) implies that the support of $F$ is contained in the closed set $V$ of $\mathbb{R}^{n+N}$ defined by the $N$ equations
\[ V = \{(x, p, q) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{N-n} | u_j(x, q) = p_j, \ u_\beta(x, q) = 0, 1 \leq j \leq n, \beta \in B \}. \]

Being the graph of a map $x \mapsto (p, q)$, $V$ is a $n$-dimensional submanifold of $\mathbb{R}^{n+N}$.

**Definition 2** A polynomial function $h(x, q)$ on $\mathbb{R}^n \times \mathbb{R}^{N-n}$ is called **harmonic** if
\[ u_j(\partial_x, \partial_q)h = 0, \ u_\beta(\partial_x, \partial_q)h = 0 \text{ for } j = 1, \ldots, n \text{ and } \beta \in B. \]

It is called **homogeneous of degree** $d$ if $h(tx, t^m q) = t^d h(x, q)$ for all $t \in \mathbb{R}$ (thus each $x_j$ has degree 1 and each $q_\beta$ has degree $m$).

**Proposition 3** Let $D := \sum_{\beta \in B} q_\beta \partial_x^\beta$. Then $u_\beta(\partial_x, \partial_q) = - e^D \circ \partial_{q_\beta} \circ e^{-D}$.

The space of harmonic polynomials is $m^n$-dimensional. Its elements are given by
\[ h = e^D f \]
where $f$ is an arbitrary polynomial of the following form
\[ f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \text{ with } 0 \leq \alpha_j \leq m - 1 \text{ for } j = 1, \ldots, n \text{ and } a_\alpha \in \mathbb{C}. \]

Besides $h = e^D f$ is homogeneous of degree $d$ (in the sense of Definition 4) if and only if $f$ is homogeneous of degree $d$.

**Proof.** Since $u_\beta(\partial_x, \partial_q) = \partial_x^\beta - \partial_{q_\beta}$ we have $[D, u_\beta(\partial_x, \partial_q)] = \partial_x^\beta$ and $[D, \partial_x^\beta] = 0$, thus $(\text{ad} \ D)^2 u_\beta(\partial_x, \partial_q) = 0$ and
\[ e^{-D} u_\beta(\partial_x, \partial_q) e^D = e^{-\text{ad} \ D} u_\beta(\partial_x, \partial_q) = (1 - \text{ad} \ D) u_\beta(\partial_x, \partial_q) \]
\[ = u_\beta(\partial_x, \partial_q) - \partial_x^\beta = - \partial_{q_\beta}. \]

[This proof may also be written without any Lie formalism, by computing the derivative with respect to $t$ of $e^{-tD} u_\beta(\partial_x, \partial_q) e^{tD}$.]

Since $e^D$ is a linear isomorphism of the space of polynomials onto itself, a polynomial $h(x, q)$ is harmonic if and only if
\[ \partial_x^m h = 0, \ \partial_{q_\beta} (e^{-D} h) = 0 \text{ for } j = 1, \ldots, n \text{ and } \beta \in B. \]

The latter equations imply $h = e^D f$ for some polynomial $f$ in the $x$ variables. Since $[D, \partial_x^m] = 0$ the former equations imply $\partial_x^m f = 0$ for $j = 1, \ldots, n$ whence our claim about $f$.

The operator $D$ preserves homogeneity in $(x, q)$ and the last statement follows. ■
Examples. Let us write down, as an example, a basis of homogeneous harmonic polynomials for $n = 2$ and $m = 4$. Here $N = 5$, $\beta = (\beta_1, \beta_2)$ with $0 \leq \beta_j \leq 3$, $\beta_1 + \beta_2 = 4$, $q = (q_{13}, q_{22}, q_{31})$ and $D = \sum q_{3j} \partial_x^j \partial_y^j$. The $16$ monomials $f(x) = x^n_1 x^n_2$, $0 \leq a \leq 3$, $0 \leq b \leq 3$, make up a basis of the relevant polynomials $f$. Since the degree of $f$ is 6 at most we have $D^2 f = 0$ and the $16$ corresponding harmonic polynomials are $h = f + Df$, that is

\begin{align*}
1, x_1, x_2, x_1 x_2, x_2^2, x_1^2 x_2, x_1 x_2^2, x_2^3, \\
x_1^3 x_2 + 6 q_{31}, x_1^2 x_2^2 + 4 q_{22}, x_1 x_2^3 + 6 q_{31}, \\
x_1^3 x_2^2 + 12 q_{22} x_1 + 12 q_{31} x_2, x_1^2 x_2^3 + 12 q_{13} x_1 + 12 q_{22} x_2, \\
x_1^3 x_2^3 + 18 q_{13} x_1^2 + 36 q_{22} x_1 x_2 + 18 q_{31} x_2^2.
\end{align*}

For $m = n = 2$ (already considered) we have $N = 3$, $q \in \mathbb{R}$, and the corresponding basis of harmonic polynomials is

$$1, x_1, x_2, x_1 x_2 + q.$$ 

More generally, let $A$ denote the set of all $\alpha \in \mathbb{N}^n$ such that $0 \leq \alpha_j \leq m - 1$ for $j = 1, \ldots, n$. By Proposition 5 the $h_{\alpha} := e^P x^\alpha$, $\alpha \in A$, make up a basis of the space of harmonic polynomials.

Proposition 4 For any polynomial $P(x, q)$ on $\mathbb{R}^n \times \mathbb{R}^{N-n}$ there exists a family of $m^n$ polynomials $Q_\alpha$, $\alpha \in A$, on $\mathbb{R}^N$ such that

$$P(x, q) = \sum_{\alpha \in A} Q_\alpha(u_1(x, q), \ldots, u_N(x, q)) h_\alpha(x, q),$$

where $u_1, \ldots, u_N$ denote the polynomials defined by (21).

Proof. Let $(a, b) = a(\partial) \overline{b}(0)$ be the Fischer inner product on the space of polynomials on $\mathbb{R}^n \times \mathbb{R}^{N-n}$. Then $h$ is harmonic if and only if $u_k(\partial_x, \partial_q) h = 0$ for $k = 1, \ldots, N$, i.e. $\langle au_k, h \rangle = 0$ for all polynomials $a$. The space of harmonic polynomials is thus the orthogonal complement of the ideal $\{ \sum_{k=1}^N a_k(x, q)u_k(x, q) \}$ generated by the $u_k$’s (where the $a_k$’s are arbitrary polynomials).

A given $P(x, q)$ now has a unique decomposition as

$$P = h + \sum_{k=1}^N a_k u_k$$

with $h$ harmonic. Separating homogeneous components we may assume $P$ is homogeneous of degree $d$ (in the sense of Definition 2). Since $u_k$ is homogeneous, each homogeneous component of a harmonic polynomial is harmonic. We may therefore assume $h$ and all $a_k u_k$ homogeneous of degree $d$, therefore $a_k$ is homogeneous of degree $d - m$. Writing similar decompositions for each $a_k$ the result easily follows.

---

1 Cf. [5] p. 312, where the coefficients 16 should be replaced, I think, by 18.
Example. For \( m = n = 2 \) the generators and harmonic polynomials are respectively
\[
\begin{align*}
  u_1 &= x_1^2, \quad u_2 = x_2^2, \quad u_3 = x_1x_2 - q \\
  h_0 &= 1, \quad h_1 = x_1, \quad h_2 = x_2, \quad h_3 = x_1x_2 + q
\end{align*}
\]
and the first non-trivial examples of decomposition in Proposition 4 are:
\[
\begin{align*}
  2x_1x_2 &= u_3h_0 + h_3, \quad 2q = -u_3h_0 + h_3 \\
  x_1q &= -u_3h_1 + u_1h_2, \quad x_2q = -u_3h_2 + u_2h_1 \\
  q^2 &= u_1u_2h_0 - u_3h_3, \quad 2x_1x_2q = (2u_1u_2 - u_3^2)h_0 - u_3h_3.
\end{align*}
\]
Replacing \( x_j \) by \(-i\partial_{\xi_j}\) and \( q_\beta \) by \(-i\partial_{\mu_\beta}\) we infer from Proposition 4 an equality of differential operators. Applying them to \( e^f \) we obtain
\[
P(-i\partial_{\xi}, -i\partial_{\mu})\tilde{f} = \sum_{\alpha \in A} Q_\alpha \left((-i\partial_{\xi})^m (\alpha - i\partial_{\mu})^\beta (-i\partial_{\mu})\right) h_\alpha (-i\partial_{\xi}, -i\partial_{\mu}) \tilde{f}
\]
\[
= \sum_{\alpha \in A} Q_\alpha (-i\partial_\lambda, 0) h_\alpha (-i\partial_{\xi}, -i\partial_{\mu}) \tilde{f}
\]
in view of (19) and the commutativity of differential operators. In particular all derivatives \( \partial^\rho \partial^\sigma \tilde{f} \) may be written in this form with polynomials \( Q_\alpha \) depending on \( \rho, \sigma \) whence, by Taylor’s formula on the variables \( (\xi, \mu) \),
\[
\tilde{f}(\xi, \lambda, \mu) = \sum_{\alpha \in A} Q_{\alpha \rho \sigma} (-i\partial_\lambda, 0) h_\alpha (-i\partial_{\xi}, -i\partial_{\mu}) \tilde{f}(0, \lambda, 0) \frac{\xi^\rho \mu^\sigma}{\rho! \sigma!} \tag{22}
\]
where \( \sum \) runs over all \( \rho \in \mathbb{N}^n, \sigma \in \mathbb{N}^{N-n} \) and \( \alpha \in A \). Remembering (12) (13) \(-i\partial_{\xi} \tilde{f} = x_1 \tilde{f}, -i\partial_{\mu_\beta} \tilde{f} = x^\beta \tilde{f}\) we have \( h_\alpha (-i\partial_{\xi}, -i\partial_{\mu}) \tilde{f} = (h_\alpha (x, q) \tilde{f}) \) with \( q_\beta = x^\beta \) for \( \beta \in B \).

**Lemma 5** For all \( \alpha \) there exists a positive integer \( C_\alpha \) such that, when replacing each \( q_\beta \) by \( x^\beta \) for \( \beta \in B \),
\[
h_\alpha (x, q) = h_\alpha (x, (x^\beta)_{\beta \in B}) = C_\alpha x^\alpha.
\]
**Proof.** For \( \alpha \in \mathbb{N}^n \) we have
\[
D^\alpha x^\alpha = \sum_{\beta \in B} q_\beta \partial^\beta x^\alpha = \sum_{\beta \in B} \frac{\alpha!}{(\alpha - \beta)!} q_\beta x^{\alpha - \beta}
\]
\[
D^2 x^\alpha = \sum_{\beta, \gamma \in B} \frac{\alpha!}{(\alpha - \beta - \gamma)!} q_\beta q_\gamma x^{\alpha - \beta - \gamma}
\]
etc (the coefficients being 0 unless \( \beta \leq \alpha \), resp. \( \beta + \gamma \leq \alpha \)). When replacing \( q_\beta \) by \( x^\beta, q_\gamma \) by \( x^\gamma \) etc, the polynomials \( D^\alpha x^\alpha, D^2 x^\alpha \) etc thus become \( x^\alpha \) times a positive integer coefficient. The same holds for \( h_\alpha = e^D x^\alpha \), whence the lemma. ■
Going back to (22) we have \( h_\alpha (-i\partial_\xi, -i\partial_\mu) \tilde{f} = C_\alpha \hat{x}_\alpha f \) and we conclude that, for \((\xi, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{N-n}\),

\[
\tilde{f}(\xi, \lambda, \mu) = \sum_{\rho, \sigma, \alpha} C_\alpha Q_{\alpha \rho \sigma} (-i\partial_\lambda, 0)(\hat{x}_\alpha f)(0, \lambda, 0) \frac{\varepsilon^\rho \mu^\sigma}{\rho! \sigma!}.
\]

Therefore the restriction to all \((0, \lambda, 0)\) of the \(m^n\) functions \(\hat{x}_\alpha f\), \(\alpha \in A\), determines \(f\). In other words, the Cauchy problem for (19) is well-posed with the Cauchy data \(h_\alpha (-i\partial_\xi, -i\partial_\mu) \tilde{f} = C_\alpha \hat{x}_\alpha f\) on the \(n\)-plane of \(\mathbb{R}^{n+N}\) defined by \(\xi = \mu = 0\).

In terms of Radon transforms we obtain the following result.

**Theorem 6** A function \(f \in \mathcal{D}(\mathbb{R}^n)\) is uniquely determined by the \(m^n\) nonlinear Radon transforms \(R(x_\alpha^m)(t, \lambda, 0)\) (with \(\alpha \in \mathbb{N}^n\), \(0 \leq \alpha_j < m\), \(t \in \mathbb{R}\), \(\lambda \in \mathbb{R}^n \setminus \{0\}\)), that is by the integrals of each \(x_\alpha^m f\) on the hypersurfaces

\[
\lambda_1 x_1^m + \cdots + \lambda_n x_n^m = t.
\]

## 5 Inversion Formulas

Let us now look for an inversion formula for the nonlinear Radon transform. The nonlinear Fourier transform \(\tilde{f}\) is greatly overdetermined, with \(n+N\) variables \((\xi, \lambda)\) instead of \(n\) for \(f\). As in Section 3 we shall restrict \(\tilde{f}\) to \(\xi = 0\) and, assuming the monomials \(x_\alpha^m\) are ordered as \(x_1^m, \ldots, x_n^m\) first, followed by the other \(x_\beta^m\)'s, it turns out that (as in the final remark of Section 3) we can also restrict to \(\lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)\), written as \(\lambda \in \mathbb{R}^n\) for short. Then

\[
\tilde{f}(0, \tau \lambda) = \int_{\mathbb{R}^n} e^{i\tau \sum_1^n \lambda_j x_j^m} f(x) dx = \tilde{Rf}(\tau, \lambda) \quad \text{with} \quad \tau \in \mathbb{R}, \lambda \in \mathbb{R}^n.
\]

### 5.1 First Case: \(m\) odd

Let \(U\) denote the dense open subset of \(\mathbb{R}^n\) defined by \(x_j \neq 0\) for all \(j\). For \(m\) odd the map \(\psi : x \mapsto y = x^m := (x_1^m, \ldots, x_n^m)\) is a diffeomorphism of \(U\) onto itself. Then

\[
\tilde{Rf}(\tau, \lambda) = \int_{\mathbb{R}^n} e^{i\lambda \cdot y} g(y) dy \equiv \hat{g}(\tau \lambda)
\]

with \(\lambda \cdot y = \sum_1^n \lambda_j y_j\) and

\[g(y) := m^{-n} |y_1 \cdots y_n|^{(1/m)-1} f(y^{1/m}) , \quad y \in \mathbb{R}^n.\]

As above \(\hat{g}\) denotes the classical \(n\)-dimensional Fourier transform and \(\tilde{Rf}\) is the \(1\)-dimensional Fourier transform with respect to \(t\).

The change \(x \mapsto y\) thus reduces the nonlinear Radon transform \(R\) to the linear one considered in the introduction: \(Rf(t, \lambda) = R_0 g(t, \lambda)\). But \(g\) is not necessarily smooth, \(\hat{g}(\lambda) = \tilde{Rf}(1, \lambda)\) is not necessarily rapidly decreasing and the inversion formula (2) may become invalid here. However \(g\) is integrable on \(\mathbb{R}^n\) and vanishes outside a compact set, therefore defines a tempered distribution. Denoting by \(\mathcal{F}\) the
inverse Fourier transform for tempered distributions on $\mathbb{R}^n$ we have $g = \mathcal{F}\hat{g}$ hence, for any $u \in \mathcal{D}(U)$,

$$
\int_U f(x) u(x^m) dx = \int_U g(y) u(y) dy = \langle \mathcal{F}\hat{g}(y), u(y) \rangle \\
= \langle (\psi^*\mathcal{F}\hat{g})(x), \det \psi'(x) u(\psi(x)) \rangle \\
= \langle m^n(x_1 \cdots x_n)^{m-1} (\psi^*\mathcal{F}\hat{g})(x), u(x^m) \rangle,
$$

using the pullback by $\psi$ of the distribution $\mathcal{F}\hat{g}$ on $U$ (cf. [6] p. 80). The absolute value may be skipped here since $m - 1$ is even and $\det \psi' > 0$. Therefore, for $f \in \mathcal{D}(\mathbb{R}^n)$,

$$f(x) = m^n(x_1 \cdots x_n)^{m-1} (\psi^*\mathcal{F}\hat{R}f(1, \cdot))(x), \quad (25)$$

an equality of distributions on $U$.

5.2 Second Case: $m$ even

The above map $\psi : x \mapsto y$ is no more a bijection: given $y$ with all $y_j > 0$, the equations $y = \frac{x^m}{m}$ now have $2^n$ solutions $x = \left( \pm y_1^{1/m}, \ldots, \pm y_n^{1/m} \right)$.

For $x, y \in \mathbb{R}^n$ we write $xy := (x_1 y_1, \ldots, x_n y_n)$. Let $E := \{1, -1\}^n$ denote the set of all $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_j = \pm 1$ and

$$\mathbb{R}_+^n := \{ x \in \mathbb{R}^n \mid x_j > 0 \text{ for } 1 \leq j \leq n \}. $$

Viewing the integral (23) as a sum of integrals over the quadrants $\varepsilon\mathbb{R}_+^n$, $\varepsilon \in E$, we obtain, by the change of variables $x \mapsto y$ with $x_j = \varepsilon_j y_j^{1/m}$, $y_j > 0$, on $\varepsilon\mathbb{R}_+^n$,

$$\mathcal{F}\hat{R}f(\tau, \lambda) = \hat{f}(0, \tau \lambda) = \int_{\mathbb{R}_+^n} e^{i\tau \lambda \cdot y} g(y) dy
$$

with $\tau \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$ and, for $y \in \mathbb{R}_+^n$,

$$g(y) := m^{-n} (y_1 \cdots y_n)^{(1/m)-1} \sum_{\varepsilon \in E} f(\varepsilon y^{1/m}).$$

Let $H$ denote the Heaviside function $H(y) = 1$ if $y \in \mathbb{R}_+^n$, $H(y) = 0$ otherwise. Equation (24) is now replaced by

$$\mathcal{F}\hat{R}f(\tau, \lambda) = \int_{\mathbb{R}_+^n} e^{i\tau \lambda \cdot y} H(y) g(y) dy = \mathcal{F}\hat{H}_g(\tau \lambda).$$

Again $H_g$ is integrable and vanishes outside a compact set, hence tempered on $\mathbb{R}^n$, and as above the Fourier inversion $H_g = \mathcal{F}\mathcal{F}\hat{H}_g$ implies the following equality of distributions on $\mathbb{R}_+^n$

$$\sum_{\varepsilon \in E} f(\varepsilon x) = m^n (x_1 \cdots x_n)^{m-1} (\psi^*\mathcal{F}\hat{R}f(1, \cdot))(x). \quad (26)$$
This gives \( f \) if its support is contained in some quadrant \( \mathbb{R}^n_+ \). Otherwise we must separate the components \( f(\varepsilon x) \), which can be achieved by replacing \( f \) with \( x^\alpha f \) for suitably chosen \( \alpha \)'s as follows.

With each \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E \) we associate the monomial

\[
p_\varepsilon(x) := x_{i_1} \cdots x_{i_k}
\]

where \( 1 \leq i_1 < \cdots < i_k \leq n \) is the (ordered) set of indices \( i \) such that \( \varepsilon_i = -1 \); for instance, \( n = 4 \) and \( \varepsilon = (-1, 1, -1, 1) \) yield \( p_\varepsilon(x) = x_1 x_3 \). The map \( \varepsilon \mapsto p_\varepsilon \) is a bijection of \( E \) onto the set of divisors of \( x_1 \cdots x_n \).

Let \( \varepsilon, \eta \in E \). A minus sign occurs in \( p_\varepsilon(\eta x) = p_\varepsilon(\eta_1 x_1, \ldots, \eta_n x_n) \) each time there is a factor \( x_i \), that is \( \varepsilon_i = -1 \), and the corresponding \( \eta_i \) is \(-1\). Therefore

\[
p_\varepsilon(\eta x) = a_{\varepsilon, \eta} p_\varepsilon(x) \quad \text{with} \quad a_{\varepsilon, \eta} := (-1)^{k(\varepsilon, \eta)},
\]

where \( k(\varepsilon, \eta) \) denotes the number of indices \( i \) such that \( \varepsilon_i = \eta_i = -1 \).

**Example.** For \( n = 2 \) the matrix \( (a_{\varepsilon, \eta}) \) is given by the table:

\[
\begin{array}{cccccc}
\varepsilon & 1 & x_1 & x_2 & x_1 x_2 & \\
\eta & ++ & -+ & ++ & -- & \\
++ & 1 & 1 & 1 & 1 & \\
-+ & 1 & -1 & 1 & -1 & \\
+- & 1 & 1 & -1 & -1 & \\
-- & 1 & -1 & -1 & 1 & \\
\end{array}
\]

Our inversion formula for \( R \) will be inferred from the following combinatorial lemma.

**Lemma 7** The set \( E = \{1, -1\}^n \) being provided with some ordering, the \( 2^n \times 2^n \) matrix \( A = (a_{\varepsilon, \eta})_{\varepsilon, \eta \in E} \) is symmetric and \( A^2 = 2^n I \) (where \( I \) is the unit matrix).

**Proof.** The symmetry is clear by the definition of \( k(\varepsilon, \eta) \).

For \( \varepsilon, \eta, \zeta \in E \) we have \( k(\varepsilon, \eta \zeta) = k(\varepsilon, \eta) + k(\varepsilon, \zeta) \) since \( \varepsilon_i = \eta_i \zeta_i = -1 \) is equivalent to \( \varepsilon_i = -1 \) and \( \eta_i = -1 \), \( \zeta_i = 1 \) or (exclusive or) \( \varepsilon_i = -1 \) and \( \eta_i = 1 \), \( \zeta_i = -1 \). Therefore

\[
a_{\varepsilon, \eta} a_{\varepsilon, \zeta} = a_{\varepsilon, \eta \zeta}.
\]

Besides, for fixed \( \eta \in E \),

\[
\prod_{i=1}^n (1 + \eta_i x_i) = 1 + \sum_i \eta_i x_i + \sum_{i<j} \eta_i \eta_j x_i x_j + \cdots + \eta_1 \cdots \eta_n x_1 \cdots x_n = \sum_{\varepsilon \in E} p_\varepsilon(\eta x) = \sum_{\varepsilon \in E} a_{\varepsilon, \eta} p_\varepsilon(x).
\]

Taking \( x_1 = \cdots = x_n = 1 \) this gives the sum of elements in each column (or row) of \( A \):

\[
\sum_{\varepsilon \in E} a_{\varepsilon, \eta} = \prod_{i=1}^n (1 + \eta_i) = \begin{cases} 2^n & \text{if } \eta = (1, \ldots, 1) \\
0 & \text{otherwise.} \end{cases}
\]
Now (28) implies
\[\sum_{\zeta \in E} a_{\varepsilon, \eta} a_{\varepsilon, \zeta} = \begin{cases} 2^n & \text{if } \eta \zeta = (1, \ldots, 1) \\ 0 & \text{otherwise.} \end{cases}\]

But \(\eta \zeta = (1, \ldots, 1)\) is equivalent to \(\eta_i = \zeta_i\) for all \(i\), that is \(\eta = \zeta\). Remembering the symmetry of \(A\), we infer that \(A^2 = 2^n I\).

Let us consider \(S f(x) := \sum_{\eta \in E} f(\eta x)\). Replacing \(f\) by \(p \cdot f\) we obtain, in view of (27),
\[S(p \cdot f)(x) = \sum_{\eta \in E} (p \cdot f)(\eta x) = p \cdot (x) \sum_{\eta} a_{\varepsilon, \eta} f(\eta x),\]
which can be inverted by \(A^{-1} = 2^{-n} A\) (Lemma 7) as
\[f(\eta x) = 2^{-n} \sum_{\zeta \in E} a_{\varepsilon, \eta} p \cdot (x)^{-1} S(p \cdot f)(x)\]
for each \(\eta \in E\). By (26) applied to each \(p \cdot f\) we have
\[S(p \cdot f)(x) = m^n (x_1 \cdots x_n)^{m-1} \psi^*(\mathcal{F}R_{p \cdot f}(1, \cdot))(x)\]
on \(\mathbb{R}^n_+\) and the latter equations show that \(f\) can be reconstructed in each quadrant of \(\mathbb{R}^n\) from the \(2^n\) nonlinear Radon transforms \(R f, R(x_1 f), R(x, x_2 f), \ldots, R(x_1 \cdots x_n f)\).

Summarizing we have proved the following theorem. Let us recall our notation: \(\tilde{R} f = Rf(1, \lambda)\) is given by (23) with \(\lambda \in \mathbb{R}^n\), \(\mathcal{F}\) is the inverse Fourier transform of tempered distributions on \(\mathbb{R}^n\), \(\psi^*\) is the pullback of distributions by \(\psi(x) = (x_1^m, \ldots, x_n^m)\), \(E = \{1, -1\}^n\) and \(p \cdot, a_{\varepsilon, \eta}\) are defined before Lemma 7.

**Theorem 8** The nonlinear Radon transform (7) is inverted by the following formulas, where \(f \in \mathcal{D}(\mathbb{R}^n)\).

(i) if \(m\) is odd
\[f(x) = m^n (x_1 \cdots x_n)^{m-1} (\psi^* \mathcal{F} \tilde{R} f)(x)\]

(equalities of distributions on the open set \(x_1 \neq 0, \ldots, x_n \neq 0\));

(ii) if \(m\) is even: for \(\eta \in E\),
\[f(\eta x) = \left(\frac{m}{2}\right)^n \sum_{\varepsilon \in E} a_{\varepsilon, \eta} p \cdot (x)^{-1} (x_1 \cdots x_n)^{m-1} (\psi^* \mathcal{F} \tilde{R} p \cdot f)(x)\]

(equalities of distributions on the open set \(x_1 > 0, \ldots, x_n > 0\)).

**References**


