

Positivity of the cotangent sheaf of singular Calabi-Yau varieties

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Abstract

We prove that the tangent and the reflexivized cotangent sheaves of any normal projective klt Calabi-Yau or irreducible holomorphic symplectic variety are not pseudoeffective, generalizing results of A. Höring and T. Peternell [HP19]. We provide examples of Calabi-Yau varieties of small dimension with singularities in codimension 2.

1 Introduction

Complex algebraic varieties with trivial canonical class are of great importance in birational geometry. Indeed, they appear naturally as possible minimal models in the Minimal Model Program (MMP), and come in quite diverse geometrical families. Since higher-dimensional MMP is generally gives rise to singular minimal models, understanding singular projective varieties with trivial canonical class is particularly relevant. Recently, three papers [HP19, Thm.1.5], [GGK19], [Dru18] achieved a singular decomposition result for these varieties :

Theorem 1.1. *Let X be a normal projective variety with klt singularities, with K_X numerically trivial. Then there exists a normal projective variety \tilde{X} with at most canonical singularities, which comes with a quasiétale finite cover $f : \tilde{X} \rightarrow X$ and decomposes as a product:*

$$\tilde{X} \cong A \times \prod_i Y_i \times \prod_j Z_j,$$

where A is a smooth abelian variety, the Y_i are singular Calabi-Yau varieties and the Z_j are singular irreducible holomorphic symplectic (IHS) varieties, as defined in Section 5.

May it seem an expected generalization of the smooth Beauville-Bogomolov decomposition result [Bea84], [Bea83], this theorem however relies on serious results from each paper: [GGK19] introduces algebraic holonomy and studies infinitesimal decompositions of the tangent sheaf \mathcal{T}_X ; [Dru18] deals with the abelian part in the infinitesimal decomposition and proves an integrability criterion for the remaining subsheaves of \mathcal{T}_X ; [HP19] establishes positivity results which add up to Druel's criterion to finish the proof.

Interestingly enough, the singular decomposition for a klt variety X may not be the same as the singular decomposition of its terminalisation. The typical example is a singular Kummer surface, which resolves by 16 blow-ups into a smooth K3 surface, but has the Beauville-Bogomolov type of an abelian surface. Other such intriguing examples are given in [GGK19, Sect.14]. Compatibility of the singular Beauville-Bogomolov decomposition with terminalisation nevertheless holds for *some* klt varieties with trivial canonical class [Dru18, Lem.4.6]. This license to terminalise is essential in the current proof of [HP19, Thm.1.5], as it involves positivity results [HP19, Thm.1.1] for klt varieties which are smooth in codimension 2: any klt variety is not, but its terminalisation surely is.

Since these positivity results have a wider scope than the mere proof of the singular decomposition theorem, it is worth extending them to normal projective klt varieties. Our main theorem is:

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Theorem 1.2. *Let X be a normal projective variety with klt singularities and numerically trivial K_X . If its tangent or reflexivized cotangent sheaf is pseudoeffective, then there is a quasiétale finite cover $\tilde{X} \rightarrow X$ such that $q(\tilde{X}) \neq 0$. Equivalently, the singular Beauville-Bogomolov decomposition of X has an abelian factor of positive dimension.*

In particular, if X is a singular Calabi-Yau or IHS variety in the sense of Def.5.1, then neither \mathcal{T}_X nor its dual $\Omega_X^{[1]}$ is pseudoeffective.

Importantly enough, this theorem does not boil down to [HP19, Thm.1.6] on a terminalisation of X ; we inevitably have to deal with codimension 2 quotient singularities on X . In this perspective, we resort to the theory of orbifold Chern classes. It has been developed in the late eighties in connection to the abundance problem for threefolds [Kol92], and we will extensively use some of its most recent developments, *inter alia* [LT18], [GKPT20], [GKT].

Let us present a brief outline of the proof, say for a variety X with pseudoeffective tangent sheaf.

The fact that \mathcal{T}_X is pseudoeffective pullsback and restricts to one factor in the Beauville-Bogomolov decomposition of X . Supposing by contradiction that X has no abelian part, we can reduce to a Calabi-Yau or IHS factor Z such that \mathcal{T}_Z is pseudoeffective. The work of [GGK19] also establishes that \mathcal{T}_Z and all its symmetric powers are stable of slope zero with respect to any polarisation H . Finally, since Z is not abelian, its orbifold second Chern class satisfies $\hat{c}_2(\mathcal{T}_Z) \cdot H^{\dim X - 2} \neq 0$. This contradicts the following generalization of [HP19, Thm.1.1]:

Theorem 1.3. *Let X be a normal projective variety with klt singularities of dimension n , H a \mathbb{Q} -Cartier ample divisor on X . Consider \mathcal{E} a reflexive sheaf on X such that:*

- $\hat{c}_1(\mathcal{E}) \cdot H^{n-1} = 0$;
- the sheaves \mathcal{E} and $S^{[l]}\mathcal{E}$, for some $l \geq 6$, are H -stable;
- \mathcal{E} is pseudoeffective.

Then $\hat{c}_1(\mathcal{E})^2 \cdot H^{n-2} = \hat{c}_2(\mathcal{E}) \cdot H^{n-2} = 0$.

Moreover, there is a finite Galois covering $\nu : \tilde{X} \rightarrow X$, étale in codimension 1, such that $\nu^{[*]}\mathcal{E}$ is locally-free, has a numerically trivial determinant, and is $\text{Gal}(\tilde{X}/X)$ -equivariantly flat on \tilde{X} , ie comes from a $\text{Gal}(\tilde{X}/X)$ -equivariant representation of $\pi_1(\tilde{X})$. In particular, $\nu^{[*]}\mathcal{E}$ is numerically flat, and, as symmetric multilinear forms on $NS(X)$:

$$c_1(\nu^{[*]}\mathcal{E}) \equiv 0, \quad c_2(\nu^{[*]}\mathcal{E}) \equiv 0.$$

The hard part here is the first assertion on the vanishing of orbifold Chern classes, the rest follows from [LT18].

In Section 2, we recall and prove basics to reduce the proof of Thm.1.3 to working on a normal projective klt surface S . A crucial ingredient is that orbifold Chern classes behave well under certain restrictions [GKPT20, Prop.3.11]. In Section 3, we introduce an *unfolding* $p : \hat{S} \rightarrow S$, obtained by gluing together local finite Galois quasiétale resolutions of the singularities of S . The surface \hat{S} may be as singular as S ; importantly enough though, any reflexive sheaf \mathcal{E} on S reflexively pulls back to a locally-free sheaf $\hat{\mathcal{E}}$ on \hat{S} . We investigate the relationship of \mathcal{E} and $\hat{\mathcal{E}}$. The key of the proof of Thm.1.3 is then to establish the nefness of $\hat{\mathcal{E}}$, which yields the Chern classes vanishing for $\hat{\mathcal{E}}$, hence for \mathcal{E} . Note that \mathcal{E} may very well not be nef itself: see Remark 2.7.

As a conclusive remark, note that investigating pseudoeffectivity of the tangent and reflexivized cotangent sheaves of a variety with trivial canonical class requires knowledge of its singular Beauville-Bogomolov decomposition. To that extent, Thm.1.2 cannot be used on an explicit variety before knowing a bare minimum about its geometry. In Sections 6 and 7, we discuss small dimensional explicit varieties that are Calabi-Yau varieties with singularities in codimension 2, hence examples for Thm.1.2. In dimension 2, we go through a systematical treatment of normal canonical surfaces with trivial canonical class, listing the 10 singular types of those with an abelian Beauville-Bogomolov type. In dimension 3, we exhibit a family of 2409 varieties among the 7555 wellformed quasismooth hypersurfaces of trivial canonical sheaf in weighted projective 4-dimensional spaces [KS92], that are Calabi-Yau threefolds with singularities in codimension 2. These examples stay out of the range of the earlier pseudoeffectivity result of [HP19, Thm.1.6], but are covered by our Thm.1.2.

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2 Notation and basic facts

Finite morphisms. We will deal with various types of finite maps.

Definition 2.1. Unless otherwise stated, all finite morphisms we speak about are surjective; we may well refer to them as finite coverings, without saying anything about how étale they are. We refer say that a finite morphism is *quasi-étale* if it is étale in codimension 1. Following [GKP16a], we call a finite morphism of normal varieties $Y \rightarrow X$ *Galois* if it is the quotient map of Y by a finite group action.

Reflexive sheaves. Let \mathcal{E} be a reflexive sheaf on a variety X . Recall the reflexivization functor $\mathcal{F} \mapsto \mathcal{F}^{**}$ enables to perform general algebraic operations in the category of reflexive sheaves.

Notably, we will denote by:

- $S^{[l]}\mathcal{E}$ the reflexivization of the l -th symmetric power of \mathcal{E} (for $l \in \mathbb{N}$),
- $\nu^{[*]}\mathcal{E}$ the reflexivization of the pullback of \mathcal{E} (for $\nu : Y \rightarrow X$ a morphism).

Note that, by [Har80, Prop.1.6], reflexive sheaves are normal: if \mathcal{E} is a reflexive sheaf on X , then for all open sets $V \subset U \subset X$ such that $\text{codim}_U U \setminus V \geq 2$, the restriction map $\mathcal{E}(U) \rightarrow \mathcal{E}(V)$ is an isomorphism. In particular, a morphism between two reflexive sheaves which is an isomorphism when restricted to a *big open set* (ie an open set whose complementary has codimension at least 2) is a global isomorphism.

Lemma 2.2. *Let $p : X \rightarrow Y$ be a finite morphism between normal projective varieties. The functor $p^{[*]}$ from the category of reflexive sheaves on Y to that of reflexive sheaves on X is left-exact.*

Proof. Let $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G}$ be an exact sequence of reflexive sheaves on Y , and denote by $Z \subset Y$ a closed subscheme of codimension at least 2 such that our reflexive sheaves are locally-free on $Y \setminus Z \subset Y_{\text{reg}}$. Reflexive pullback a priori only gives morphisms

$$p^{[*]}\mathcal{E} \rightarrow p^{[*]}\mathcal{F} \rightarrow p^{[*]}\mathcal{G},$$

whose composition is zero. By [SPA, Lem.31.12.7], the kernel K of the morphism $p^{[*]}\mathcal{F} \rightarrow p^{[*]}\mathcal{G}$ is reflexive. There is a natural morphism from $p^{[*]}\mathcal{E}$ to the kernel K , which restricts to an isomorphism over $X \setminus p^{-1}(Z)$. As both sheaves are reflexive and $p^{-1}(Z)$ has codimension 2, they are isomorphic over all X . \square

Some pullbacks of reflexive sheaves are automatically reflexive [GKP16a, Prop.5.1.2]:

Proposition 2.3. *Let X be a normal projective variety of dimension $n \geq 2$, let H be an ample \mathbb{Q} -Cartier divisor on X , and \mathcal{E} be a reflexive sheaf on X . Then, for m big and divisible enough, a general element D in $|mH|$ is a normal variety and $\mathcal{E}|_D$ is a reflexive sheaf.*

2.1 Nefness and pseudoeffectivity

Let us recall that a coherent sheaf \mathcal{E} on a normal variety X has a projectivization $\mathbb{P}(\mathcal{E})$ with a canonical, so-called tautological, line bundle ζ on it and a natural morphism $p : \mathbb{P}(\mathcal{E}) \rightarrow X$ with a natural sheaf quotient map: $p^*\mathcal{E} \twoheadrightarrow \zeta$. An account on this set-up is given in [DG65, Chapt.4]. We simply recall the universal property of this construction: for any scheme $q : C \rightarrow X$, to give an X -morphism $\nu : C \rightarrow \mathbb{P}(\mathcal{E})$ is equivalent to giving a line bundle L over C together with a sheaf surjection $q^*\mathcal{E} \twoheadrightarrow L$.

Projectivizations are standardly used for generalizing positivity notions of line bundles to coherent sheaves, as follows.

Definition 2.4. Let \mathcal{E} be a coherent sheaf on a normal variety X . It is called *nef* if the tautological line bundle ζ on $\mathbb{P}(\mathcal{E})$ is nef.

Remark 2.5. This coincides with [Laz03, Def.6.1.1] when the sheaf \mathcal{E} is locally-free. Note that for a torsion-free coherent sheaf \mathcal{E} , the scheme $\mathbb{P}(\mathcal{E})$ may well have several irreducible components. Somehow, several of these components may be relevant for studying the positivity of \mathcal{E} : not only the mere one which is dominant onto X , but also components which may be contracted to a non-zero dimensional locus of X . Such components don't exist for a reflexive sheaf on a normal projective surface: so in this case, nefness is easier to study.

Proposition 2.6. *We have the following properties:*

- if $Y \subset X$ is a normal subvariety, and \mathcal{E} is a nef coherent sheaf on X , then $\mathcal{E}|_Y$ is nef;
- conversely, nefness of a coherent sheaf \mathcal{E} is enough to be checked on all curves of $\mathbb{P}(\mathcal{E})$;
- if $f : Y \rightarrow X$ is a finite dominant morphism of normal varieties and \mathcal{E} is a coherent sheaf on X , \mathcal{E} is nef if and only if $f^*\mathcal{E}$ is;
- if $f : Y \rightarrow X$ is a proper birational morphism resolving the singularities of a normal variety X and \mathcal{E} is a coherent sheaf on X such that $f^*\mathcal{E}$ is nef, then \mathcal{E} is nef;
- any coherent sheaf which is a quotient of a nef coherent sheaf is nef.

These are simple consequences of the universal property of $\mathbb{P}(E)$ and of the fact [DG65, 4.1.3.1] that for a dominant morphism $f : Y \rightarrow X$ and a coherent sheaf \mathcal{E} in X ,

$$\mathbb{P}(f^*\mathcal{E}) = \mathbb{P}(\mathcal{E}) \times_X Y.$$

Interestingly enough, nefness does not behave well through reflexive pullbacks.

Remark 2.7. Let X be a *singular Kummer surface*, ie the finite quotient of an abelian surface A by the involution $i : a \mapsto -a$. Since $p : A \rightarrow X$ is a finite quasiétale cover and \mathcal{T}_X is locally-free on a big open set, the reflexive sheaves $p^{[*]}\mathcal{T}_X$ and \mathcal{T}_A are the same. In particular,

$$p^{[*]}\mathcal{T}_X = \mathcal{O}_A \oplus \mathcal{O}_A \text{ is nef.}$$

We are going to prove that \mathcal{T}_X itself is not nef. We first compute it.

Recalling [GKKP11, App.B], we consider the functor taking the invariant direct image of a \mathbb{Z}_2 -coherent sheaf on A : $\mathcal{E} \mapsto p_*\mathcal{E}^{\mathbb{Z}_2}$. It sends reflexive sheaves to reflexive sheaves, so that the following equality, which is clear on the big open étale locus of p , extends to a global sheaf isomorphism:

$$\text{for any } \mathcal{E} \text{ reflexive sheaf on } X, p^{[*]}\mathcal{E} \text{ is naturally } \mathbb{Z}_2\text{-equivariant and } (p_*p^{[*]}\mathcal{E})^{\mathbb{Z}_2} \cong \mathcal{E}. \quad (1)$$

Moreover, it is an exact functor. Still from [GKKP11, App.B], if \mathcal{E} is a \mathbb{Z}_2 -equivariant coherent sheaf on A , then the sheaf $p_*\mathcal{E}^{\mathbb{Z}_2}$ is a direct summand of $p_*\mathcal{E}$. Note that a given coherent sheaf \mathcal{E} on A may have several structures of \mathbb{Z}_2 -equivariant object. For example, \mathcal{O}_A comes with a natural and a reversed action, defined on an affine open set U by:

$$\begin{aligned} f \in \mathcal{O}_A(U) &\mapsto f \circ i \in \mathcal{O}_A(i(U)), \\ f \in \mathcal{O}_A(U) &\mapsto -f \circ i \in \mathcal{O}_A(i(U)). \end{aligned}$$

So $p_*\mathcal{O}_A$ is the direct sum of two reflexive sheaves of rank 1, \mathcal{O}_X (the invariants by the natural action) and \mathcal{F} (the invariants by the reversed action).

We know that $\mathcal{T}_X \cong p_*(\mathcal{O}_A \oplus \mathcal{O}_A)^{\mathbb{Z}_2}$ with Eq.1. We note that the natural \mathbb{Z}_2 -equivariant structure on \mathcal{T}_A acts diagonally, and reversely on each trivial summand. So $\mathcal{T}_X \cong \mathcal{F} \oplus \mathcal{F}$.

Let us finally check that \mathcal{F} is not nef. We compute locally: let $V \subset X, U = p^{-1}(V) \subset A$ be affine open sets with local coordinates $(x, y) \in \mathbb{C}^2 \simeq U$ so that $p|_U$ ramifies only at $(0, 0)$. The quotient map $p : U \rightarrow V$ rewrites:

$$\begin{aligned} \mathbb{C}[u, v, w]/(uv - w^2) &\cong \mathcal{O}_X(V) \rightarrow \mathbb{C}[x, y] \cong \mathcal{O}_A(U) \\ u, v, w &\mapsto x^2, y^2, xy, \end{aligned}$$

so its image $\mathbb{C}[x^2, y^2, xy]$ identifies with the local ring $\mathcal{O}_X(V)$. Hence,

$$\mathcal{F}(V) \simeq \{f \in \mathbb{C}[x, y] \mid \forall x, y, f(x, y) = -f(-x, -y)\} = x\mathbb{C}[x^2, y^2, xy] \oplus y\mathbb{C}[x^2, y^2, xy],$$

so that $\mathcal{F}^{\otimes 2}(V) \simeq u\mathcal{O}_X(V) \oplus v\mathcal{O}_X(V) \oplus w\mathcal{O}_X(V) = \mathcal{I}_{\text{Sing}(X)}(V)$. This isomorphism is actually global:

$$\mathcal{F}^{\otimes 2} \cong \mathcal{I}_{\text{Sing}(X)}.$$

Ideal sheaves are not nef, so $\mathcal{F}^{\otimes 2}$ is not nef, so by [Kub70, Prop.2], \mathcal{F} is not nef.

Pseudoeffectivity is standardly defined for locally-free sheaves through projectivisation too:

Definition 2.8. Let \mathcal{E} be a locally-free sheaf on a normal projective variety X . It is considered *pseudoeffective* if it satisfies one of the following equivalent conditions:

- the tautological line bundle on $\mathbb{P}(\mathcal{E})$ is pseudoeffective;
- there is an ample Cartier divisor H on X such that for all $c > 0$, there are integers i, j such that $i > cj > 0$ and

$$h^0(X, \mathrm{Sym}^i \mathcal{E} \otimes \mathcal{O}_X(jH)) \neq 0.$$

Remark 2.9. Equivalence of both conditions comes from [Dru18, Lem.2.7].

Generalizing this definition to a meaningful pseudoeffectivity notion for any coherent sheaf seems intricate. For reflexive sheaves, we will use [HP19, Def.2.1]:

Definition 2.10. Let X be a normal projective variety and H an ample Cartier divisor on X . A reflexive sheaf \mathcal{E} on X is said *pseudoeffective* if, for all $c > 0$, there are numbers $i, j \in \mathbb{N}$ with $i > cj$ such that:

$$h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) \neq 0.$$

Example 2.11. The sheaf \mathcal{T}_X in Remark 2.7 is pseudoeffective, as $\mathcal{T}_X = \mathcal{F} \oplus \mathcal{F}$ with $S^{[2]}\mathcal{F} \cong \mathcal{O}_X$.

Definition 2.12. Let \mathcal{E} be a reflexive sheaf on a normal projective variety X . Denote by $\mathbb{P}'(\mathcal{E})$ the normalization of the unique dominant component of $\mathbb{P}(\mathcal{E})$ onto X . Let P be a resolution of $\mathbb{P}'(\mathcal{E})$, such that the birational morphism $r : P \rightarrow \mathbb{P}'(\mathcal{E})$ over X is an isomorphism precisely over the open locus $X_0 \subset X_{\mathrm{reg}}$ where \mathcal{E} is locally-free.

Denoting by π the morphism $P \rightarrow \mathbb{P}(\mathcal{E})$ and by $\mathcal{O}_P(1)$ the pullback of the tautological bundle of $\mathbb{P}(\mathcal{E})$ by π , [Nak04, V.3.23] asserts that one can choose (often not uniquely) an effective divisor Λ supported in the exceptional locus of r such that

$$\zeta := \mathcal{O}_P(1) \otimes \mathcal{O}_P(\Lambda)$$

satisfies $\pi_* \zeta^{\otimes m} \simeq S^{[m]} \mathcal{E}$ for all $m \in \mathbb{N}$. Such ζ is called a *tautological class* of \mathcal{E} .

As said in [HP19, Lem.2.3],

Lemma 2.13. *With the same notations as previously, ζ is pseudoeffective on P if and only if \mathcal{E} is pseudoeffective as a reflexive sheaf.*

This point of view notably shows the independency of Definition 2.10 of the choice of an ample Cartier divisor H .

Proposition 2.14. *Let X be a normal projective variety, H an ample \mathbb{Q} -Cartier divisor, \mathcal{E} a pseudoeffective reflexive sheaf on X . Then for m big and divisible enough, for a general element $D \in |mH|$, the sheaf $\mathcal{E}|_D$ is reflexive and pseudoeffective.*

Proof. Let $U \subset X_{\mathrm{reg}}$ be a big open set on which \mathcal{E} is locally-free. For m big and divisible enough and for a general element D in $|mH|$, $U \cap D$ is a big open set of D . By Proposition 2.3, we can assume D is a normal subvariety of X and $\mathcal{E}|_D$ is reflexive.

Let us fix a $c > 0$ and take i, j integers such that $i > cj > 0$ and $h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) > 0$. Up to taking a smaller j (which may be negative if needed), we can assume that

$$h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X((j-m)H)) = 0.$$

By normality of reflexive sheaves,

$$\begin{aligned} h^0(D, S^{[i]}(\mathcal{E}|_D) \otimes \mathcal{O}_D(jH)) &= h^0(U \cap D, S^i(\mathcal{E}|_{U \cap D}) \otimes \mathcal{O}_{U \cap D}(jH)) \\ &\geq h^0(U, S^i(\mathcal{E}|_U) \otimes \mathcal{O}_U(jH)) - h^0(U, S^i(\mathcal{E}|_U) \otimes \mathcal{O}_U((j-m)H)) \\ &= h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) - h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X((j-m)H)) > 0, \end{aligned}$$

where the second equality comes from tensoring by $S^i(\mathcal{E}|_U) \otimes \mathcal{O}_U(jH)$ and going to cohomology in the following exact sequence:

$$0 \rightarrow \mathcal{O}_U(-mH) \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_{U \cap D} \rightarrow 0.$$

□

Proposition 2.15. *Let \mathcal{E} be a reflexive sheaf on a normal projective variety X , and $f : Y \rightarrow X$ be a finite dominant morphism of normal projective varieties. If \mathcal{E} is pseudoeffective, then $f^{[*]} \mathcal{E}$ is.*

For a more general statement, we refer the reader to [HP, Lem.3.15].

Proof. Since f is finite dominant, the preimage of a big open set by f remains a big open set. In particular, there are two big open sets $U \subset X_{\text{reg}}$ and $f^{-1}(U) =: V \subset Y_{\text{reg}}$ such that $\mathcal{E}|_U$ is locally-free, and hence $(f^{[*]} \mathcal{E})|_V = f|_V^* \mathcal{E}|_U$.

Fix H an ample Cartier divisor on X , H' on Y its ample pullback. For $c > 0$, the pseudoeffectivity of \mathcal{E} grants the existence of integers $i > cj > 0$ such that $S^{[i]} \mathcal{E} \otimes \mathcal{O}_X(jH)$ has non-zero global sections. By normality of reflexive sheaves, we can restrict such non-zero sections to U , then pull them back by $f|_V$. As $f|_V : V \rightarrow U$ is dominant, this gives rise to non-zero sections of

$$f|_V^*(S^i \mathcal{E}|_U) \otimes \mathcal{O}_V(jH') = S^i(f|_V^* \mathcal{E}|_U) \otimes \mathcal{O}_V(jH') = (S^{[i]} f^{[*]} \mathcal{E})|_V \otimes \mathcal{O}_V(jH'),$$

which extend to global sections on Y , by normality again. \square

Lemma 2.16. *Let \mathcal{E} be a nef reflexive sheaf on a normal projective variety X . Then it is pseudoeffective.*

Proof. Let $\pi : P \rightarrow \mathbb{P}(\mathcal{E})$ be a modification as in Definition 2.12, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the tautological bundle of \mathcal{E} and Λ an effective divisor on P such that $\zeta = \pi^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_P(\Lambda)$ is a tautological class of \mathcal{E} .

Since \mathcal{E} is nef, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef, so is its pullback. Hence ζ is pseudoeffective on P , ie \mathcal{E} is pseudoeffective as a reflexive sheaf on X . \square

Definition 2.17. Let D be a \mathbb{Q} -Cartier divisor on a normal projective variety X . We define its *stable base locus*:

$$\mathbb{B}(D) := \bigcap_{m \in M} \text{Bs}(mD),$$

where $M \subset \mathbb{N}$ is the set of all m such that mD is Cartier and $\text{Bs}(mD)$ is the base locus of the linear system $|mD|$.

We then define its *restricted base locus*:

$$B_-(D) := \bigcup_{n \in \mathbb{N}^*} \mathbb{B}\left(D + \frac{1}{n}A\right),$$

where A is an arbitrary very ample divisor. Note that the union is strictly decreasing.

Of course, a nef \mathbb{Q} -divisor has an empty restricted base locus. To that extent, the restricted base locus measures the non-nefness of a pseudoeffective line bundle. However, not all curves of a restricted base locus $B_-(D)$ must be D -non-positive, even in the simpler case where D is a line bundle on a smooth surface and $B_-(D)$ is the negative part of its Zariski decomposition.

2.2 Stability

Definition 2.18. Let \mathcal{E} be a torsion-free coherent sheaf on a normal projective variety X with an ample \mathbb{Q} -Cartier divisor H . Then there exists an open set $U \subset X_{\text{reg}}$ such that $\text{codim}_X(X \setminus U) \geq 2$ and $\mathcal{E}|_U$ is locally-free. For some m big and divisible enough, $n-1$ general members of $|mH|$ cut out a smooth curve C lying in U . The *H-slope* of \mathcal{E} is then well-defined as:

$$\mu_H(\mathcal{E}) := \frac{c_1(\mathcal{E}|_C)}{m^{n-1} \text{rk}(\mathcal{E})}.$$

The sheaf \mathcal{E} is said *stable*, respectively *semistable* with respect to H , if all non-zero torsion-free coherent subsheaves \mathcal{F} satisfy:

$$\mu_H(\mathcal{F}) < \mu_H(\mathcal{E}), \text{ respectively } \mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E}).$$

A generalization of a well-known Mehta-Ramanathan result says that stability behaves well under some well-chosen restrictions; we recall it as it is stated in [HP19, Lem.2.11]:

Lemma 2.19. *Let X be a normal projective variety of dimension n , and H an ample Cartier divisor on X . Let \mathcal{E} be a torsion-free coherent sheaf on X , that is stable with respect to H . Then there is m_0 such that, for all $m \geq m_0$, and for D_1, \dots, D_k general elements of $|mH|$ with $k \in \llbracket 1, n-1 \rrbracket$, if we denote by Y the complete intersection $D_1 \cap \dots \cap D_k$, $\mathcal{E}|_Y$ is stable with respect to $H|_Y$.*

Remark 2.20. Note that the converse is clearly true.

Stability a priori weakens through finite Galois reflexive pullbacks:

Lemma 2.21. *Let $p : Y \rightarrow X$ be a finite Galois cover of normal projective varieties of dimension n , G its Galois group, H an ample \mathbb{Q} -Cartier on X , \mathcal{E} be a reflexive sheaf on X . Let $\mathcal{F} := p^{[*]}\mathcal{E}$. Then, if \mathcal{E} is H -stable, \mathcal{F} is p^*H -semistable.*

Proof. Suppose that \mathcal{E} is H -stable. By Lemma 2.19, on a smooth curve C cut out by $n-1$ very general elements of the linear system defined by a suitable multiple of H , the now locally-free sheaf $\mathcal{E}|_C$ is still $H|_C$ -stable. In particular, [Laz03, Lem.6.4.12] applies; so the pullback sheaf $\mathcal{F}|_{p^{-1}(C)}$ is $p^*H|_C$ -semistable. Hence, \mathcal{F} is H -semistable. \square

Note that positivity and stability of a zero-slope locally-free sheaf are related by Miyaoka's result [Miy87], [Laz03, Prop.6.4.11]:

Proposition 2.22. *Let \mathcal{E} be a vector bundle on a smooth curve C . If \mathcal{E} is semistable and $c_1(\mathcal{E}) = 0$, then \mathcal{E} is nef.*

More subtle than the mere stability of \mathcal{E} is the stability of \mathcal{E} and some of its symmetric powers:

Remark 2.23. We recall an interesting fact stated in [BK08, Cor.6, following Rmk.]. If \mathcal{E} is a locally-free stable sheaf on a smooth projective variety X , then the following are equivalent:

- $S^r \mathcal{E}$ is stable for some $r \geq 6$;
- $S^r \mathcal{E}$ is stable for any $r \geq 6$.

Whether or not the stability of all $S^{[l]}\mathcal{E}$ for $l \in \mathbb{N}$ could boil down to the stability of some $S^{[l]}\mathcal{E}$ for a finite amount of l 's remains an open question, when asked about a reflexive sheaf \mathcal{E} on a smooth projective variety X or about a locally-free sheaf \mathcal{E} on a normal projective variety X .

Nevertheless, this remark allows us to rewrite the key result [HP19, Prop.1.3] in the following way:

Lemma 2.24. *Let \mathcal{E} be a locally-free sheaf on a smooth curve C . Assume that \mathcal{E} and $S^l \mathcal{E}$ for some $l \geq 6$ are stable and that $c_1(\mathcal{E}) = 0$. Denoting by ζ the tautological bundle on $\mathbb{P}(\mathcal{E})$, ζ is nef and satisfies:*

$$\zeta^{\dim Z} \cdot Z > 0,$$

for any closed proper subvariety $Z \subset \mathbb{P}(\mathcal{E})$.

Despite that the reflexive pullback $p^{[*]}\mathcal{E}$ of a H -stable reflexive sheaf \mathcal{E} by a finite dominant morphism p is merely p^*H -semistable and a priori not stable (let alone his reflexive symmetric powers), the conclusive property of Lemma 2.24 is preserved by $p^{[*]}$:

Remark 2.25. Let \mathcal{E} be a reflexive sheaf on a normal projective variety X , and $C \subset X$ a smooth curve such that \mathcal{E} is locally-free in an analytical neighborhood of C , such that the tautological bundle ζ on $\mathbb{P}(\mathcal{E}|_C)$ is nef and such that it holds:

$$\zeta^{\dim Z} \cdot Z > 0,$$

for any closed proper subvariety $Z \subset \mathbb{P}(\mathcal{E}|_C)$.

Let $p : \hat{X} \rightarrow X$ be a finite dominant morphism, where \hat{X} is a normal projective variety. Denote $\hat{C} := p^{-1}(C)$, $\hat{\mathcal{E}} := p^{[*]}\mathcal{E}$ and $\hat{\zeta}$ the tautological bundle of $\mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}})$. If we have that $p^*(\mathcal{E}|_C) = \hat{\mathcal{E}}|_{\hat{C}}$, then the following diagram is Cartesian with tautological compatibility $\hat{\zeta} = q^*\zeta$:

$$\begin{array}{ccc} \mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}}) & \xrightarrow{q} & \mathbb{P}(\mathcal{E}|_C) \\ \downarrow \hat{\pi} & & \downarrow \pi \\ \hat{C} & \xrightarrow{p} & C \end{array}$$

Hence, $\hat{\zeta}$ is nef and satisfies, for any closed proper subvariety $Z \subset \mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}})$:

$$\hat{\zeta}^{\dim Z} \cdot Z > 0.$$

Remark 2.26. The case in which this remark will be relevant for us is when X is a normal projective surface with an ample \mathbb{Q} -Cartier divisor H , C is a smooth curve arising as a very general element of $|mH|$, for m big and divisible enough, and $p : \hat{X} \rightarrow X$ is the morphism constructed in Section 2.3, so that $\hat{\mathcal{E}}$ is locally-free. In this set-up, [GKPT20, Prop.3.11.1] grants the additional assumption $p^*(\mathcal{E}|_C) = \hat{\mathcal{E}}|_{\hat{C}}$.

2.3 Orbifold Chern classes

As we already saw in Def.2.18, torsion-free sheaves on a normal projective variety have a notion of a first Chern class. Simultaneously, locally-free sheaves on a normal projective variety have notions of Chern classes of any degree, which can be seen as acting on algebraic cycles or equivalently on homology classes. For a thorough treatment of these notions, we refer to [MS57, XI] and [Ful98, Chapt.3, Chapt.19].

On a variety X of dimension n that is smooth in codimension 2 (as considered in [HP19]), reflexive sheaves can be found locally-free on an open set U with $\text{codim}_X(X \setminus U) \geq 3$. Hence, one can define a second Chern class $c_2(\mathcal{E}) \in H^4(X; \mathbb{Z})$ just by extending $c_2(\mathcal{E}|_U) \in H^4(U; \mathbb{Z})$.

However, reasonably mild classes of singular varieties, such as klt varieties, may be burdened with codimension 2 singularities. Luckily enough, as [GKKP11, Proposition 9.4] states, codimension 2 singularities of klt varieties necessarily are quotient singularities. There are standard constructions for orbifold first and second Chern classes of a reflexive sheaf \mathcal{E} on a normal projective variety X , whose singularities in codimension 2 are all quotient singularities. References for them include [Kol92], [LT18], [GKPT20], [GKT].

Let X be a normal projective klt variety, H an ample \mathbb{Q} -Cartier divisor and \mathcal{E} a reflexive sheaf on X . The foundation of the theory of orbifold Chern classes is the existence of an open subvariety $Y \subset X$ whose complementary has codimension at least 3, and of a normal quasiprojective klt variety \hat{Y} , with a finite Galois morphism $p : \hat{Y} \rightarrow Y$ of Galois group G , such that $\hat{\mathcal{E}} := p^*\mathcal{E}$ is a locally-free G -equivariant sheaf on \hat{Y} . The various covers which may arise from the construction for a given $(Y \subset X, \mathcal{E})$ are, in some sense, all related [GKPT20, Lem.3.4]. Let us call the whole data $(Y \subset X, \hat{Y}, p)$ an *unfolding* of X . This construction is slightly simpler if launched on a normal projective klt surface S , because then $Y = X = S$.

In this set-up, we can define a first, respectively a squared first and a second orbifold Chern class of \mathcal{E} as multilinear forms on $NS(X)^{n-1}$, respectively $NS(X)^{n-2}$. These forms are defined on ample \mathbb{Q} -classes H_1, \dots, H_{n-1} by:

$$\begin{aligned} \hat{c}_1(\mathcal{E}) \cdot H_1 \cdots H_{n-1} &= \frac{1}{m^{n-1} \cdot |G|} c_1(\hat{\mathcal{E}}) \cdot p^*(mH_1) \cdots p^*(mH_{n-1}), \\ \hat{c}_1^2(\mathcal{E}) \cdot H_1 \cdots H_{n-2} &= \frac{1}{m^{n-2} \cdot |G|} c_1(\hat{\mathcal{E}})^2 \cdot p^*(mH_1) \cdots p^*(mH_{n-2}), \\ \hat{c}_2(\mathcal{E}) \cdot H_1 \cdots H_{n-2} &= \frac{1}{m^{n-2} \cdot |G|} c_2(\hat{\mathcal{E}}) \cdot p^*(mH_1) \cdots p^*(mH_{n-2}), \end{aligned}$$

where m is big and divisible enough that general elements of $p^*(mH_1), \dots, p^*(mH_{n-1})$ cut out a complete intersection smooth curve in \hat{Y} and general elements of $p^*(mH_1), \dots, p^*(mH_{n-2})$ a complete intersection normal klt surface in \hat{Y} .

As stated in [GKPT20, Thm.3.13.2], these orbifold Chern classes are compatible with general restrictions as well as the unfolding construction is [GKPT20, Prop.3.11].

3 Restricting to a general surface

We prove the following proposition in Section 3.2:

Proposition 3.1. *Let S be a normal projective klt surface, and H an ample \mathbb{Q} -Cartier divisor on S . Let \mathcal{E} be a reflexive sheaf on S such that:*

- $\hat{c}_1(\mathcal{E}) \cdot H = 0$;
- \mathcal{E} and $S^{[l]}\mathcal{E}$, for some $l \geq 6$, are stable with respect to H ;
- \mathcal{E} is pseudoeffective.

Then there is an unfolding $p : \hat{S} \rightarrow S$ as in Section 2.3 on which the locally-free sheaf $\hat{\mathcal{E}} = p^{[*]}\mathcal{E}$ is nef.

In Section 3.1, we explain how this result implies the first part of Theorem 1.3, namely the vanishing of the squared first and second orbifold Chern classes.

3.1 Consequences of Proposition 3.1

We are going to combine Proposition 3.1 with this lemma:

Lemma 3.2. *Let S be a normal projective surface, H an ample \mathbb{Q} -Cartier divisor on S and \mathcal{E} a locally-free sheaf on S . Assume that \mathcal{E} is nef and $c_1(\mathcal{E}) \cdot H = 0$. Then:*

$$c_1(\mathcal{E})^2 = c_2(\mathcal{E}) = 0.$$

Proof. Let $\tilde{S} \xrightarrow{\varepsilon} S$ be the minimal resolution of S , $\tilde{H} = \varepsilon^*H$. Writing $\tilde{\mathcal{E}} := \varepsilon^*\mathcal{E}$, we get a nef locally-free sheaf on a smooth surface. The functoriality of Chern classes of locally-free sheaves by continuous pullbacks [MS57, XI-Lem.1] guarantees $c_i(\tilde{\mathcal{E}}) = \varepsilon^*c_i(\mathcal{E})$ for $i = 1, 2$. In particular, $c_1(\tilde{\mathcal{E}}) \cdot \tilde{H} = 0$. By nefness, $c_1(\tilde{\mathcal{E}})^2 \geq 0$. Hence, by Hodge Index Theorem, $c_1(\tilde{\mathcal{E}})^2 = 0$ which yields, by [DPS94, Prop.2.1, Thm.2.5], $c_2(\tilde{\mathcal{E}}) = 0$. So we obtain:

$$c_1(\mathcal{E})^2 = c_2(\mathcal{E}) = 0.$$

□

Proof of the first assertion in Theorem 1.3. Let a variety X , an ample \mathbb{Q} -Cartier divisor H , and a reflexive sheaf \mathcal{E} be as in the assumptions of Theorem 1.3. By Proposition 2.14, Lemma 2.19 and [GKPT20, Prop.3.11], we can consider an integer m big and divisible enough that $n - 2$ general members of $|mH|$ cut out a complete intersection normal projective klt surface S in X on which:

- $\mathcal{E}|_S$ and $(S^{[l]}\mathcal{E})|_S$, for some $l \geq 6$, are still reflexive;
- as a consequence, $S^{[l]}(\mathcal{E}|_S) = (S^{[l]}\mathcal{E})|_S$;
- both $\mathcal{E}|_S$ and $S^{[l]}(\mathcal{E}|_S)$ remain $H|_S$ -stable of zero slope;
- $\mathcal{E}|_S$ is pseudoeffective.

Then, by Proposition 3.1, there is a finite Galois cover $p : \hat{S} \rightarrow S$ such that the reflexive pullback $\hat{\mathcal{E}} := p^{[*]}\mathcal{E}|_S$ is a nef locally-free sheaf of zero slope. Lemma 3.2 yields:

$$c_1(\hat{\mathcal{E}})^2 = c_2(\hat{\mathcal{E}}) = 0,$$

so that, by construction, $\hat{c}_1^2(\mathcal{E}|_S) = \hat{c}_2(\mathcal{E}|_S) = 0$ and hence:

$$\hat{c}_1^2(\mathcal{E}) \cdot H^{n-2} = \hat{c}_2(\mathcal{E}) \cdot H^{n-2} = 0.$$

The first assertion in Theorem 1.3 is established. □

3.2 Proof of Proposition 3.1

Let S be a normal projective klt surface, and H an ample \mathbb{Q} -Cartier divisor on S . Let \mathcal{E} be a reflexive sheaf on S such that:

- $\hat{c}_1(\mathcal{E}) \cdot H = 0$;
- \mathcal{E} and $S^{[l]}\mathcal{E}$, for some $l \geq 6$, are stable with respect to H ;
- \mathcal{E} is pseudoeffective.

As in Section 2.3, we denote by $p : \hat{S} \rightarrow S$ a finite Galois cover on which the sheaf $\hat{\mathcal{E}} = p^{[*]} \mathcal{E}$ is locally-free. Let $\hat{H} := p^* H$ be an ample \mathbb{Q} -Cartier divisor on \hat{S} , $\hat{\pi} : \mathbb{P}(\hat{\mathcal{E}}) \rightarrow \hat{S}$ be the natural map and $\hat{\zeta}$ be the tautological bundle on $\mathbb{P}(\hat{\mathcal{E}})$.

Abiding by [HP19, Sect.3.2], we prove two lemmas. The first lemma uses the stability of \mathcal{E} and of $S^{[l]} \mathcal{E}$, for some $l \geq 6$, to prove the ampleness of $\hat{\zeta}$ on certain subvarieties of $\mathbb{P}(\hat{\mathcal{E}})$.

Lemma 3.3. *Keep the notations. For any closed proper subvariety $Z \subset \mathbb{P}(\hat{\mathcal{E}})$ such that the image $\hat{\pi}(Z)$ is not a point in \hat{S} , for m big and divisible enough and for a very general curve $\hat{C} \in p^* |mH|$, the restricted tautological $\hat{\zeta}|_{Z \cap \hat{\pi}^{-1}(\hat{C})}$ is ample.*

Proof. Let $Z \subset \mathbb{P}(\hat{\mathcal{E}})$ be a closed proper subvariety whose image $\hat{\pi}(Z)$ has dimension 1 or 2 in \hat{S} . Since p is finite, $p(\hat{\pi}(Z))$ has dimension 1 or 2 in S . Hence, for m big and divisible enough, a very general curve $C \in |mH|$ satisfies:

- C is a smooth curve inside the locus $S_0 \subset S_{\text{reg}}$ where \mathcal{E} is locally-free;
- $\hat{C} := p^{-1}(C)$ is still very general in $p^* |mH|$ and hence smooth too;
- consequentially, we have locally-free sheaf isomorphisms $\hat{\mathcal{E}}|_{\hat{C}} = p^* \mathcal{E}|_C$ and $S^l(\mathcal{E}|_C) = (S^{[l]} \mathcal{E})|_C$;
- since mH is ample, $Z \cap \hat{\pi}^{-1}(\hat{C}) \neq \emptyset$;
- since Z is proper in $\mathbb{P}(\mathcal{E})$, $Z \cap \hat{\pi}^{-1}(\hat{C})$ is proper in $\hat{\pi}^{-1}(\hat{C})$;
- both $\mathcal{E}|_C$ and $S^6(\mathcal{E}|_C)$ remain $H|_C$ -stable of zero slope, by Lemma 2.19.

Apply now Lemma 2.24 and Remark 2.25: they establish that $\hat{\zeta}|_{\hat{\pi}^{-1}(\hat{C})}$ is nef and that, for any closed proper variety $W \subset \hat{\pi}^{-1}(\hat{C}) = \mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}})$,

$$\left(\hat{\zeta}|_{\hat{\pi}^{-1}(\hat{C})} \right)^{\dim W} \cdot W > 0.$$

Using this formula for any closed subvariety W of $Z \cap \hat{\pi}^{-1}(\hat{C})$, the Nakai-Moishezon criterion shows that $\hat{\zeta}|_{Z \cap \hat{\pi}^{-1}(\hat{C})}$ is ample. \square

The second lemma is set at the higher level of $(\hat{S}, \hat{\mathcal{E}})$ directly. It uses the pseudoeffectivity and \hat{H} -semistability of the locally-free sheaf $\hat{\mathcal{E}}$, inferred by Lemmas 2.15 and 2.21, but no other property of \mathcal{E} .

Lemma 3.4. *Keep the notations. If $\hat{\zeta}$ is not nef, then there is a closed proper subvariety W of $\mathbb{P}(\hat{\mathcal{E}})$ such that, for m big and divisible enough and for a very general curve $\hat{C} \in p^* |mH|$:*

$$\emptyset \neq W \cap \hat{\pi}^{-1}(\hat{C}) \subsetneq \hat{\pi}^{-1}(\hat{C}), \quad \hat{\zeta}|_{W \cap \hat{\pi}^{-1}(\hat{C})} \text{ is nef and not big.}$$

This result essentially relies on [HP19, Lem.3.4].

Proof. Denote by $\mu : \tilde{S} \rightarrow \hat{S}$ the minimal resolution of \hat{S} , by $\tilde{\mathcal{E}} := \mu^* \hat{\mathcal{E}}$, by $\tilde{\zeta}$ the tautological bundle of $\mathbb{P}(\tilde{\mathcal{E}})$. We have a Cartesian diagram with compatibility of tautological bundles:

$$\begin{array}{ccc} \mathbb{P}(\tilde{\mathcal{E}}) & \xrightarrow{\mu'} & \mathbb{P}(\hat{\mathcal{E}}) \\ \downarrow \tilde{\pi} & & \downarrow \hat{\pi} \\ \tilde{S} & \xrightarrow{\mu} & \hat{S} \end{array}$$

Note that $\mathbb{P}(\tilde{\mathcal{E}})$ with its tautological $\tilde{\zeta}$ is a smooth modification of $\mathbb{P}(\hat{\mathcal{E}})$ just as in Definition 2.12. By Lemma 2.13, $\tilde{\zeta}$ is pseudoeffective.

Suppose that $\hat{\zeta}$ is not nef. In particular, $\tilde{\zeta}$ is not nef, though it is $\tilde{\pi}$ -ample. Let $Z \subset B_-(\tilde{\zeta})$ be an irreducible component of maximal dimension. Note that Z contains a $\tilde{\zeta}$ -negative curve N : its image $\mu'(N)$ must be a $\hat{\zeta}$ -negative curve, hence it is not in a fiber of $\hat{\pi}$. So $\hat{\pi}(\mu'(Z))$ is not a point in \hat{S} . Moreover, since $\tilde{\zeta}$ is pseudoeffective, $Z \neq \mathbb{P}(\tilde{\mathcal{E}})$.

Now, for a very general curve $\hat{C} \in p^* |mH|$ for m big and divisible enough,

- $\hat{C} \subset \hat{S}_{\text{reg}}$; in particular, μ is an isomorphism over \hat{C} ;
- $\emptyset \neq Z \cap \mu'^{-1}(\hat{\pi}^{-1}(\hat{C})) \subsetneq \mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))$;
- $\hat{\mathcal{E}}|_{\hat{C}}$ is nef by Lem.2.19 and Prop.2.22 and it has $\hat{H}|_{\hat{C}}$ -slope zero;
- hence, $\tilde{\zeta}|_{\mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))}$ is nef too, and moreover its top power is zero;
- hence, by [HP19, Lem.3.4] (which applies since Z was chosen with minimal codimension):

$$\emptyset = \left(\tilde{\zeta}|_{\mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))} \right)^{\dim \mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))} \geq \left(\tilde{\zeta}|_{Z \cap \mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))} \right)^{\dim Z \cap \mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))} \geq 0.$$

As μ' is an isomorphism over \hat{C} , $W := \mu'(Z)$ works well as the closed proper subvariety of $\mathbb{P}(\hat{\mathcal{E}})$ we wanted to construct. \square

We now combine these lemmas to establish Proposition 3.1.

Proof of Proposition 3.1. Suppose by contradiction that $\hat{\mathcal{E}}$ is not nef. Then Lemma 3.4 yields a closed proper subvariety W of $\mathbb{P}(\hat{\mathcal{E}})$ which satisfies, for m big and divisible enough and for a very general curve $\hat{C} \in p^*|mH|$:

$$0 \neq W \cap \hat{\pi}^{-1}(\hat{C}) \subsetneq \hat{\pi}^{-1}(\hat{C}) \text{ and } \hat{\zeta}|_{W \cap \hat{\pi}^{-1}(\hat{C})} \text{ is nef and not big.}$$

The first condition shows that $\hat{\pi}(W)$ is not a point. So Lemma 3.3 applies, hence $\hat{\zeta}|_{W \cap \hat{\pi}^{-1}(\hat{C})}$ is ample, contradiction! \square

4 Proof of Theorem 1.3

As it follows from the discussion in Section 3.1, Theorem 1.3 is halfway. Here is what remains to prove:

Theorem 4.1. *Let X be a normal projective klt variety of dimension n with an ample \mathbb{Q} -Cartier divisor H . Let \mathcal{E} be a reflexive sheaf on X , such that:*

- \mathcal{E} is H -semistable;
- the following equalities hold:

$$\hat{c}_1(\mathcal{E}) \cdot H^{n-1} = \hat{c}_1^2(\mathcal{E}) \cdot H^{n-2} = \hat{c}_2(\mathcal{E}) \cdot H^{n-2} = 0.$$

Then there is a finite Galois morphism $\nu : \tilde{X} \rightarrow X$, étale in codimension 1, such that $\nu^{[]}\mathcal{E}$ is locally-free with a numerically trivial determinant and $\text{Gal}(\tilde{X}/X)$ -equivariantly flat. Consequentially, $\nu^{[*]}\mathcal{E}$ is numerically flat and its first and second Chern classes are numerically trivial.*

Proof. We apply [LT18, Thm.1.4] to obtain a finite Galois morphism $\nu : \tilde{X} \rightarrow X$, étale over X_{reg} , such that $\nu^{[*]}\mathcal{E}$ is locally-free with a numerically trivial determinant and $\text{Gal}(\tilde{X}/X)$ -equivariantly flat.

Let then $\varepsilon : \tilde{X}' \rightarrow \tilde{X}$ be a resolution of \tilde{X} and $\mathcal{E}' := \varepsilon^* \nu^{[*]}\mathcal{E}$, which is a flat locally-free sheaf with a numerically trivial determinant on \tilde{X}' . As shown in [HP19, Rmk.2.6], \mathcal{E}' is then numerically flat and its Chern classes vanish (as cohomological classes on \tilde{X}'). By Prop.2.6, $\nu^{[*]}\mathcal{E}$ is nef, hence numerically flat. Moreover, for any \mathbb{Q} -Cartier divisors D_1, \dots, D_{n-2} ,

$$c_2(\nu^{[*]}\mathcal{E}) \cdot D_1 \cdots D_{n-2} = c_2(\mathcal{E}') \cdot \varepsilon^* D_1 \cdots \varepsilon^* D_{n-2} = 0,$$

so the Chern classes of $\nu^{[*]}\mathcal{E}$ are trivial, which completes the proof of the theorem. \square

5 Proof of Theorem 1.2

We give a few definitions along the lines of Theorem 1.1:

Definition 5.1. Let X be a normal projective klt variety of dimension $n \geq 2$. It is called:

- a *Calabi-Yau* variety if $h^0(Y, \Omega_Y^{[q]}) = 0$ for all integers $1 \leq q \leq n - 1$ and all quasiétale finite covers $Y \rightarrow X$;
- an *irreducible holomorphic symplectic (IHS)* variety if there is a reflexive form $\sigma \in H^0(X, \Omega_X^{[2]})$ such that, for any quasiétale finite cover $f : Y \rightarrow X$, the reflexive form $f^{[*]}\sigma$ generates $H^0(X, \Omega_Y^{[1]})$ as an algebra for the wedge product.

We use the terms singular Calabi-Yau (resp. IHS) variety and Calabi-Yau (resp. IHS) variety interchangeably, unless explicitly said otherwise. They may both accidentally denote smooth varieties.

Definition 5.2. For the sake of a consistent terminology, let us call a *singular K3 surface*, or for short a *K3 surface*, a normal projective klt surface which has no finite quasiétale cover by an abelian variety. Equivalently, it is a Calabi-Yau variety or an IHS variety of dimension 2.

Definition 5.3. For the sake of a convenient vocabulary, let us define the *augmented irregularity* $\tilde{q}(X)$ of a normal projective klt variety X with trivial canonical class as the maximum of all irregularities $q(Y)$ of finite quasiétale covers Y of X . Note that it is precisely the dimension of the abelian part in the singular Beauville-Bogomolov decomposition of X .

Let us now proceed to prove Theorem 1.2.

Proof of Theorem 1.2. Let X be a normal projective klt variety of dimension at least 2 with trivial canonical class. Suppose that $\Omega_X^{[1]}$ is pseudoeffective (the same whole argument works just alike for the tangent sheaf \mathcal{T}_X) and assume by contradiction that $\tilde{q}(X) = 0$.

The singular Beauville-Bogomolov decomposition then reads:

$$f : \tilde{X} \rightarrow X \text{ and } \tilde{X} \cong \prod_i Y_i \times \prod_j Z_j,$$

with the same notations as in Theorem 1.1.

Remember that $f^{[*]}\Omega_X^{[1]} = \Omega_{\tilde{X}}^{[1]}$, since reflexive sheaves are normal and there is a big open set over which f is just a finite étale cover. By Lemma 2.15, $\Omega_{\tilde{X}}^{[1]}$ is pseudoeffective; it splits according to the product defining \tilde{X} . So there is a factor Y (Calabi-Yau or IHS) of \tilde{X} such that $\Omega_Y^{[1]}$ is pseudoeffective [HP19, inductive argument in Proof of Thm.1.6]. Now, $\Omega_Y^{[1]}$ satisfies all hypotheses of Theorem 1.3, the stability assumptions coming from [GKP16b, Prop.8.20] and [GGK19, Rmk.8.3].

As a consequence, for some ample polarization H on Y , $\hat{c}_2(\Omega_Y^{[1]}) \cdot H^{\dim Y} = 0$, so that Y has a finite quasiétale cover by an abelian variety by [LT18, Thm.1.4], contradiction! \square

Remark 5.4. This pseudoeffectiveness result can be considered as an interesting improvement of the effectiveness result [GGK19, Thm.11.1], which says that $\tilde{q}(X) = 0$ if and only if, for all $m \in \mathbb{N}$, $h^0(X, S^{[m]}\Omega_X^{[1]}) = 0$.

Examples for Theorem 1.2 are to search among normal projective klt varieties with trivial canonical class singularities in codimension 2, which are plethoric. But singular varieties whose decomposition is known are not so numerous; and, for sure, one shall understand the Beauville-Bogomolov type of a given variety before telling anything about the positivity of its reflexivized cotangent sheaf.

Example 5.5. A first example to which Theorem 1.2 applies is the following [GGK19, Par.14.2.2]: let F be a Fano manifold on which a finite group G acts freely in codimension 1. Suppose there is a smooth G -invariant element Y in the linear system $|-K_F|$. Then, Y is a smooth Calabi-Yau variety with a G -action. If the volume form on Y is preserved by this action, then $X := Y/G$ is a normal projective klt variety with trivial canonical class, and the morphism $Y \rightarrow X$ has no ramification divisor, hence it is quasiétale. The fact that the decomposition of X consists of a smooth Calabi-Yau manifold Y guarantees that X is a singular Calabi-Yau variety, as presented in Definition 5.1.

Although X may well have singularities in codimension 2, they merely stem from its global quasi-étale quotient structure. In particular, [HP19, Thm.1.6] actually proves the non-pseudoeffectiveness of \mathcal{T}_X and $\Omega_X^{[1]}$, namely because it applies to Y and converts onto X through Proposition 2.15. Hence, the example is quite shallow: it has no real need for the machinery dealing with singularities in codimension 2 that Theorem 1.2 is about.

In the next two sections, we present deeper examples for Theorem 1.2. We stay in small dimensions ($n = 2, 3$) to better monitor the Beauville-Bogomolov type of our examples, and so we present K3 surfaces and Calabi-Yau threefolds with singularities in codimension 2 that are not constructed as global quasi-étale quotients of varieties which are smooth in codimension 2.

6 Surfaces in Theorems 1.1 and 1.2

We already discussed in the Introduction the delicate fact that some degenerations of smooth K3 surfaces, such as singular Kummer surfaces, are of abelian Beauville-Bogomolov type. But in dimension 2, the alternative remains simple: a normal projective klt surface with trivial canonical class is either a singular K3 surface, or it has a finite quasi-étale cover by an abelian surface. Moreover, in the second case, the surface appears to be a finite quasi-étale *quotient* of an abelian surface, as proven in Proposition 7.3. There is an effective criterion [LT18, Thm.1.4] to distinguish between the two cases:

$$S \text{ is a K3 surface if and only if } \hat{c}_2(S) \neq 0.$$

Let us explain in examples how to compute $\hat{c}_2(S)$ from the knowledge of the singularities of S .

Example 6.1. We show that a normal projective klt surface S with trivial canonical class and exactly 16 nodes of type A_1 as singular points has a finite quasi-étale cover by an abelian surface. From [Kol92, Def.10.7, Thm.10.8] we have

$$\hat{c}_2(S) = e_{\text{top}}(S) - 16 \left(1 - \frac{1}{2}\right) = e_{\text{top}}(S) - 8,$$

Note that the crepant resolution \tilde{S} of S is a K3 surface. Indeed, as there are smooth rational curves in \tilde{S} , no finite étale cover of it is a torus. By the smooth Beauville-Bogomolov decomposition theorem, \tilde{S} has a finite étale cover of degree d which is a K3 surface. This cover should contain $16d$ numerically independent smooth rational (-2) -curves and have a Picard rank smaller than 20, so $d = 1$, ie \tilde{S} is a K3 surface. Additivity of topological Euler numbers [Ful93, pp.141-142] yields:

$$e_{\text{top}}(S) = e_{\text{top}}(\tilde{S}) - 16 e_{\text{top}}(\mathbb{P}^1) + 16 e_{\text{top}}(\text{point}) = 24 - 16 \times 2 + 16 = 8.$$

Hence $\hat{c}_2(S) = 0$, as expected.

More generally, let S be a normal projective surface with trivial canonical class and canonical singularities. For $x \in \text{Sing}(S)$, denote by k_x the number of (-2) -curves over x in the minimal crepant resolution \tilde{S} and by r_x the order of the local quotient map standing for the singularity at x . For the curious reader, these parameters are tabulated for the classified A_n, D_n, E_6, E_7, E_8 singularities in [Rei, Fig.1, Exercise 10]. Then, we have an explicit formula:

$$\hat{c}_2(S) = 24 - \sum_{x \in \text{Sing}(S)} k_x + 1 - \frac{1}{r_x}, \quad (2)$$

under the mild assumption that \tilde{S} is a K3 surface, which is automatically true if it contains at least 11 (-2) -curves, *a fortiori* if S is a finite quasi-étale quotient of a torus by Cor.A.3.

Proof. Note that \tilde{S} is a smooth surface with trivial canonical class. As there are smooth rational curves in \tilde{S} , no finite étale cover of it is a torus. So by the smooth Beauville-Bogomolov decomposition theorem, \tilde{S} has a finite étale cover of degree d which is a K3 surface. This cover should contain at least $11d$ numerically independent smooth rational (-2) -curves and have a Picard rank smaller than 20, so $d = 1$, ie \tilde{S} is a K3 surface. \square

Lemma 6.2. *If S is a finite quasi-étale quotient of an abelian surface, then $\sum_{x \in \text{Sing}(S)} k_x \geq 16$.*

Given the list of the singularities of S , it is then simple to check whether or not it is a singular K3 surface.

Note that we can not prescribe any list of du Val singularities satisfying Eq.2 to be realized by a normal canonical projective surface with trivial canonical bundle. As the following examples show, this may happen for different reasons, and one has to think in terms of finite quotients of abelian surfaces to identify the global obstructions.

Example 6.3. There is no normal canonical projective surface S with trivial canonical class which singularities are 5 points of type A_4 . Suppose by contradiction that there were such a surface S . Then the resolution \tilde{S} must be a K3 surface, so

$$5 \times 4 = \sum_{x \in \text{Sing}(S)} k_x \leq \rho(\tilde{S}) - \rho(S) \leq 19,$$

contradiction!

Note that group theory on abelian surfaces gives a deeper understanding of that peculiar example. Indeed, up to isogeny, the only abelian surface on which there is an automorphism of order 5 which is not a translation is \mathbb{C}^2/Λ , where $\Lambda = \mathbb{Z}[(\zeta_5, \zeta_5^2)]$ and ζ_5 a primitive 5-th root of unit [BL04, Cor.13.3.6], and the cyclic group \mathbb{Z}_5 acts on it by multiplication :

$$\mathbb{Z}_5 \simeq \langle \sigma \rangle, \quad \sigma = \begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^2 \end{pmatrix} \notin SL(2, \mathbb{C}).$$

So A/\mathbb{Z}_5 does have 5 quotient singularities, but they are klt and not canonical. A minimal resolution of this surface has relative Picard number 10, each singular point giving rise to two exceptional divisors E_i, F_i with $E_i^2 = -3, E_i F_i = 1, F_i^2 = -2$ [Bri68, Satz 2.11].

Example 6.4. There is no normal canonical projective surface S with trivial canonical class which singularities are 2 points of type A_3 and 11 points of type A_1 . Suppose, by contradiction, that there were such surface S . Then it holds $\hat{c}_2(S) = 0$, so that S is a quasiétale quotient of an abelian variety A by a finite group G . Up to isogeny, we can suppose that G contains no translation of A . The 11 points of type A_1 must stem from $\frac{11|G|}{2}$ points on A , grouped in orbits of size $\frac{|G|}{2}$, with stabilisers isomorphic to \mathbb{Z}_2 .

By Lemma A.4, these points all have the same stabilisor $\langle \text{id}, \sigma \rangle \subset G$. As σ is an automorphism of order 2 on an abelian surface, it precisely has 16 fixed points [BL04, Ex.13.2.7], among which already our $\frac{11|G|}{2}$ points. Hence $|G| = 2$, and we can not get A_3 singularities in A/G , contradiction.

Let us just reassure the reader, that there are (finitely many, but still) several examples of normal canonical surfaces arising as finite quasiétale quotients of abelian varieties. In fact, we can exhaustively list them:

Proposition 6.5. *A normal canonical surface with trivial canonical class is of abelian Beauville-Bogomolov type of and only if its singularities are :*

- 16 singularities A_1 : realized by an abelian variety A quotiented by the involution $-\text{id}$;
- 9 singularities A_2 : realized by $E_3 \times E_3$ quotiented by the diagonal action of $\mathbb{Z}_3 = \left\langle z \mapsto \begin{pmatrix} j & 0 \\ 0 & j^{-1} \end{pmatrix} z \right\rangle$, where $E_3 = \mathbb{C}/\langle 1, j \rangle$ and j is a primitive 3-rd root of unit;
- 4 singularities A_3 and 6 singularity A_1 : realized by the square of an elliptic curve $E \times E$ quotiented by the action of $\mathbb{Z}_4 = \left\langle b : z \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z \right\rangle$;
- 1 singularity A_5 , 4 singularities A_2 and 5 singularities A_1 : realized by $E_3 \times E_3$ quotiented by the diagonal action of $\mathbb{Z}_6 = \left\langle d : z \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} z \right\rangle$, where $E_3 = \mathbb{C}/\langle 1, \omega \rangle$ and ω is a primitive 6-th root of unit;
- 6 singularities A_3 and 1 singularity A_1 : realized by \mathbb{C}^2/Λ_8 quotiented by the following action of the binary dihedral group

$$BD_8 = \left\langle a : z \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} z, b' : z \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z + (1, 0) \right\rangle,$$

where $\Lambda_8 = \langle (1, 1), (1, -1), (i - 1, 0), (0, i - 1) \rangle$;

- 2 singularities D_4 , 3 singularities A_3 and 2 singularities A_1 : realized by $E_4 \times E_4$ quotiented by the linear action of $BD_8 = \langle a, b \rangle$, where $E_4 = \mathbb{C}/\mathbb{Z}[i]$;
- 4 singularities D_4 and 3 singularities A_1 : realized by \mathbb{C}^2/Λ_8 quotiented by the linear action of BD_8 ;
- 1 singularity D_5 , 3 singularities A_3 , 2 singularities A_2 , 1 singularity A_1 : realized by $E_3 \times E_3$ quotiented by the linear action of the binary dihedral group $BD_{12} = \langle b, d \rangle$;
- 1 singularity A_5 , 2 singularities A_3 , 4 singularities A_2 : realized by \mathbb{C}^2/Λ_8 quotiented by the following action of the binary tetrahedral group

$$BT_{24} = \left\langle a, b', c' : z \mapsto \frac{1}{2} \begin{pmatrix} i+1 & i-1 \\ i+1 & 1-i \end{pmatrix} z + \left(\frac{1}{2}, \frac{-i}{2} \right) \right\rangle$$

- 1 singularity E_6 , 1 singularity D_4 , 4 singularities A_2 , 1 singularity A_1 : realized by \mathbb{C}^2/Λ_8 quotiented by the linear action of

$$BT_{24} = \left\langle a, b, c : z \mapsto \frac{1}{2} \begin{pmatrix} i+1 & i-1 \\ i+1 & 1-i \end{pmatrix} z \right\rangle$$

For a presentation of these groups as embedded in $SL(2, \mathbb{C})$, we refer to [Rei, Exercise 10]. As the proof of this proposition concerns much rather group theory than birational geometry, we postpone it to Appendix A.

Now that we understood the singular Beauville-Bogomolov decomposition well-enough for surfaces, applying Theorem 1.2 leads to a nice simple dichotomy in dimension 2. Let S be a normal projective klt surface. Then the following are equivalent:

- S is not a singular K3 surface;
- S is a finite quasiétale quotient of an abelian variety;
- $\hat{c}_2(S) = 0$;
- \mathcal{T}_S is pseudoeffective;
- $\Omega_S^{[1]}$ is pseudoeffective.

7 Threefolds in Theorems 1.1 and 1.2

In the Section 5, we defined singular Calabi-Yau and IHS varieties. They can also be defined by means of their algebraic holonomy, an approach which [GGK19] uses thoroughly. This notably enables one to prove that IHS varieties must have even dimension [GGK19, Thm.12.1, Prop.12.10]. In particular, the singular Beauville-Bogomolov decomposition for a normal projective klt variety X of dimension 3 is quite simple: \tilde{X} has to be one of the following:

- a smooth abelian variety;
- a product $S \times E$, where S is a K3 surface as in Definition 5.2 and E is a smooth elliptic curve;
- a Calabi-Yau variety.

The aforementioned [LT18, Thm.1.4] provides a criterion for identifying the purely abelian case by computing $\hat{c}_2(X)$.

One is then left with two cases: the singular threefold X may arise from a product $S \times E$, in which case \mathcal{T}_X and $\Omega_X^{[1]}$ are pseudoeffective because of the abelian factor E ; alternatively, X can be a genuine singular Calabi-Yau threefold. This second possibility is hard to identify, but, when it happens, it may give new examples for Theorem 1.2.

The next subsection is devoted to providing a necessary condition for a normal projective klt threefold to be finitely quasiétale covered by a product $S \times E$.

7.1 Products of a K3 surface and an elliptic curve

We are going to prove the following result:

Proposition 7.1. *Let X be a normal projective klt threefold with trivial canonical class. Suppose its Beauville-Bogomolov decomposition is of the form*

$$\tilde{X} = S \times E,$$

where S is a K3 surface and E a smooth elliptic curve. Then $\rho(X) \geq 2$.

Let us first state a weak uniqueness result, guaranteeing that the statement of Proposition 7.1 makes sense.

Proposition 7.2. *Let X be a normal projective klt variety with trivial canonical class. Then the number, types and dimensions of the factors of a finite quasiétale covering $\tilde{X} \rightarrow X$ as in Theorem 1.1 do not depend on the choice of that covering.*

Proof. It is straightforward from the proof of the decomposition theorem. The Beauville-Bogomolov decomposition comes from a unique [GGK19, Rmk.2.12] infinitesimal splitting by reflexive sheaves satisfying various stability, positivity, holonomy action and integrability conditions:

$$\mathcal{T}_X = \mathcal{G} \oplus \bigoplus_i \mathcal{E}_i \oplus \bigoplus_j \mathcal{F}_j.$$

This infinitesimal splitting behaves well through reflexive pullbacks by finite quasiétale maps. In particular, for any finite quasiétale covering $f : \tilde{X} \rightarrow X$ as in Theorem 1.1, one can arrange by an adequate renaming:

$$f^{[*]} \mathcal{G} \cong p_A^* \mathcal{T}_A, \quad f^{[*]} \mathcal{E}_i \cong p_{Y_i}^* \mathcal{T}_{Y_i}, \quad f^{[*]} \mathcal{F}_j \cong p_{Z_j}^* \mathcal{T}_{Z_j},$$

with the usual notations for the factors of \tilde{X} , p_M being the projection on the factor M . Dimensions of the factors of \tilde{X} are then the ranks of the sheaves in the infinitesimal decomposition of \mathcal{T}_X , whereas types of the factors of \tilde{X} are given by the holonomy groups acting irreducibly on the sheaves in the infinitesimal decomposition. \square

A finite quasiétale morphism is not necessarily a quotient map by a finite group action free in codimension 1. In the smooth case however, [Bea83, Lem.p.9] allows us to assume that the finite étale decomposition morphism $p : \tilde{X} \rightarrow X$ is Galois. Let us state a partial singular analog:

Proposition 7.3. *Let X be a normal projective klt variety with trivial canonical class. Take a finite quasiétale covering $f : \tilde{X} \rightarrow X$ as in Theorem 1.1. Suppose that all Calabi-Yau factors of \tilde{X} have even dimension. Then there is a finite quasiétale Galois morphism $f' : \tilde{Z} \rightarrow X$, so that \tilde{Z} splits into factors in the same number, types, and dimensions as \tilde{X} .*

Proof. By [GKP16a, Thm.1.5], we can take a finite quasiétale Galois covering $g : Y \rightarrow X$ such that any finite morphism $Z \rightarrow Y$ étale over Y_{reg} is étale over Y . By purity of the branch locus, any quasiétale morphism $Z \rightarrow Y$ is then étale.

Note that Y is still a normal projective klt variety with trivial canonical class, hence has a singular Beauville-Bogomolov decomposition $h : Z \rightarrow Y$. By Proposition 7.2, the factors of Z have the same type as those of \tilde{X} . It writes:

$$Z = A \times \prod_i Y_i \times \prod_j Z_j,$$

where A is an abelian variety, Y_i Calabi-Yau varieties and Z_j IHS varieties. Since all Y_i and, of course, all Z_j have even dimension, by [GGK19, Cor.13.3], they are simply connected.

Hence, finite étale fundamental groups equal: $\widehat{\pi}_1(Z) \simeq \widehat{\pi}_1(A)$. That is to say, any finite étale cover of Z actually stems from a finite étale cover of A .

We now use [GKP16a, Thm.3.16]: there is a finite Galois morphism $\gamma : \tilde{Z} \rightarrow Z$ such that $\Gamma = g \circ h \circ \gamma : \tilde{Z} \rightarrow X$ is finite Galois and ramifies where $g \circ h$ does. So Γ is still quasiétale, in particular $h \circ \gamma : \tilde{Z} \rightarrow Y$ is quasiétale too. By construction of Y , $h \circ \gamma$ is then étale, so that γ is étale. By construction of Z , one has:

$$\tilde{Z} = A' \times \prod_i Y_i \times \prod_j Z_j,$$

where A' is a finite étale cover of the abelian variety A . Finally, $\Gamma : \tilde{Z} \rightarrow X$ is finite Galois quasi-étale, and \tilde{Z} splits as mandated. \square

Remark 7.4. The main obstacle for generalizing this proposition is the fact that fundamental groups of odd-dimensional Calabi-Yau varieties are poorly understood [GGK19, Sect.13.2]; most notably, they may not be finite.

Here is the last ingredient for the proof of Proposition 7.1:

Lemma 7.5. *Let S be a K3 surface as in Definition 5.2, E a smooth elliptic curve. Then:*

$$\text{Aut}(S \times E) \cong \text{Aut}(S) \times \text{Aut}(E).$$

Proof. Let \tilde{S} be the minimal resolution of S . It is a smooth K3 surface, so $\text{Aut}(\tilde{S})$ is discrete. Moreover, the uniqueness of minimal resolution implies that any automorphism of S lifts to an automorphism of \tilde{S} , and this is obviously an injection. Hence, $\text{Aut}(S)$ is discrete.

Let us now copy the argument by [Bea83, Lem.p.8]. Let $u \in \text{Aut}(S \times E)$. Since the projection $p_E : S \times E \rightarrow E$ is the Albanese map of $S \times E$, we can factor $p_E \circ u$ by it: there is $v \in \text{Aut}(E)$ such that $p_E \circ u = v \circ p_E$. Hence, there is a map $w : E \rightarrow \text{Aut}(S)$ which decomposes:

$$u : (s, e) \in S \times E \mapsto (w_e(s), v(e)).$$

Since $\text{Aut}(S)$ is discrete, the map w is constant, so $u = (w_0, v)$. \square

Proof of Proposition 7.1. Let X be a normal projective variety of dimension 3 with trivial canonical class. Suppose that there is a finite quasi-étale cover $f : S \times E \rightarrow X$, where S is a singular K3 surface and E a smooth elliptic curve. By Proposition 7.3, we can assume that there is a finite group G acting on $S \times E$ such that f is the induced quotient map. By Lemma 7.5, G can be considered a subgroup of $\text{Aut}(S) \times \text{Aut}(E)$. As it acts diagonally, we have the following diagram:

$$\begin{array}{ccccc} S \times E & \xrightarrow{p_S} & S & & \\ & \searrow f & \searrow & \searrow & \\ E & \xrightarrow{p_E} & X & \longrightarrow & S/G_S \\ & \searrow & \downarrow & & \\ & & E/G_E & & \end{array}$$

so that $\rho(X)$ is at least 2. \square

7.2 Hypersurfaces with trivial canonical class in weighted \mathbb{P}^4

The aim of this last part is to provide examples of Calabi-Yau threefolds that are singular along curves, by establishing the following result:

Proposition 7.6. *Let $\mathbb{P} = \mathbb{P}(w_0, \dots, w_4)$ be a weighted projective space and $d = w_0 + \dots + w_4$ such that there is a general wellformed quasismooth hypersurface X of degree d in \mathbb{P} . Suppose that X contains no edge of \mathbb{P} . Then X is a singular Calabi-Yau in the sense of Definition 5.1.*

Before proving that, we recall a few definitions.

Introduction. A general exposition to complete intersections in weighted projective spaces can be found in [IF00]. We recall what we need here, swapping definitions and equivalent properties if relevant.

- We have the usual convention for tuples with a few elements left out: $(a_0, \dots, \hat{a}_i, \dots, a_n)$ is the tuple $(a_j)_{j \in [0, n] \setminus \{i\}}$.

- We denote by $\mathbb{P}(w_0, \dots, w_n)$ a weighted projective space of dimension n , where $\gcd(w_0, \dots, w_n) = 1$. It is a projective variety with only quotient singularities. It is normal if and only if $\gcd(w_0, \dots, \hat{w}_i, \dots, w_n) = 1$ for all i . When not ambiguous, we may just write it \mathbb{P} .
- There is a ramified quotient map: $p : \mathbb{P}^n \rightarrow \mathbb{P}$, by the finite diagonal group action of $\bigoplus_i \mathbb{Z}_{w_i}$ on \mathbb{P}^n . With homogeneous coordinates on either side, we can write:

$$p : [x_0 : \dots : x_n] \in \mathbb{P}^n \mapsto [y_0 = x_0^{w_0} : \dots : y_n = x_n^{w_n}] \in \mathbb{P}.$$

We denote by $\mathcal{O}_{\mathbb{P}}(1)$ the ample \mathbb{Q} -Cartier divisor on \mathbb{P} whose pullback by p is $\mathcal{O}_{\mathbb{P}^n}(1)$.

- A general hypersurface of degree d in \mathbb{P} is a general element of $|\mathcal{O}_{\mathbb{P}}(d)|$, if there is one. Equivalently, it is the zero-locus of a linear combination with general coefficients of monomials $y_0^{\alpha_0} \cdots y_n^{\alpha_n}$, with $\alpha_0 w_0 + \dots + \alpha_n w_n = d$, if there are enough such possible monomials that for each i , there is such a monomial that is not divisible by y_i . It is not necessarily reduced irreducible, but we will soon give a sufficient condition on d and \mathbb{P} for it to be.
- There is a quotient map $q : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}$ defined by $q(y_0, \dots, y_n) = [y_0 : \dots : y_n]$ with fiber isomorphic to \mathbb{C}^* . For any closed subvariety X in \mathbb{P} , we define the *cone* over X as the Zariski closure of $q^{-1}(X)$ in \mathbb{C}^{n+1} , and we denote it by C_X .

Definition 7.7. Let X be a closed subscheme in \mathbb{P} . It is called *quasismooth* [IF00, Def.6.3] or *nondegenerate* [KS92, Sec.2.1,Par.2] if its cone C_X is smooth at any point but its vertex $0 \in \mathbb{C}^{n+1}$.

Remark 7.8. By [IF00, 3.1.6], any quasismooth closed subscheme of \mathbb{P} is actually a subvariety, and even a full suborbifold of \mathbb{P} .

There is an arithmetical criterion [IF00, Thm.8.1], [KS92, Thm.1] for quasismoothness of a general hypersurface of given degree:

Definition 7.9. We say that a positive integer d is *partitionable* in positive integers w_0, \dots, w_n , or that w_0, \dots, w_n *partition* d , if there are non-negative integers $\alpha_0, \dots, \alpha_n$ such that

$$d = \sum_{i=0}^n \alpha_i w_i.$$

Proposition 7.10. Let X_d be a general hypersurface of degree d in $\mathbb{P}(w_0, \dots, w_4)$. It is quasismooth if and only if:

- for any $i \in \llbracket 0, 4 \rrbracket$, there is $j \in \llbracket 0, 4 \rrbracket$, which may equal i , such that w_i divides $d - w_j$;
- for any two different $i_1, i_2 \in \llbracket 0, 4 \rrbracket$, there are two different j_1, j_2 , which may coincide with i_1, i_2 , such that w_{i_1}, w_{i_2} partition both $d - w_{j_1}$ and $d - w_{j_2}$;
- for any $k \geq 3$ different indices $i_1, \dots, i_k \in \llbracket 0, 4 \rrbracket$, d is partitionable in w_{i_1}, \dots, w_{i_k} .

If so, X_d is a closed reduced irreducible variety of dimension 3, by Rem. 7.8.

Example 7.11. A general X_{120} in $\mathbb{P}(3, 7, 20, 40, 50)$ is quasismooth because:

- 120 is divisible by 3,20,40; $120 - 50 = 70$ is divisible by 7; $120 - 20$ is divisible by 50;
- the only case to check for $k = 2$ is $i_1, i_2 = 1, 4$, in which 7,50 do partition 120;
- nothing to check for $k \geq 3$.

Example 7.12. A general X_{56} in $\mathbb{P}(2, 4, 9, 13, 28)$ is quasismooth because:

- 56 is divisible by 2,4,28; $56 - 2 = 54$ is divisible by 9; $56 - 4 = 52$ is divisible by 13;
- the only case to check for $k = 2$ is $i_1, i_2 = 2, 3$, and taking $j_1 = 0, j_2 = 1$ is fine ;
- all cases with $k \geq 3$ are clear.

We want to know whether a given hypersurface in \mathbb{P} has trivial canonical class. For that, we would expect an adjunction formula to hold. It is true under some additional assumptions:

Definition 7.13. Let $X_d \subset \mathbb{P}(w_0, \dots, w_n)$ be a general quasismooth hypersurface of degree d in a weighted projective space. It is said to be *wellformed* if it satisfies one of these two equivalent conditions:

- the adjunction formula holds: $K_X \sim \mathcal{O}_X(d - \sum w_i)$ as \mathbb{Q} -Cartier divisors ;
- the following arithmetical conditions hold:
 - for any i , $\gcd(w_0, \dots, \hat{w}_i, \dots, w_n) = 1$,
 - and for any $i < j$, $\gcd(w_0, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, w_n) \mid d$.

The Examples 7.11 and 7.12 present general wellformed quasismooth hypersurfaces.

For now on, we restrict our attention to hypersurfaces in a 4-dimensional weighted projective space \mathbb{P} , unless stated otherwise.

Singularities of general quasismooth hypersurfaces of dimension 3. Note that if X is a general quasismooth hypersurface of degree d and of dimension 3 in the 4-dimensional weighted projective space \mathbb{P} , then X is a full suborbifold of \mathbb{P} (see [BB12, Def.5] for a definition, [Dol82, Thm.3.1.6] for a proof). In particular, $X_{\text{sing}} = X \cap \mathbb{P}_{\text{sing}}$, and at any point $x \in X \cap \mathbb{P}_{\text{sing}}$, writing that \mathbb{P} is locally isomorphic to a quotient \mathbb{C}^4/G_x , X is locally isomorphic to \mathbb{C}^3/G_x in a compatible way with inclusions. Hence, X has only quotient singularities, so it is klt. The locus X_{sing} is a finite union of curves and points, which may be of various types:

- a *vertex* in \mathbb{P} is a point with $y_i = 1$ for a single $i \in \llbracket 0, 4 \rrbracket$ and $y_j = 0$ for all $j \neq i$. If $w_i \neq 1$, this vertex is a singular point in \mathbb{P} . It gives rise to a singular point in X if and only if it lies in it, ie w_i does not divide d .
- an *edge* in \mathbb{P} is a line with equation $y_j = 0$ for all $j \in J$, for a certain $J \subset \llbracket 0, 4 \rrbracket$ of cardinal 3. If $\gcd(w_j)_{j \notin J} \neq 1$, the edge is in \mathbb{P}_{sing} . Recall that X is taken general in its linear system. Hence, an edge in \mathbb{P} lies entirely in X if and only if $(w_j)_{j \notin J}$ do not partition d , in X_{sing} if and only if $(w_j)_{j \notin J}$ do not partition d and have a non-trivial common divisor. If an edge in \mathbb{P}_{sing} does not lie entirely in X , it gives a finite amount of points in X_{sing} .
- a *2-face* in \mathbb{P} is a 2-plane with equation $y_j = 0$ for all $j \in J$, for a certain $J \subset \llbracket 0, 4 \rrbracket$ of cardinal 2. If $\gcd(w_j)_{j \notin J} \neq 1$, the 2-face is in \mathbb{P}_{sing} . By quasismoothness, no 2-face lies entirely in X . Hence, any 2-face intersects X along an effective 1-cycle. In this way, 2-faces in \mathbb{P}_{sing} may produce curves in X_{sing} .

Under the additional hypothesis that X contains no edge of \mathbb{P} , we can say more about its singular locus, thanks to the following lemmas:

Lemma 7.14. *Let X be a general quasismooth hypersurface of degree d in the weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \dots, w_4)$. Suppose that it contains no edge of \mathbb{P} . Then the base locus $\text{Bs}(\mathcal{O}_{\mathbb{P}}(d))$ has dimension 0.*

Proof. Let Z be an irreducible component of the base locus of $\mathcal{O}_{\mathbb{P}}(d)$, let us prove by induction on $\dim \mathbb{P}$ that it is a point. Suppose we are at the induction step where the ambient space \mathbb{P}' has local coordinates y_0, y_1, y_2, \dots and dimension 4, 3 or 2.

Denote by H_i the hyperplane $\{y_i = 0\}$ in \mathbb{P}' , by \mathbb{P}'_i the isomorphic weighted projective space $\mathbb{P}'(\dots, \hat{w}_i, \dots)$. By [BR86, Prop.4.A.3], we have an isomorphism between the restriction $\mathcal{O}_{\mathbb{P}'}(d) \otimes \mathcal{O}_{H_i}$ and the \mathbb{Q} -Cartier divisor $\mathcal{O}_{\mathbb{P}'_i}(d)$. This translates to global sections as a surjection:

$$H^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(d)) \twoheadrightarrow H^0(\mathbb{P}'_i, \mathcal{O}_{\mathbb{P}'_i}(d)), \quad (3)$$

which is given by setting $y_i = 0$ when considering the global sections as certain polynomials in the local coordinates of \mathbb{P}' .

The quasismoothness of X in \mathbb{P} and the way the composite surjection

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \twoheadrightarrow H^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(d)),$$

writes in local coordinates yield a global section of $\mathcal{O}_{\mathbb{P}'}(d)$ of the form $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2}$. In particular, there is an $i = 0, 1$ or 2 such that $Z \subset H_i \simeq \mathbb{P}'_i$. Moreover, by Eq.3, Z sits in the base locus of $\mathcal{O}_{\mathbb{P}'_i}(d)$.

Induction propagates from $\mathbb{P}' = \mathbb{P}$ down to when we obtain that Z is contained in an edge H_{ijk} of \mathbb{P} and in the base locus $\text{Bs}(\mathcal{O}_{\mathbb{P}_{ijk}}(d)) \subset \text{Bs}(\mathcal{O}_{\mathbb{P}}(d)) \subset X$. Since X contains no edge of \mathbb{P} , Z is in $X \cap H_{ijk}$ of dimension 0, so it is a point. \square

Remark 7.15. With the same notations and hypotheses, the intersection of X with any 2-face of \mathbb{P}_{sing} is a reduced curve.

Proof. As in the proof of Lemma 7.14, the intersection is scheme-theoretically defined by a general section of $\mathcal{O}_{\mathbb{P}_{ij}}(d)$. We are to show that such general section of $\mathcal{O}_{\mathbb{P}_{ij}}(d)$ is quasismooth in the weighted projective space \mathbb{P}_{ij} , hence it is a variety.

We use the arithmetical criterion 7.10: since X contains no edge of \mathbb{P} , each pair w_a, w_b partitions d . We are left to check the criterion for $k = 1$: fix any $a \neq i, j$, we want to find $b \neq i, j$ such that w_a divides $d - w_b$. It is clear that there is a $b \in \llbracket 0, 4 \rrbracket$ satisfying that. As H_{ij} is a 2-face in \mathbb{P}_{sing} , the greatest common divisor of all weights except w_i, w_j is non-trivial, divides d but neither w_i nor w_j (by wellformedness). In particular, since this greatest common divisor divides $w_b = d - \alpha w_a$, $b \neq i, j$, as wished. \square

It is worth noticing that the quasismoothness of X does not yield the smoothness of the preimage $p^{-1}(X)$. This preimage is interesting since the restricted quotient map $p^{-1}(X) \rightarrow X$ is an unfolding of X , as defined in Section 2.3. Let us discuss the possible singularities of this $\hat{X} := p^{-1}(X)$.

Proposition 7.16. *Let X be a general quasismooth hypersurface of degree d in a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \dots, w_4)$, p the natural quotient $\mathbb{P}^4 \rightarrow \mathbb{P}$, $\hat{X} = p^{-1}(X)$. Suppose that X contains no edge of \mathbb{P} . Then \hat{X} is smooth in codimension 2.*

Proof. The threefold \hat{X} is general in the linear system $p^*|\mathcal{O}_{\mathbb{P}}(d)|$, whose base locus has dimension 0 by Lemma 7.14. By Bertini's theorem, \hat{X} is smooth in codimension 2. \square

Remark 7.17. The converse of Proposition 7.16 does not hold: for instance, the general quasismooth X_7 in $\mathbb{P}(1, 1, 1, 2, 2)$ contains the edge of equation $y_0 = y_1 = y_2 = 0$, but its unfolding is nevertheless smooth in codimension 2.

Example 7.18. The hypothesis of Proposition 7.16 is not that X contains no edge of \mathbb{P}_{sing} , but that it contains no edge of \mathbb{P} at all: for instance, consider the general $X = X_{56}$ in $\mathbb{P}(2, 4, 9, 13, 28)$. It contains a single edge of \mathbb{P} , namely e of equation $y_0 = y_1 = y_4 = 0$. This edge does not actually lie in \mathbb{P}_{sing} , as 9 and 13 are coprime, but one can check that \hat{X} has the curve $p^{-1}(e)$ in its singular locus (by computing the derivatives of the equation defining \hat{X} in \mathbb{P}^4 along the curve $p^{-1}(e)$).

Example 7.19. Let us consider the general $X = X_{1734}$ in $\mathbb{P}(91, 96, 102, 578, 867)$. Its polynomial is a general linear combination of the following monomials:

$$y_4^2, y_3^3, y_2^{17}, y_1^{17} y_2, y_0 y_1 y_2 y_3 y_4, y_0^2 y_1^2 y_2^2 y_3^2, y_0^3 y_1^3 y_2^3 y_4, y_0^4 y_1^4 y_2^4 y_3, y_0^6 y_1^6 y_2^6, y_0^{18} y_1.$$

With this description, one easily checks that X is quasismooth. As $y_4^2, y_3^3, y_2^{17}, y_0^{18} y_1$ appear with general coefficients in the equation of X , we know that X contains no edge of \mathbb{P} . In particular, \hat{X} is smooth in codimension 2 by Proposition 7.16.

Moreover, the curves of X_{sing} are precisely the intersections of X with all 2-faces of \mathbb{P}_{sing} , which we can list:

- $y_0 = y_1 = 0$ of type $\frac{1}{17}(6, 11)$,
- $y_0 = y_3 = 0$ of type $\frac{1}{3}(1, 2)$,
- $y_0 = y_4 = 0$ of type $\frac{1}{2}(1, 1)$.

Example 7.20. One can check similarly that the hypersurface X_{120} of Example 7.11 contains no edge of its ambient projective space. It has precisely one curve in its singular locus, of type $\frac{1}{10}(3, 7)$.

It is possible to check the type of singularities of a general hypersurface of a given degree in a given weighted projective space by a simple computer program.

We move to the proof of Proposition 7.6. It relies on the following lemma and sublemma:

Lemma 7.21. *Let X be a general wellformed quasismooth hypersurface of dimension 3 in a weighted projective space \mathbb{P} not isomorphic to \mathbb{P}^4 . Assume that X has trivial canonical class and that it contains no edge of \mathbb{P} . Then $\hat{c}_2(X) \cdot \mathcal{O}_X(1) > 0$.*

Lemma 7.22. *Let X be a general wellformed quasismooth hypersurface of dimension 3 in a weighted projective space \mathbb{P} . Assume that X has trivial canonical class and contains no edge of \mathbb{P} . Then there are at most 10 curves in X_{sing} , with different cohomological classes in the list of the*

$$[\mathcal{O}_X(w_i) \cdot \mathcal{O}_X(w_j)] \in H^4(X; \mathbb{Q}), \text{ for } 0 \leq i < j \leq 4.$$

Proof of Lemma 7.22. By Remark 7.15, each curve in X_{sing} is scheme-theoretically the complete intersection of X with a 2-face H_{ij} of \mathbb{P}_{sing} . This association being bijective, there are as many curves in X_{sing} as 2-faces in \mathbb{P}_{sing} , so at most 10. The curve that corresponds to the 2-face H_{ij} has cohomological class $[\mathcal{O}_X(w_i) \cdot \mathcal{O}_X(w_j)]$. \square

Proof of Lemma 7.21 using Lemma 7.22. Let $p : \mathbb{P}^4 \rightarrow \mathbb{P}$ be the natural quotient map. Writing $\mathbb{P} = \mathbb{P}(w_0, \dots, w_4)$ with (w_0, \dots, w_4) not colinear to $(1, \dots, 1)$, the morphism p has degree $w_0 \cdots w_4$, which we denote by N , and X has degree $w_0 + \dots + w_4$, which we denote by d . We may also write s for the symmetric elementary polynomial of degree 2 in the weights and q for the sum of their squares: $d^2 = q + 2s$.

Since X is a full suborbifold of \mathbb{P} , $\hat{X} := p^{-1}(X) \rightarrow X$ is an unfolding of X as defined in Section 2.3. Applying the left-exact functor of reflexive pullback (see Lemma 2.2) to the exact sequence:

$$0 \rightarrow \mathcal{T}_X \rightarrow \mathcal{T}_{\mathbb{P}|_X} \rightarrow -K_{\mathbb{P}},$$

we get another exact sequence:

$$0 \rightarrow p^{[*]} \mathcal{T}_X \rightarrow p^{[*]} \mathcal{T}_{\mathbb{P}|_X} \rightarrow p^{[*]}(-K_{\mathbb{P}}) \rightarrow \mathcal{Z} \rightarrow 0,$$

where the coherent sheaf \mathcal{Z} is supported on the locus $p^{-1}(\text{Sing } X) \subset \hat{X}$ of codimension at least 2.

Because of the last surjection, $\dim_{k(p)} \mathcal{Z} \otimes \mathcal{O}_p \leq 1$ for any closed point $p \in \hat{X}$.

By Proposition 7.16, the unfolding \hat{X} is smooth in codimension 2, so the usual second Chern class $c_2(\mathcal{Z})$ makes sense. Since usual Chern classes are additive, and $c_1(\mathcal{T}_X) = 0, c_1(\mathcal{Z}) = 0$:

$$\hat{c}_2(\mathcal{T}_X) \cdot \mathcal{O}_X(1) = \hat{c}_2(\mathcal{T}_{\mathbb{P}|_X}) \cdot \mathcal{O}_X(1) + \frac{1}{N} c_2(\mathcal{Z}) \cdot \mathcal{O}_{\hat{X}}(1).$$

By the Miyaoka-Yau inequality [GKT, Thm.1.5], we have a positive contribution:

$$\hat{c}_2(\mathcal{T}_{\mathbb{P}|_X}) \cdot \mathcal{O}_X(1) = \hat{c}_2(\mathcal{T}_{\mathbb{P}}) \cdot (-K_{\mathbb{P}}) \cdot \mathcal{O}_{\mathbb{P}}(1) \geq \frac{4}{10} (-K_{\mathbb{P}})^3 \cdot \mathcal{O}_{\mathbb{P}}(1) = \frac{4d^3}{10N}.$$

Let us estimate the other summand. Take m big and divisible enough that $p^* \mathcal{O}_X(m) = \mathcal{O}_{\hat{X}}(m)$ is very ample and S a general element in $|\mathcal{O}_{\hat{X}}(m)|$. By [Kol92, Lem.10.9],

$$c_2(\mathcal{Z}) \cdot \mathcal{O}_{\hat{X}}(1) = \frac{1}{m} c_2(\mathcal{Z}|_S) = -\frac{1}{m} \deg(\mathcal{Z}|_S)$$

Denote by C_1, \dots, C_k the curves in X_{sing} . By Lemma 7.22, we can bound:

$$\begin{aligned} \deg(\mathcal{Z}|_S) &\leq \text{Card} \left(S \cap \bigcup_{i=1}^k p^{-1}(C_i) \right) = \sum_{i=1}^k N \mathcal{O}_X(m) \cdot C_i \\ &\leq Nm \mathcal{O}_X(1)^3 \sum_{0 \leq i < j \leq 4} w_i w_j \\ &= mNs(-K_{\mathbb{P}}) \cdot \mathcal{O}_{\mathbb{P}}(1)^3 \\ &= msd. \end{aligned}$$

Finally putting the positive and negative part together,

$$\begin{aligned} \hat{c}_2(X) \cdot \mathcal{O}_X(1) &> \frac{4d^3 - 10sd}{10N} \\ &= \frac{d(4q - 2s)}{10N} \\ &= \frac{d}{10N} \sum_{0 \leq i < j \leq 4} (w_i - w_j)^2 > 0. \end{aligned}$$

\square

Proof of Proposition 7.6. Consider X a general wellformed quasismooth hypersurface of degree $d = w_0 + \dots + w_4$ in a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \dots, w_4)$. Suppose that X contains no edge of \mathbb{P} . If \mathbb{P} is \mathbb{P}^4 , X is smooth and there is nothing to prove. Let us assume $\mathbb{P} \not\cong \mathbb{P}^4$. By Lemma 7.21, $\hat{c}_2(X) \cdot \mathcal{O}_X(1) \neq 0$, hence by [LT18, Thm.1.4], X is not a finite quotient of an abelian threefold. Moreover, one has $\text{Pic}(X) \simeq \mathbb{Z}$ [Dol82, Thm.3.2.4(i)], so Proposition 7.1 applies to X : it is not covered by a product of a K3 surface and an elliptic curve. So X is a singular Calabi-Yau threefold. \square

Examples for Proposition 7.6. General wellformed quasismooth hypersurfaces with trivial canonical class in 4-dimensional weighted projective spaces are classified in [KS92]. There is an explicit exhaustive list of the 7555 of them. In this list, 7238 elements are not smooth in codimension 2, and 2409 elements that are not smooth in codimension 2 also contain no edge of their ambient weighted projective space. These elements fulfill the hypotheses for Proposition 7.6, just as Examples 7.11 and 7.19 did: so they are singular Calabi-Yau threefolds to which Theorem 1.2 applies.

The exhaustive enumerations of elements of the [KS92] classification satisfying additional properties were done by running a simple computer program on the database [KS].

Remark 7.23. Let X be one of these singular Calabi-Yau hypersurfaces. As the natural quotient map $p : \mathbb{P}^4 \rightarrow \mathbb{P}$ is ramified along divisors, the unfolding $\hat{X} = p^{-1}(X) \rightarrow X$ results in a variety \hat{X} that certainly has a non-trivial canonical class. Hence, X is not at all constructed as a finite quasiétale global quotient, contrarily to the unsatisfying Example 5.5.

Remark 7.24. For the sake of transparent terminology, let us explain why the varieties studied in [KS92] are the same as general quasismooth wellformed hypersurfaces of trivial canonical class in a 4-dimensional weighted projective space.

First, any variety that [KS92] calls a nondegenerate Calabi-Yau hypersurface is sitting in an open set of nondegenerate Calabi-Yau hypersurfaces of a given linear system. This is precisely what we referred to as a general quasismooth hypersurface of trivial canonical class in a weighted projective space.

The paper [KS92] classifies “tuples” of positive integers $d, \{\{w_0, \dots, w_N\}\}$ such that:

- there are no “trivial variables”, ie $\frac{d}{2} \notin \{\{w_0, \dots, w_N\}\}$,
- $N = 3$ or 4 ,
- there is a nondegenerate Calabi-Yau hypersurface of degree d in $\mathbb{P}(w_0, \dots, w_N)$ with condition “ $c = 9$ ”.

Here, we use $\{\{\cdot\}\}$ to denote tuples where order does not matter, or equivalently sets where elements may appear with a certain multiplicity.

We claim that the map f :

$$d, \{\{w_0, \dots, w_N\}\} \mapsto \begin{cases} d, \{\{w_0, \dots, w_3, \frac{d}{2}\}\} & \text{if } N = 3 \\ d, \{\{w_0, \dots, w_4\}\} & \text{else,} \end{cases}$$

is a one-to-one correspondence between the data of [KS92] and all tuples $d, \{\{w_0, \dots, w_4\}\}$ such that $X_d \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ is a general quasismooth wellformed hypersurface with trivial canonical class. To prove this claim, let us compute the image of this injective map f : a tuple $d, \{\{w_0, \dots, w_4\}\}$ is in the image of f if and only if at most one of the w_i equals $\frac{d}{2}$ and the general hypersurface of degree d in the projective space of weights $\{\{w_i \mid 2w_i \neq d\}\}$ is quasismooth, has trivial canonical class and satisfies the rewritten condition $c = 9$:

$$\sum_{i=0}^4 1 - \frac{2w_i}{d} = 3, \text{ ie } \sum_{i=0}^4 w_i = d.$$

So, it is clear that the image by f of the [KS92] tuples with $N = 4$ is made of all $d, \{\{w_0, \dots, w_4\}\}$ such that $X_d \subset \mathbb{P}(w_0, \dots, w_4)$ is a general quasismooth wellformed hypersurface of trivial canonical class, and for all i , $2w_i \neq d$.

The image by f of the tuples with $N = 3$ is easily checked to stand for quasismooth hypersurfaces in weighted projective spaces of dimension 4. We check that the quasismooth hypersurfaces arising

in that way are wellformed by a careful application of the criterion [IF00, 6.13], [Dim86, Prop.2], together with elementary arithmetic. Notably, the fact that

$$\sum_{i=0}^3 w_i = \frac{d}{2}$$

together with the quasismoothness conditions implies that either for all i , $\gcd(w_0, \dots, \hat{w}_i, \dots, w_3) = 1$, or $d \equiv 2 \pmod{4}$ and for all i , $\gcd(w_0, \dots, \hat{w}_i, \dots, w_3) = 1$ or 2 helps to apply the criterion. As adjunction formula holds, these general quasismooth wellformed hypersurfaces in weighted projective spaces of dimension 4 have trivial canonical class. Conversely, precisely those general quasismooth wellformed hypersurfaces X_d of trivial canonical class in a $\mathbb{P}(w_0, \dots, w_3, \frac{d}{2})$ are in the image by f of the $N = 3$ data.

Example 7.25. The wellformed quasismooth Calabi-Yau hypersurface X_{1734} in $\mathbb{P}(91, 96, 102, 578, 867)$ comes from the $N = 3$ data (originally denoted $n = 4$) in [KS92], since $1734 = 2 \times 867$. The wellformed quasismooth Calabi-Yau hypersurface X_{120} in $\mathbb{P}(3, 7, 20, 40, 50)$ comes from the $N = 4$ data (originally denoted $n = 5$).

A Finite quasiétale quotients of abelian surfaces

Here we prove Proposition 6.5. When describing the effective examples, we will use the various notations of Proposition 6.5 for denoting some lattices, elliptic curves, complex roots of unit and affine maps $\mathbb{C}^2 \rightarrow \mathbb{C}^2$.

Lemma A.1. *Let G be a finite group acting non-trivially on an abelian surface A , such that the quotient A/G is a normal canonical surface. Then, up to quotienting A by an isogeny, one can assume that G acts faithfully and contains no translation. The group G injects in $SL(2, \mathbb{C})$. Each non-trivial element of G has order $d \in \{2, 3, 4, 6\}$ and fixes a finite number of points.*

Proof. First, it does not change anything to the geometry of S to assume that G acts faithfully on A . Second, as translations are precisely the automorphisms of A with no fixed point [BL04, Prop.13.1.3], those that may be in G form a normal subgroup N of G ; up to considering $G' = G/N$ and $A' = A/N$, we can assume that G contains no translation.

Recall now that any element of G acts by an automorphism on A of the form

$$g : [z] \in A \mapsto [M(g)z + T(g)] \in A,$$

where $M(g)$ is a matrix in $GL(2, \mathbb{C})$ and $T(g)$ is an element of \mathbb{C}^2 . On one hand, M is a group representation, on the other hand, T is not even a group homomorphism. As only finitely many points in S are in the singular locus, finitely many points in A have a non-trivial stabiliser in G . So each element g of G has finitely many fixed points, and at least one. Near a fixed point z_g for g , the action is locally modeled after the multiplication by $M(g)$ in \mathbb{C}^2 near the point $(0, 0)$. As $[z_g] \in S$ is a canonical singularity, we have $M(g) \in SL(2, \mathbb{C})$ [Rei]. Hence a group homomorphism $g \in G \mapsto M(g) \in SL(2, \mathbb{C})$, which is injective because G acts faithfully and contains no translation.

Finally, let g be an element of G . Of course it has finite order d , and by [BL04, Prop.13.2.5], the Euler indicator $\varphi(d)$ is 1, 2 or 4, ie $d \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$. But the values $d \in \{5, 8, 10, 12\}$, ie those with $\varphi(d) = 4$, are excluded for the following reason: by [BL04, Cor.13.3.6], an automorphism of A with finite order $d \in \{5, 8, 10, 12\}$ can not have its associated linear action in $SL(2, \mathbb{C})$ (recall Example 6.3). \square

From now on, we stick to this set-up and denote by S the quotient A/G .

Lemma A.2. *If G has odd order, then it is isomorphic to \mathbb{Z}_3 , and S has precisely 9 singular points of type A_2 . Moreover, this situation can occur.*

Proof. If G has odd order, by Lagrange theorem, each element of G has order 1 or 3. Moreover, finite subgroups of $SL(2, \mathbb{C})$ are classified in [Rei, Exercise 10], and they have odd order if and only if they are cyclic. So G is isomorphic to \mathbb{Z}_3 . Let $\sigma \in \text{Aut}(A)$ be a generator of G . It has 9 fixed points [BL04, Ex.13.2.7]. Other points in A have trivial stabiliser under G , so they don't yield singular points on S . Hence, S has 9 A_2 points.

The situation occurs, since we can have \mathbb{Z}_3 act diagonally on the product of elliptic curves $E_3 \times E_3$. \square

Corollary A.3. *Let S be a normal canonical surface arising as a finite quasiétale quotient of an abelian surface A . Then its crepant resolution \tilde{S} contains at least 16 (-2) -curves in its exceptional locus.*

Proof. If the group G acting on A has odd order, we are done by Lem.A.2. Else, there is an automorphism of order 2 of A in G . It has 16 fixed points, which generate 16 singular points in S , hence at least 16 (-2) -curves in the exceptional locus of \tilde{S} . \square

We are left to assume that G has even order.

Lemma A.4. *If G has even order, there is only one element of order 2 in G .*

Proof. Suppose we have σ, σ' two elements of order 2 in G . They both act on A by an automorphism of the form $g : [z] \mapsto [M(g)z + T(g)]$, where $M(g)$ is a matrix in $SL(2, \mathbb{C})$ and $T(g)$ is an element of A . As they have order 2, $M(\sigma) = M(\sigma') = -\text{id}$. Moreover, since G contains no translation, and $\sigma \circ \sigma' : z \mapsto z + T(\sigma) - T(\sigma')$, we have $T(\sigma) = T(\sigma')$. We conclude that $\sigma = \sigma'$. \square

We can now classify all possible singularities for the surface S in the simplest case where G is a cyclic group.

Lemma A.5. *Suppose that G is cyclic with even order. Then the singularities of $S := A/G$ can only be:*

- 16 singularities A_1 ,
- 4 singularities A_3 and 6 singularity A_1 ,
- 1 singularity A_5 , 4 singularities A_2 and 5 singularities A_1 ,

and these three possibilities are effective.

Proof. Let σ be the generator of G . Up to conjugating the whole action of G on A by a well-chosen translation, we can assume that σ fixes $0 \in A$. So 0 has stabilisor G . The order of σ distinguishes between three possible cases:

- σ has order 2, ie $G = \{\pm \text{id}\}$: it has precisely 16 fixed points and other points of A have trivial stabilisor.
- σ has order 4. It has 4 fixed points. The 12 remaining fixed points for $\sigma^2 = -\text{id}$ have stabilisor \mathbb{Z}_2 and are paired in orbits under the action of G . So S has 4 singularities A_3 and 6 singularities A_1 .
- σ has order 6. It has one fixed point, hence one singularity A_5 in S . The 15 remaining fixed points for $\sigma^3 = -\text{id}$ have stabilisor \mathbb{Z}_2 and are arranged in orbits of length 3: they yield 5 singularities A_1 in S . The 8 remaining fixed points for σ^2 (which has indeed order 3) have stabilisor \mathbb{Z}_3 and are paired in orbits: hence 4 singularities A_2 in S . The other points have trivial stabilisor, so they add no singularity to the picture.

Reciprocally, we can provide examples for these cases by defining actions of:

- \mathbb{Z}_2 on any abelian surface (this is the singular Kummer surface again),
- \mathbb{Z}_4 by $b : z \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z$ on any product of an elliptic curve with itself,
- \mathbb{Z}_6 diagonally on the product $E_3 \times E_3$.

\square

We are left to study what happens if G is not a cyclic group. As it is a finite subgroup of $SL(2, \mathbb{C})$ with elements of order 2, 3, 4 or 6 only, it has to be isomorphic to one of the following groups: BD_8, BD_{12}, BT_{24} . We are going to examine these cases one by one and show that each of them yields precisely one possible type of singularities for the quotient surface S . These 3 last cases finish the proof of Proposition 6.5.

The binary dihedral group BD_8 . Suppose that $G \simeq BD_8$ acts on an abelian surface A . Let $a, b \in G$ be the two generators such that

$$M(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Without loss of generality, we can assume that $T(a) = 0$. Both a and b have order 4, so they have 4 fixed points each. As $b^2 = a^2 = -\text{id}$, b acts with order 2 on the set of fixed points of a , we can distinguish between three cases:

First case: a and b have no common fixed point.

Second case: a and b have the same 4 fixed points.

Third case: a and b have precisely 2 common fixed points.

Let us show that each case leads to precisely one possible list of singularities for A/G , and give an effective example where that list of singularities actually occurs.

Analysis of the first case: each point in A has stabilisor isomorphic to a proper subgroup of BD_8 , that is \mathbb{Z}_4 or \mathbb{Z}_2 . Among the automorphisms a, b, ab of order 4, no two of them have a common fixed point. So there are at least 12 distinct points in A stabilized by \mathbb{Z}_4 , and these are stable by $-\text{id}$, so they are the only ones. They are grouped by orbits of length 2. The 4 remaining points fixed by $-\text{id}$ have stabilisor \mathbb{Z}_2 and all are in the same BD_8 -orbit. Hence, the singularities of A/G are 6 A_3 points and 1 A_1 point.

Analysis of the second case: we use Eq.2 on the normal canonical surface A/G , which already has 4 singular points of type D_4 and can have a points of type A_3 , b points of type A_1 . It yields

$$24 = 4 \left(5 - \frac{1}{8} \right) + a \left(4 - \frac{1}{4} \right) + b \left(2 - \frac{1}{2} \right),$$

so $a = 0, b = 3$ is the only integral solution.

Analysis of the third case: we use Eq.2 on the normal canonical surface A/G , which already has precisely 2 singular points of type D_4 and can have a points of type A_3 , b points of type A_1 . It yields that $a = 2a' + 1$ is odd and that

$$10 + \frac{1}{2} = a' \left(8 - \frac{1}{2} \right) + b \left(2 - \frac{1}{2} \right),$$

so $a' = 1, b = 2$ or $a' = 0, b = 7$. But $-\text{id} \in G$ fixes 16 points on A , among which the $4b$ points which go to the A_1 singular points of A/G . Hence, $a' = 1, b = 0$.

Synthesis of the first case: take the notations of Proposition 6.5. As the multiplications by $a, M(b')$ preserve the lattice Λ_8 in \mathbb{C}^2 , and as \mathbb{C}^2/Λ_8 is indeed an abelian surface A , we have a group $G = \langle a, b' \rangle$ acting on A . We are left to check two non-obvious facts: that G is indeed isomorphic to BD_8 and that a, b' fix no common point on A .

Using the standard presentation of BD_8 , it is enough for the first claim to check that $a^2 = b'^2 = (b'a)^2$. As $g \mapsto M(g)$ is a group morphism and $T(a^2) = T(-\text{id}) = 0$, we are left to prove that $T(b'^2) = T((b'a)^2) = 0$. But

$$\begin{aligned} T(b'^2) &= b'.(1, 0) = (0, -1) + (1, 0) = 0 \text{ in } \Lambda_8, \\ T((b'a)^2) &= b'.(i, 0) = (0, -i) + (1, 0) = 0 \text{ in } \Lambda_8, \end{aligned}$$

as wished. For the second claim, note that the four fixed points of a are $(0, 0), (1, 0), \left(\frac{i+1}{2}, \frac{i+1}{2}\right)$ and $\left(\frac{i+1}{2}, \frac{i-1}{2}\right)$ and that b' does not fix any of them. So we have an example for the first case.

Synthesis of the second case: the linear action of the group $\langle a, b \rangle$ quotients down to an action on $A = \mathbb{C}^2/\Lambda_8$. Moreover, as b , contrarily to b' , has no translation part, it is clear that the group $\langle a, b \rangle$ is BD_8 . Finally, we already computed the four fixed points of a , and it is now easy to check that b fixes all of them. Hence an example for the second case.

Synthesis of the third case: as E_4 supports multiplication by i , the linear action of $BD_8 = \langle a, b \rangle$ on $A = E_4 \times E_4$ is well-defined. Let us compute the fixed points of a on this abelian surface: they are $(0, 0), \left(\frac{i+1}{2}, 0\right), \left(0, \frac{i+1}{2}\right), \left(\frac{i+1}{2}, \frac{i+1}{2}\right)$. Among them, b fixes $(0, 0)$ and $\left(\frac{i+1}{2}, \frac{i+1}{2}\right)$, and switches the two others. This is an example for the third case.

The binary dihedral group BD_{12} . Suppose that $G \simeq BD_{12}$ acts on an abelian surface A . Let $b, d \in G$ be the two generators such that

$$M(d) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad M(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We can assume that $T(d) = 0$, so that the only automorphism of order 2 in G is $d^3 = -\text{id}$. As $bdb^{-1} = d^5$, the single fixed point of d (which is 0) is fixed by b^{-1} . Now we have precisely one singular point of type D_5 in A/G , and remaining a points of type A_3 , b points of type A_2 , c points of type A_1 satisfying Eq.2:

$$24 = 6 - \frac{1}{12} + a \left(4 - \frac{1}{4}\right) + b \left(3 - \frac{1}{3}\right) + c \left(2 - \frac{1}{2}\right),$$

and the additional constraint that the 16 fixed points of $-\text{id}$ on A are precisely the points whose stabiliser has an even order, ie $1 + 3a + 6c = 16$, so $a + 2c = 5$. The only solution is $a = 3, b = 2, c = 1$.

Reciprocally, as the multiplications by ω and ω^{-1} act on the elliptic curve E_3 (with notations from Proposition 6.5), the natural linear action of BD_{12} on \mathbb{C}^2 quotients down to an action on $E_3 \times E_3$. So BD_{12} can indeed act on an abelian surface in the way we just described.

The binary tetrahedral group BT_{24} . Suppose that $G \simeq BT_{24}$ acts on an abelian surface A . Let $a, b, c \in G$ be the two generators such that

$$M(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M(c) = \frac{1}{2} \begin{pmatrix} i+1 & i-1 \\ i+1 & 1-i \end{pmatrix}.$$

Note that, by definition of the binary tetrahedral group, any multiplicative relation between these matrices has to be satisfied by the automorphisms a, b, c too, despite they may have a translation part. In particular, we have $a^2 = b^2 = (ba)^2 = c^3$, $acb = c$, $abc = ca$. Moreover, it is easy to check that these relations give a presentation of the group BT_{24} . Note that accordingly, every element in BT_{24} can be written as $a^\alpha b^\beta c^\gamma$ with $\alpha \in \{0, 1\}$ and $\beta, \gamma \in \{0, 1, 2\}$.

As c has order 6, it fixes precisely one point x of A . Let us conjugate the whole action of G on A by a translation, so that $x = 0$, so $T(c) = 0$. If 0 is fixed by any element of G which is not a power of c , it is stabilized by the whole group. Hence, we can distinguish between two cases:

First case: the point 0 is stabilized by the whole group.

Second case: no point is stabilized by the whole group.

Let us show that each case leads to precisely one possible list of singularities for A/G , and give an effective example where that list of singularities actually occurs.

Analysis of the first case: as 0 has stabiliser BT_{24} , it is the common fixed point of all elements of order 6 in G . So A/G has precisely one singularity of type E_6 , and no singularity of type D_5 or A_5 . The quotient can have additional a points of type D_4 , b points of type A_3 , c points of type A_2 , d points of type A_1 . By Eq.2, we have:

$$24 = 7 - \frac{1}{24} + a \left(5 - \frac{1}{8}\right) + b \left(4 - \frac{1}{4}\right) + c \left(3 - \frac{1}{3}\right) + d \left(2 - \frac{1}{2}\right),$$

and by discussing the 16 fixed points of $-\text{id}$, we get $1 + 3a + 6b + 12d = 16$, so $a + 2b + 4d = 5$. We check by hand that the only possibility is $a = 1, b = 0, c = 4, d = 1$.

Analysis of the second case: the point 0 fixed by c has a BT_{24} -orbit of cardinal 4, containing all points of A stabilised by \mathbb{Z}_6 . The stabiliser of any point of A which is not in this orbit is $BD_8, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_2$ or trivial. It gives precisely one singular point of type A_5 in the quotient A/G , and remaining a points of type D_4 , b points of type A_3 , c points of type A_2 , d points of type A_1 , with:

$$24 = 6 - \frac{1}{6} + a \left(5 - \frac{1}{8}\right) + b \left(4 - \frac{1}{4}\right) + c \left(3 - \frac{1}{3}\right) + d \left(2 - \frac{1}{2}\right),$$

and $4 + 3a + 6b + 12d = 16$, so $a + 2b + 4d = 4$. Again, there is only one possibility: $a = 0, b = 2, c = 4, d = 0$.

Synthesis of the first case: clearly a, b, c preserve the lattice Λ_8 , so $BT_{24} = \langle a, b, c \rangle$ acts on the abelian surface $A = \mathbb{C}^2/\Lambda_8$. The common fixed point is 0, so this example falls in the first case.

Synthesis of the second case: still $a, M(b'), M(c')$ preserve the lattice Λ_8 , so we have an action of the group $G = \langle a, b', c' \rangle$ on \mathbb{C}^2/Λ_8 . We have to check that G is the binary tetrahedral group. For that, it is enough to check the translation parts of the relations between a, b', c' that we previously gave as a presentation BT_{24} :

$$0 = T(a^2) = T(b^2) = T((b'a)^2) = T(c'^3), T(ac'b') = T(c') \text{ and } T(ab'c') = T(c'a).$$

But indeed:

$$\begin{aligned} T(c'^3) &= c'^2 \cdot \left(\frac{1}{2}, \frac{-i}{2}\right) = c' \cdot \left(\frac{i}{2}, \frac{i}{2}\right) = 0, \\ T(ac'b') &= ac' \cdot (1, 0) = a \cdot \left(\frac{i}{2}, \frac{-1}{2}\right) = \left(\frac{-1}{2}, \frac{i}{2}\right) = \left(\frac{1}{2}, \frac{-i}{2}\right) = T(c'), \\ T(ab'c') &= ab' \cdot \left(\frac{1}{2}, \frac{-i}{2}\right) = a \cdot \left(\frac{-i}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{-i}{2}\right) = T(c'a). \end{aligned}$$

So $G = BT_{24}$. Finally, we checked earlier that a, b' have no common fixed points, so this example does not fall in the first case, but in the second case as we wished.

And with these two last constructions, the classification is done.

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