Advection in Discrete Exterior Calculus

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Abstract

When solving non-stationary advection-diffusion problems, stability is a major issue. In cases where the diffusive part is very small, only implicit schemes can guarantee this stability. We use the notion of exterior derivative $d$, Lie derivative $L_{\gamma}$ and Hodge operator $\nabla$ to introduce the time dependent advection-diffusion problem of a differential form $\omega$:

$$\frac{d\omega}{dt} + (\mathbf{v} \cdot \nabla) \omega = \nabla \cdot \mathbf{J}$$

This formulation includes not only the usual scalar case, but also other interesting problems like the magnetic advection-diffusion problem related to magnetohydrodynamics. A discretization along the ideas of Discrete Exterior Calculus (DEC) helps reveal fundamental structural properties shared by all advection-diffusion problems.

1. Introduction

A huge amount of research has been directed at numerical methods for transient 2nd-order advection-diffusion problems for an unknown scalar function $u = u(x,t)$ on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v} u) = 
\nabla \cdot \mathbf{D} \nabla u + f \quad \text{in} \quad \Omega \times (0,T],$$

Here:
- the diffusion coefficient $\alpha(x) > 0$ is a positive semi-definite tensor field.
- $\mathbf{v} \in \mathbb{R}^n$ is a given Lipschitz continuous velocity field, $f \in L^2(\Omega)$.
- the boundary $\Gamma$ splits according to the figure.

Another important advection-diffusion problem is the so-called magnetic advection-diffusion problem for the vector field $\mathbf{u} \in \mathbb{R}^n$:

$$\frac{d\mathbf{u}}{dt} + \mathbf{v} \times \mathbf{u} = \nabla \times \mathbf{A}$$

2. Linear Advection of Differential Forms

A Lipschitz continuous vector field $\mathbf{v} : \Omega \to \mathbb{R}^n$ induces a flow $X(x,t) = \Phi(x,t)$, $X(x,0) = x$.

The pullback $\Phi^* \omega = X^* \omega$, $X^* = (X^T \partial X)^{-1}$, of that flow map is used to define the Lie derivative, the advection operator for differential forms $\omega \in \Gamma^k(\Omega)$:

$$L_{\gamma} = \Phi^* \omega = \sum_{i=1}^n (\partial_i \Phi^j) \omega^{ij} + d \gamma.$$ 

The representation in terms of contraction $\varepsilon_i$ and exterior derivative $\varepsilon_i$ is due to Cartan.

3. General Solution Strategies

The approximation spaces are the conforming full and reduced polynomial space $P_m^2$ and $P_m^2$, in simple meshes introduced in [1] but also the non-conforming, globally discontinuous, piecewise polynomial spaces $P^m^2, \Omega$ used in discontinuous Galerkin methods.

The general solution strategies for advection-diffusion with small diffusion ($\varepsilon < 1$) are the following:

- Eulerian Methods/Methods of Lines:
  - Explicit Euler:
    - $M_{\Phi^2} \omega^{n+1} = M_{\Phi^2} \omega^n + \tau \mathbf{D} \nabla u^n + f^n$, $\vdash \omega^n$ on $\Gamma$.
  - Implicit Euler:
    - $M_{\Phi^2} \omega^{n+1} = M_{\Phi^2} \omega^n + \tau \mathbf{D} \nabla u^n + f^n$.

- Semi-Lagrangian Methods:
  - Implicit and $\Phi = \nabla \times \mathbf{A}$:
    - $M_{\Phi^2} \omega^{n+1} = M_{\Phi^2} \omega^n + \tau \mathbf{D} \nabla u^n + f^n$.

4. Eulerian Methods

A standard discretization for advection-diffusion allows $H^1$-conformity only in $\mathbb{R}^2$. So-called stabilized Galerkin discretization, such as discontinuous Galerkin, Galerkin/Least-Squares methods or subgrid viscosity techniques, allow for stability in a certain, stronger mesh-dependent norm. Further, we recall the following:

- Stable methods build on upwind methods for pure advection.
- Natural spaces for DEC are discontinuous approx. spaces for pure advection.
- Discrete approx. spaces for DEC allow for upwind fluxes.

Hence we consider the following upwind discretization:

$$\phi(x,0), \hspace{1cm} \omega(x,0), \hspace{1cm} \int_0^T \omega(x,t) \ dt$$

Theorem 1: Order of convergence for stationary advection in $H^1(\Omega)$:

$$\|L_{\gamma} - \omega\|_{H^1(\Omega)} = \mathcal{O}(\tau + h)$$

Outline of proof:

- Hodge rule (1) establishes stability:
  $$\|L_{\gamma} - \omega\|_{H^1(\Omega)} \leq \mathcal{O}(\tau + h).$$

Introduce piecewise constant approximation $\mathbf{v}$ of flow, i.e. $(\beta(\mathbf{v} - \delta h \mathbf{v})) = 0 \forall h$ and gain powers of mesh size $h$ in continuity estimates:

- Case $P_{m}^2$, $\chi(\mathbf{v}) = h$ (as for $h = 0$):
  $$\|L_{\gamma} - \omega\|_{H^1(\Omega)} \leq \mathcal{O}(\tau + h).$$

5. Semi-Lagrangian Methods

The semi-Lagrangian Galerkin timestepping scheme for a transient advection problem constructs sequences $(\mathbf{u}^n)^{N} = \mathbf{u}_0$ approximating $(\mathbf{u}(t_0 + n\tau))$ according to:

- Find $(\mathbf{u}^n)^{N}$ such that for all $\mathbf{v}^n$:
  $$\langle \mathbf{v}^n \cdot \mathbf{D} \nabla, \mathbf{u}_n \rangle = \langle \mathbf{v}^n \cdot \nabla, \mathbf{u}_n \rangle$$

Remark 1: The classical theory for Semi-Lagrangian methods is not satisfactory for the magnetic advection-diffusion problem. A simple adaptation to the case of advection of different forms does not converge for lowest order approximation (edge elements) for the magnetic advection-diffusion problem.

References