Ch. III Sheaves and algebraic sets

3.1 Sheaves

Defn: Let $X$ be a topological space. A presheaf $\mathcal{F} \to X$ is the following data:

1) For every open set $U$ of $X$, we have a set $\mathcal{F}(U)$,

2) If $V \subseteq U$ open, there exists a (restriction) map

$$\nu_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$$

having the following properties:

i) If $W \subseteq V \subseteq U$ open sets then

$$\nu_{W,U} \circ \nu_{V,W} = \nu_{V,U}$$

ii) $\nu_{U,U} = 1_{\mathcal{F}(U)}$.

A presheaf is a sheaf if we also have the gluing property:

3) If $U = \bigcup_{i} U_i$ with $U_i \subseteq X$ open and $f_i \in \mathcal{F}(U_i)$ s.t. $\forall i,j \quad \nu_{U_i \cap U_j, U_i}(f_i) = \nu_{U_i \cap U_j, U_j}(f_j)$

then $\exists ! f \in \mathcal{F}(U)$ s.t. $\nu_{U_i,U}(f) = f_i$. 
Defn: We say that \( F \) is (pre-) sheaf of groups/rings/modules if \( F(U) \) is a group/ring for all \( U \in X \) open and the maps \( 
abla, \nabla \) are morphisms of groups/rings/modules.

Notation: we call \( F(U) \) the sections of \( F \) on \( U \) and \( F(X) \) the global sections of \( F \).

Examples:
1) \( X = \mathbb{R}^m \) with the euclidean topology
   \[ \forall U \subset X \text{ set } F(U) := \left\{ f: U \to \mathbb{R} \text{ is } C^\infty \right\} \]
   and \( \forall V \subset U \nabla_U : F(U) \to F(V) 
   f \mapsto f|_V \)
   Then \( F \) is a sheaf (even sheaf of rings since \( f_1 + f_2, f_1 \cdot f_2 \)
gives ring structure)

2) \( X \) an affine irreducible set, \( G \) a group. We set
   \[ \forall U \subset X \quad G(U) := \left\{ f: U \to G \text{ is constant} \right\} \]
   and \( \nabla, \nabla \) the restrictions.
   Then \( F \) is a sheaf of groups. Check the gluing property:
   if \( U = \bigcup_i U_i \) with \( U_i \in X \) open and
   \[ f_i: U_i \to G \text{ constant so } f_i(x) = g_i \forall x \in U_i \]
suppose \( f_i\mid u_{i,j} = f_j\mid u_{i,j} \quad \forall i,j \)

Can we guess? Since \( X \) irreducible we have \( U_i \cap U_j \neq \emptyset \quad \forall i,j \)

\[ \Rightarrow g_i = f_i(x) = f_j(x) = g_j \quad \text{for some } x \in U_i \cap U_j \]

\( \Rightarrow g_i \) does not depend on \( U_i \).

\[ \Rightarrow f: U \to \mathbb{C} \quad \text{has property } f\mid u_i = f_i. \]

\[ \S 2 \text{ Affine sets.} \]

\( X \text{ affine} \)

Goal: replace \( P(X) \) by more flexible structure sheaf \( G_X \).

**Lemma:** Let \( X \) be a topological space and \( (U_i)_{i \in I} \) a base of the topology (all unions \( U = \bigcup U_i \))

Suppose we have fixed \( \forall U_i \) a fcbn. ring

\[ \mathcal{F}(U_i) = \{ f: U_i \to \mathbb{C} \mid f \text{ has certain properties} \} \quad \text{any } x \neq \]

Suppose we have the following properties:

1) If \( U_j \subseteq U_i \) and \( f \in \mathcal{F}(U_i) \) then \( f\mid u_j \in \mathcal{F}(U_j) \)

2) If \( U_k = \bigcup_{i \in I} U_i \) and \( f_i \in \mathcal{F}(U_i) \) s.t. \( f_i\mid U_{i\setminus j} = f_j\mid U_{i\setminus j} \quad \forall i,j \)

then \( \exists! f \in \mathcal{F}(U_k) \) such that \( f\mid U_i = f_i \).

Then \( \exists! \) sheaf of rings \( \mathcal{F} \to X \) that coincides with \( \mathcal{F}(U_i) \) on \( U_i \).
Proof: If \( U \subset X \) arbitrary open set, we can write
\[
U = \bigcup_{i \in I} U_i \quad \text{with} \quad f \in I.
\]
We set
\[
\mathcal{F}(U) = \{ f \to k \mid f(U_i) \supseteq U, \forall i \in I \}
\]
and check that this is a sheaf (exercise). \( \Box \)

Definition: Let \( X \) be affine, \( \mathcal{P}(X) \) its function ring.

The sets \( (X \setminus \text{V}(f)) \) for \( f \in \mathcal{P}(X) \)
form a base of the Zariski topology.

We define the set
\[
G_x (X \setminus \text{V}(f)) = \mathcal{P}(X)_f
\]
This definition satisfies the conditions of the lemma, and we denote by \( G_x \to X \) the unique sheaf that coincides with \( G_x (X \setminus \text{V}(f)) \) on \( X \setminus \text{V}(f) \).

We call \( G_x \) the sheaf of regular functions on \( X \), the sheaf of \( R \)

Examples:

1) \( X = k^n \). Then every closed set is finite \( x_1, \ldots, x_n \)
so can be written as \( \text{V}(x_1 - a_1, \ldots, x_n - a_n) \)

\Rightarrow Every open set has form \( k \setminus \text{V}(f) \), so \( U \subset k \) open
\[
G_k (U) = k[x]_f
\]
where \( f \) polynomial on \( U \).

\[
V(f) = k \setminus U.
\]
2) If \( X \) affine, \( U = X \setminus V(f) \). Then by definition \( G_X(U) \subset P(X) \).

In particular \( G_X(X) = P(X) \setminus \Pi(X) \setminus V(f) \).

3) \( X = k^2 \) then \( k^2 \setminus \langle g_0 \rangle \) open, not of the form \( k^2 \setminus V(f) \).

We have \( G_{k^2}(k^2 \setminus \langle g_0 \rangle) = \Pi_{X_0} \), \( X_0 \in G_{k^2}(k^2) \).

Proof that our definition satisfies conditions of Lemma:

1) \( U \cap V(f) \subset U \cap V(g) \) then \( V(g) \subset V(f) \)

so \( I(UV(f)) = I(UV(g)) = \text{Ann} g \).

Nullstellensatz \( \Rightarrow f^l = gh \).

If \( \frac{a}{g} \in \Pi(U) \) then \( G_X(U) \).

\[
\frac{a}{g} \in \Pi(U) \implies \frac{a}{g} \in \Pi(U) \quad \frac{a}{g} \in \Pi(U) \quad \frac{a}{g} \in \Pi(U)
\]

2) For simplicity suppose \( X \) irreducible (see book)

Suppose \( X \setminus V(f) = (X \setminus V(g)) \cup (X \setminus V(f)) \).

Then \( \frac{a}{f} \in \Pi(U) \Rightarrow \frac{a}{f} \in \Pi(U) \), \( \frac{a}{f} \in \Pi(U) \).

so \( \frac{a}{f} \in \Pi(U) \).

(when some exponent for \( f_1, f_2 \) )

\( \equiv (X \setminus V(g)) \cup (X \setminus V(f)) \).

\( X \setminus V(f, g) \).
\[ c = \alpha_1 f_2^n = \alpha_2 f_4^n \quad \text{on} \quad X \setminus V(f_1 f_2) \quad \text{for some} \quad \alpha_1, \alpha_2 \quad \text{that are open, so dense since} \ X \ \text{is normal.} \]

\[ \iff \quad \alpha_1 f_2^n = \alpha_2 f_4^n \quad \text{on} \ X. \]

By Nullstellensatz, \( \exists \) implies \( f^m = b_1 f_1^n + b_2 f_2^n \) for some \( b_1, b_2 \).

Set now \( a = \alpha_1 b_1 + \alpha_2 b_2 \). Then \( \frac{a}{f^m} \in \mathbb{P}(X)^f \)

and \( \frac{a}{f^m} \bigg|_{x \setminus V(f_0)} = \frac{a_i}{f_i} \quad \square \)

§3 Ringel spaces and algebraic sets

Defn: A ringel space is a topological space \( X \) endowed with a sheaf of rings \( G_x \). We call \( G_x \) the structure sheaf of \( X \).

Ex:

1) \( X = \mathbb{R}^n \), \( G_x = C^\infty \)-forms is a ringel space

2) \( X \) affine, \( G_x = \text{regular forms} \)

3) \( X = C \) with \( \text{Cotangent topology} \). Many choices

- \( G_x = \text{holomorphic forms} \)
- \( G_x = \text{regular forms} \)
- \( G_x = \text{loc. const. forms with values in} \ \mathbb{C} \)

Defn: Let \( (X, G_x) \) and \( (Y, G_y) \) be ringel spaces. A morphism of ringel spaces is a continuous map \( f: X \to Y \)
s.t. $U \subset \gamma$, open and $\forall u \in G_\gamma(u)$ we have

$$g^{-1}(g) = \gamma_0 \circ \rho \circ G_x(g^{-1}(u))$$

**Ex.** $(X, \mathcal{B}_x)$, $(Y, \mathcal{B}_y)$ affine. Suppose $\phi : X \to Y$ is a regular map. Then $\phi$ is a morphism of ringed spaces.

**Ex.** If $(X, \mathcal{B}_x)$ is a ringed space, $U \subset X$ open. Then we can define $G_x$ just by setting $\forall U \subset X$ open

$$G_x(U) = G_x(U).$$

**Defn.** A ringed space $(X, \mathcal{B}_x)$ is an algebraic set if

a) $\forall x \in X \exists \gamma \subset X$ open s.t. $(\gamma, \mathcal{B}_x)$ is isomorphic to an affine set.

b) $X$ is quasi-compact, i.e. every open covering admits a finite subcovering.

**Remark:** algebraic set = obtained by gluing finitely many affines

**Examples**

1) If $X$ affine, then $(X, \mathcal{B}_x)$ is algebraic.

   a) is trivial

   b) If $x = \bigcup U_i$, then $\rho = \rho(X \setminus U_i) = \bigcap \mathcal{V}(U_i)$

   Since $\rho(X)$ noetherian, finitely many $U_i$ are sufficient.
2) If \( X \) affine, \( \mathcal{U} \) is open. Then \( U \) is algebraic.

**NB:** \( U \) might be not affine. (exercise sheet).

**Proof of (a):** Fix \( x \in X \). We have

\[
\mathcal{I}(X \setminus \{x\}) \subset \mathcal{I}(X \setminus \mathcal{U}) \subset \mathcal{I}(X \setminus \nu f f)
\]

so we can find \( f \in \mathcal{I}(X \setminus \nu f f) \) s.t.

\[
x \in X \setminus \nu f f \subset \mathcal{U}.
\]

**Claim:** \( X \setminus \nu f f \) is isom. to affine set.

**Proof:** if \( X = \mathbb{A}^n \) then define

\[
\nu f f := \{ f \mid f \in \mathbb{k}[x_1, \ldots, x_n] \}
\]

of: \( X \setminus \nu f f \rightarrow \mathbb{A}^{n+1} \)

\[
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(\frac{1}{x_{n+1}}))
\]

Then \( \text{Im} \, \Psi = \nu f f \setminus \{x_{n+1} \rightarrow 1 \} \) affine set.

and \( \Psi \) has inverse \( \Psi: \text{Im} \, \Psi \rightarrow X \setminus \nu f f \)

\[
(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)
\]

By exercise sheet 4 we have \( \mathbb{P}(X) \supset \mathbb{P}(\text{Im} \, \Psi) \)

\[
\n \rightarrow \quad \frac{X \setminus \nu f f}{\text{Im} \, \nu f f}
\]

**Proof of (b):** exercise and finish some details.
3) Projective sets are algebraic

If $z \in \mathbb{P}^n$ then $z = 0 \leq z$

where $z_i = z \cap \{ x_i \neq 0 \}$ affine ($\mathbb{C}^n - \mathbb{A}$)

Therefore show: $G_z = \pi(S_n) \cap \mathbb{A}_z$ of exercise sheet.

§ 4 Stables and local rings

Defn: let $(X, \mathcal{O}_X)$ be algebraic, for $x \in X$ set $U_1, U_2$ open such that $x \in U_1 \cap U_2$, and $f_i \in \mathcal{O}_X(U_i)$.

The couples $(U_i, f_i)$ and $(U_2, f_2)$ are equivalent if there exists $x \in U_1 \cap U_2$, $U \subseteq x$ open set such that $f_1|_U = f_2|_U$.

The equivalence classes for this relation are called the germs of functions at $x$, the set of germs is called the stable $\mathcal{O}_{x}^x$ (or the local ring of $X$ in $x$).

Prop: $\mathcal{O}_{x,x}$ has a natural structure of local ring with maximal ideal $m_{x,x} = \{ f \in \mathcal{O}_{x,x} \mid f(x) = 0 \}$.
Proof: Define a ring structure:

If \( f \in \mathcal{C}_x, x \) represented by \( f : \mathcal{U} \to \mathcal{X} \),

\[ s : \mathcal{C}_x, x \quad : \quad j : \mathcal{Y} \to \mathcal{X} \]

then \( f + g \) is represented by

\[ \forall y \in \mathcal{Y} \quad f(\mathcal{U} y) + g(\mathcal{U} y) \] / \( f(\mathcal{U} y) \cdot g(\mathcal{U} y) \)

Ex. check that definition does not depend on choice of representant.

- The maximal ideal. We define \( \mathcal{e}_x : \mathcal{C}_x, x \to \mathcal{X} \)

\[ \quad f \quad \mapsto f(x) \]

This is a morphism of rings and \( m_x = \ker \mathcal{e}_x \).

\[ \implies m_x \text{ maximal}. \]

- The ring is local: fix \( \mathcal{U} x \in \mathcal{X} \) affine. Then we have \( \mathcal{C}_x \).

\[ \mathcal{C}_x, x = \mathcal{C}_x, x \cdot \mathcal{X} \]. Now apply:

Prop: If \( x \) is affine, \( x \in \mathcal{X} \) and \( m = I(\mathcal{X}) \in \mathcal{P}(\mathcal{X}) \). Then we have

\[ \mathcal{C}_x, x \cong \mathcal{P}(\mathcal{X}) m. \]
Proof: we have a natural map

\[ \pi: \mathcal{O}(X) \rightarrow \mathcal{O}_{x,x} \]

\[ f \mapsto \text{germ of } f \text{ in } x. \]

If \( f \in \mathcal{O}(X) \setminus m \) then the germ \( f \in \mathcal{O}_{x,x} \) is invertible

(the formal \( \frac{1}{f} \) is regular on \( X \setminus V(f) \) at \( x \))

amiv. \\
\[ \Rightarrow \]

\[ \mathcal{O}(X) \rightarrow \mathcal{O}_{x,x} \]

property \\
\[ \leftarrow \]

\[ \mathcal{O}(X)_m \]

let us check the surjectivity: \( x \)

if \( f \in \mathcal{O}_{x,x} \) represented by \( f: U \rightarrow k \)

then \( U > X \setminus V(g) \) at \( x \) for some \( g \in \mathcal{O}(X) \)

so \( f \) represented by \( f: X \setminus V(g) \rightarrow k. \)

\[ \Rightarrow \]

\[ f = \frac{a}{g} \]

with \( a \in \mathcal{O}(X), g \in \mathcal{O}(X) \setminus m \)

\[ \mathcal{O}_x(X \setminus V(g)) \]

\[ \mathcal{O}(X)_m \]

\[ \Rightarrow \]

\[ \frac{a}{g} \in \mathcal{O}(X)_m \]