Chapter IV: Dimension

§ 1: Definitions

Definition: Let $X$ be a topological space. The dimension of $X$ is the supremum of lengths of chains

$$X_0 \supsetneq X_1 \supsetneq \ldots \supsetneq X_n$$

where $X \subseteq X$ are closed irreducible sets.

Examples:

1. $X = \mathbb{R}^d$

$X_0 = \mathbb{R} \supsetneq X_1 \supsetneq \ldots \supsetneq X_n = \mathbb{R}$

$\Rightarrow \dim \mathbb{R}^d = d$.

2. $X = \mathbb{R}^2$

$X_0 = \mathbb{R} \supsetneq X_1 = \text{line} \supsetneq X_2 = \mathbb{R}^2$

$\Rightarrow \dim \mathbb{R}^2 = 2$.

3. $X = \mathbb{R}^n$, and topology $\Rightarrow \dim X = 0$ (irreducible closed sets are points)

Proposition: Let $X$ be a topological space of finite dimension.

If $Y \subseteq X$ is a subset then $\dim Y \leq \dim X$.

If $X$ is irreducible and $Y \subseteq X$ is closed then $\dim Y < \dim X$.

We call $\dim Y - \dim X$ the codimension of $Y$ in $X$. 
Proof: If \( F_0 \subseteq \ldots \subseteq F_n \) chain in \( Y \) then
\[
F_0 \subseteq \ldots \subseteq F_n \text{ chain in } X \quad \text{(indeed if } F_i = F_{i+1} \text{ then } F_i = F_i \cap Y \text{)}
\]
\[
= \dim Y \leq \dim X
\]
\[
F_{i+1} = F_{i+1} \cap Y
\]
\[
\text{If } Y \text{ closed & circd. and } F_0 \subseteq \ldots \subseteq F_{\dim Y} = Y \text{ a chain of maximal length, then}
\]
\[
F_0 \subseteq \ldots \subseteq F_{\dim Y} \subseteq X = F_{\dim Y + 1} \text{ is longer chain } \quad \Box
\]

Prop: \( X \) a topd. space st.
\[
X = \bigcup_{i=1}^n X_i \text{ with } X_i \subseteq X \text{ closed.}
\]

Then \( \dim X = \max_{i=1..n} \dim X_i \).

Def: Let \( A \) be a ring. The Krull dimension is the supremum of
lengths of chains
\[
P_0 \subseteq P_1 \subseteq \ldots \subseteq P_n
\]
where \( P_i \subseteq A \) are prime ideals. We note it \( \dim_{\text{Krull}} A \).

Ex. \( A \) a field \( \Rightarrow \dim_{\text{Krull}} A = 0. \)

\( A \) a principal ring, not a field \( \Rightarrow \dim_{\text{Krull}} A = 1. \)

Indeed if \( p \in A \) prime, then \( p \) is maximal.

Prop: \( X \) a affine and \( P(X) \) its function ring.

Then \( \dim X = \dim_{\text{Krull}} P(X) \).
Proof: $\exists$ decreasing bijection $\psi: X \mapsto \mathbb{P}(X)$ prime

Thus, let $X$ be a $k$-algebra of finite type and $k = \mathbb{P}(X)$. Then $\dim_{k} k = \delta_{k} k \sim \text{transcendence degree}$

Ex.

\[ \dim_{k} k^n = n \text{ since } \psi_{k}(X, \ldots, X_n) = n. \]

If $X$ is affine, then $\dim X$ is finite. Indeed, $X \subseteq \mathbb{A}^{n}_{k}$ finite dimensional.

Defn. An algebraic variety is an irreducible algebraic set.

Prop. Let $X$ be an algebraic variety. Then $\dim X < \infty$ and

\[ \forall \beta \neq \emptyset \text{ open set we have } \dim (\emptyset) = \dim X. \]

Proof. 1) If $F_{0} \subseteq \cdots \subseteq F_{n}$ is a maximal chain (maybe infinite)

fix $V \subseteq X$ open affine set. $V \cap F_{0} \neq \emptyset$

\[ \Rightarrow V \cap F_{0} \subseteq \cdots \subseteq V \cap F_{n} \text{ is a chain in } V \Rightarrow \dim X \leq \dim V < \infty \]

2) If $X$ affine, and $U \subseteq X$ open then $U \supseteq X \setminus V(f)$

\[ \Rightarrow \dim X \setminus V(f) \leq \dim U \leq \dim X \] But

\[ \text{Fr } \Gamma(X) = \text{Fr } \Gamma(X) \text{ f } \Rightarrow \dim_{k} \mathbb{P}(X) = \dim_{k} \mathbb{P}(X) \text{ f } \]

3) corollary $(1) + (2)$: exercise
§2. Dimension and equations

Goal: show that $V(f) \cap X$ has "typically" codimension 1.

NB: sufficient to prove for $X$ affine.

Thm (Krull's Hauptsatz)

Let $X$ be an affine variety of dim $n$, $\mathcal{R}(X)$ its function ring.
Let $f \in \mathcal{R}(X)$ be a non-invertible element.

Then $V(f)$ is an affine set s.t. all the irreducible components have dimension $n-1$.

NB: $f$ invertible $\iff$ Nullstellensatz $V(f) = \emptyset$.

Proof in the case $X = \mathbb{A}^n$:

Since $V(f) = \bigcup V(f_i)$ where $f = f_1 \cdots f_r$ is the decomposition in irreducible factors.

It suffices to prove $\dim V(f_i) = n-1$. Thus $\deg f$ irreducible.

Also $\deg x_i f > 0$ (otherwise renumerates).

The ring $R = \mathbb{C}[x_1, \ldots, x_n]/(f)$ is integral and $x_1, \ldots, x_n$ are algebraically dependent in $\mathcal{R}(R)$, since $x_1, \ldots, x_n \implies \partial_x \mathcal{R}(R) \leq n-1$. 

\[ f(x_1, \ldots, x_n) = 0 \]
let us show that $\bar{x}_1, \ldots, \bar{x}_{n-1}$ are alg. independent in $\mathbb{F}(R)$.

otherwise $\exists$ polynomial $g$ \( g(\bar{x}_1, \ldots, \bar{x}_{n-1}) = 0 \)

in $R = \mathbb{K}[X_1, \ldots, X_n]/(f)$

$\Rightarrow g \in (f)$. Impossible since $\deg x_n f > 0$. \( \Box \)

Cor: Let $X$ be an affine variety of dim $n$ and $f_1, \ldots, f_t \in \mathbb{F}(X)$.

If $w \in V_X(f_1, \ldots, f_t)$ is an irreducible, then dim $W \geq n - t$.

Proof by induction on $n$:

if $w \in V_X(f_1, \ldots, f_t)$ irreducible, then $\exists w' \in V_X(f_1, \ldots, f_{t-1})$ irreducible s.t. $w \subset w'$.

$\Rightarrow$ dim $w' \geq n - (t - 1)$ and $w = V_{w'}(\bar{f}_t)$ with $\bar{f}_t \in \mathbb{F}(w')$.

If $\bar{f}_t = 0$ then $w = w'$ ok.

If $\bar{f}_t$ invertible then $w = \emptyset$ \( \Box \)

otherwise dim $w = \text{dim } w' - 1 \geq n - t$.

\textbf{Hauptsichelsetz}

\textbf{NB:} $V(f_1, \ldots, f_t)$ is not always equi-dimensional:

$V_{\mathbb{K}}(X_1 X_2, X_1 (X_1 + X_2 + 1)) = \text{line } w \text{ point.}$

Concern: if $w \cup w'$ is proper of codim $k$ then $\exists$ $k$ linear equations $f_1, \ldots, f_k = 1$. $w = V(f_1, \ldots, f_k)$

Not true if $w$ affine (sketch 3, ex 5).
Prop Let $X$ be an affine variety and $P(X)$ be factorial.

If $Y \subset X$ is irreducible of codim 1, then $\exists f \in P(X)$ s.t. $I_X(Y) = (f)$. In particular $Y = V_X(f)$.

Proof: Since codim $Y = 1$ we have

$I_X(Y) \subset P(X)$ is prime and if $p \in I_X(Y)$ is prime

then $p = I_X(Y)$.

Choose $g \neq 0 \in I_X(Y)$ and decompose $g = \prod_i f_i^{m_i}$ into irreducible factors.

$I_X(Y)$ prime $f_i \in I_X(Y) \Rightarrow (f_i) \subset I_X(Y) \Rightarrow (f_i) = I_X(Y)$ (up to renumbering)

is prime

Example/Exercise: If $X = V(x_0x_3 - x_2^2) \subset \mathbb{A}^3$ quadric cone

then line $l = V(x_0, x_2) \subset X$

and codim $l = 1$ but $I(l)(f)$

Prop: Let $X$ be an affine variety and $Y \subset X$ a subvariety of codim $\geq 1$.

Then $\forall 1 \leq s \leq n$ there exist $f_1, \ldots, f_s \in P(X)$ s.t.

1) $Y = V(f_1, \ldots, f_s)$

2) All the irreducible pts of $V(f_1, \ldots, f_s)$ have codim $s$.

In particular $Y = V(f_1, \ldots, f_s)$ is an irreducible opt.
Proof by induction: \( n = 1 \) can be handled separately.

\( n + 1 \Rightarrow n \) if \( f_1, \ldots, f_{n+1} \in \mathcal{P}(X) \) and \( V(f_1, \ldots, f_{n+1}) = \cup \mathcal{Z}_i \)

with \( \text{codim } \mathcal{Z}_i = n + 1 \) for all \( i \)

\( \text{codim } \mathcal{W} = \text{codim } \mathcal{Z}_i = \mathcal{Z}_i \neq \mathcal{W} \Rightarrow I(\mathcal{W}) \neq \bigcup_{i=1}^{\infty} I(\mathcal{Z}_i) \)

choose \( f_{n+1} \in I(\mathcal{W}) \) s.t. \( f_{n+1} \notin I(\mathcal{Z}_i) \) for all \( i \).

Then \( \mathcal{W} \subset V(f_1, \ldots, f_n) \) and \( V(f_{n+1}) \cap \mathcal{Z}_i \neq \mathcal{Z}_i \forall i \).

Projective case

Prop: \( \forall X \subset \mathbb{P}^n \) be a projective variety and \( R = \left[ \mathcal{O}_{X, x}, \mathcal{X}_x \right] / I(X) \)

let \( f \in \mathcal{O}(X) = \mathcal{O}_{X, x} \) be homogeneous s.t. \( f \in R \) is not constant

1) All the irreducible parts of \( \mathcal{V}(f) \subset X \) have codim 1.

2) If \( \text{dim } X > 0 \) then \( \mathcal{V}(f) \neq \emptyset \).

NB 2) is not true for \( X \) affine: a curve can be disjoint from a hyperplane.

Proof: 1) reduce to affine case

2) let \( \mathcal{C}(X) \subset X^{n+1} \) be the affine cone over \( X \)

then \( \text{dim } \mathcal{C}(X) = \text{dim } X + 1 \).

and \( V(f) \subset X^{n+1} \) has codim 1.
We have \( \overline{\text{V}(f) \cap \xi'(x)} \in \mathcal{D} \) non-empty!

Hence for all \( f \in \text{Aut}(X) \)
\[ \dim \text{V}(f) \cap \xi'(x) \geq \dim X + 1 - 1 > 0. \]

\[ \Rightarrow X \cap \overline{\text{V}_p(f)} = \overline{\left( \left( \text{V}(f) \cap \xi'(x) \right) \setminus \mathcal{D} \right)} \neq \emptyset \]

§ 3 Morphisms and dimension

If \( \phi: X \to Y \) is a non-dominant morphism, then no general relation between \( \dim X \) and \( \dim Y \).

E.g., \( \phi: X \to Y \) constant always exists.

Lemma: Let \( X, Y \) be (affine) varieties and \( \phi: X \to Y \) be a dominant morphism. Then \( \dim X \geq \dim Y \).

Proof: \( \phi^* \colon \mathcal{P}(Y) \to \mathcal{P}(X) \) injective, since \( \phi \) dominant.

\[ \Rightarrow \phi^* \colon \mathcal{P}(Y) \to \mathcal{P}(X) \]
\[ \Rightarrow \exists \mathfrak{m} \mathcal{P}(Y) \leq \mathfrak{m} \mathcal{P}(X) \]

Recall: if \( \phi: X \to Y \) isomorphism, \( y \subseteq Y \) closed \( \Rightarrow \phi^{-1}(y) \) is closed.

Prop: Let \( \phi: X \to Y \) be a dominant morphism between (affine) varieties and \( y \subseteq Y \) s.t. \( \phi^*(y) \neq \emptyset \).

Then every irreducible component of \( \phi^*(y) \) has \( \dim \geq \dim X - \dim Y \).
Proof. Let $m = \dim Y$. Then $\exists f_1 \cdots f_m \in \mathcal{F}(Y)$ s.t.
\[
dim V(f_1, \ldots, f_m) = 0
\]
so all $f_i$ have $\dim f_i = \dim X - m = \dim X - \dim Y$.

Thm: Let $\phi : X \to Y$ be a dominant morphism between algebraic varieties. Then $\exists U \subseteq Y$ open s.t. $U \subseteq \phi f(X)$ and

For all $u \in U$ all the irreducible components of $\phi^{-1}(u)$ have $\dim = \dim X - \dim Y$.

Proof: book. Sheet 6: this is optimal.

Cor. (Semicontinuity of fiber dimension)

Let $\phi : X \to Y$ be a closed morphism between algebraic varieties.

For all $u \in U$ we set $Y_i = \{ y \in Y \mid \dim \phi^{-1}(y) \geq i \}$. Then $Y_i \subseteq Y$ is closed.

Example: $\dim X = 3$, $\dim Y = 2$. 

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