Chapter 2: Torsion and compatible pts

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function s.t. $f(0, \ldots, 0) = 0$.

Set $S = V(f)$, then we want to compute the image of $S$ in $\Theta$.

What is the image of $S$? Analytic point of view.

Taylor expansion:

$$f(h_1, \ldots, h_n) = \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(0) + \frac{1}{2} \sum_{i,j} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

For $h_i$ small, the first term is dominant so

$$T_{S, \Theta} = \left\{ (v_1, \ldots, v_n) \in \mathbb{R}^n \mid \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(0) = 0 \right\}$$

Today: algebraic formulation

§ 1 Torsion space

We will study $X$ in a neighbourhood of point so that $X$ affine.

Define: let $X$ be an affine set and $x \in X$. A derivation of $X$ in $x$ is an additive map $v: \mathcal{P}(X) \to k$ s.t. $v(x) = 0$ and for all $f, g \in \mathcal{P}(X)$

$$v(f \cdot g) = f(x) v(g) + g(x) v(f) \quad (\text{Leibniz rule})$$

We call $v$ a tangent vector of $X$ in $x$, and denote by $T_x X$ the set of tangent vectors.
Rem: $T_{x,x}$ is a $K$-vector space and $T_x x$ is the tangent span of $x$ in $x$.

Defn-Prop: Let $f: x \to y$ be a morphism of affine sets, $x \subseteq X$ and $y = f(x)$. Then we have an induced linear map

$$T \phi_x: T_{x,x} \longrightarrow T_{y,y},$$

$$v \longmapsto \phi^* v$$

where $\phi^*: \mathcal{P}(Y) \longrightarrow \mathcal{P}(X), f \longmapsto f \circ \phi$.

We call $T \phi_x$ the tangent map.

Proof: Let us check that $\phi^* v$ is a derivation. Let $f, g \in \mathcal{P}(Y)$

$$\phi^* v (f \cdot g) = v (\phi^* f \cdot \phi^* g) = \phi^* f (x) \cdot v(\phi^* g) + \phi^* g (x) \cdot v(\phi^* f)$$

$$= f(\phi(x)) \cdot g(\phi(x)) \cdot v(\phi^* f) + g(\phi(x)) \cdot f(\phi(x)) \cdot v(\phi^* f)$$

Rem: If $x \stackrel{f}{\longrightarrow} y \stackrel{\phi}{\longrightarrow} z$ then

$$T(\phi \circ \phi) = T \phi \circ T \phi_x$$

Examples:

$\$ Let $x = \{x_i \}^{n}_{i=1}$

4) $X = \mathbb{C}^n$, $x = 0$. Let $v: \mathcal{P}(X) \longrightarrow K$ be a derivation.

By hypothesis $v(x_i x_j) = 0 \ \forall i, j$. 

$$\$$
Thus if \( f = \sum_{i=0}^{d} a_i x_i \) decompositions in homog. opts then
\[
\hat{f}_0 = 0 \quad (f(\theta) = 0)
\]
and
\[
\nu(f) = \nu(f_1) = \nu\left( \sum_{j=1}^{n} a_j x^j \right) =
\begin{align*}
&= \sum_j \nu(a_j x^j) \\
\text{Ker} &\rightarrow \sum_j a_j \nu(x^j) + 0 \cdot \nu(a_j) \\
&= \sum_j a_j \nu(x^j).
\end{align*}
\]
Thus \( \nu \) is determined by \( \nu(x_1, \ldots, x_n) \)
\[
\Rightarrow T^*_{(x_1, \ldots, x_n)\mathfrak{o}} \cong \mathfrak{o}^n.
\]

2) \( f \in k[x_1, \ldots, x_n] \) polynomial without multiple factors
so \( \theta \in \nu(f) \). Set \( X = \nu(f) \). claim:
\[
T^*_{x_j\mathfrak{o}} = \{ (b_1, \ldots, b_n) \in \mathfrak{o}^n \mid \sum_{i=1}^{n} b_i \frac{\partial F}{\partial x_i}(\theta) = 0 \}
\]
Proof: let \( (v, P(x)) \longrightarrow \mathfrak{o} \) be a derivation
\[
k[x_1, \ldots, x_n]/(f)
\]
By Kebniz \( \nu(x_i x_j) = 0 \) \( \forall \) \( i, j \), and
\( \nu \) is determined by \( \nu(x_i) = x_i \), \( i = 1, \ldots, n \).
Moreover, we have the condition \( \nu(f) = 0 \) \( (f = 0 \in P(x)) \)
\[
\text{since} \quad f = \sum_{i=1}^{n} a_i x_i + \Theta(x^2) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\theta) x_i + \Theta(x^2)
\]
The condition \( \nu(f) = 0 \) can be rewritten as

\[
0 = \nu(f) = \nu\left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \cdot x_i \right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \cdot b_i
\]

3) Here generally if \( X = k^n \) s.t.

\[
I(X) = (f_1, \ldots, f_r)
\]

then

\[
\overline{T}_{x, \mathcal{K}} = \ker J_x (f_1, \ldots, f_r)
\]

where

\[
J_x (f_1, \ldots, f_r) = \left( \frac{\partial f_i}{\partial x_j}(x) \right)_{j=1 \ldots r} \quad \text{is the Jacobian matrix}
\]

Prop: Let \( X \) be affine, \( x \in X \) and \( m_x \in P^r(X) \) be its maximal ideal. Then we have

\[
\overline{T}_{x, \mathcal{K}} \cong \left( \frac{m_x}{m_x^2} \right)^r
\]

Rem: More generally, if \( X \) algebraic and \( m_x \in \mathcal{O}_{X, x} \), the maximal ideal, we define the fiber space by \( \left( \frac{m_x}{m_x^2} \right)^r \).

2) We have \( k \cong P(X)/m_x \), so \( m_x/m_x^2 \) is a k-vector space where the scalar multiplication is defined by

\[
k \times m_x/m_x^2 \to m_x/m_x^2 \\
(f, g) \to f \cdot g
\]

\[
f, g \in m_x^2
\]
Proof: Let \( \nu : P(X) \to k \) be a derivation

\[
\begin{array}{c}
\nu \\
\downarrow \\
\nu_{\nu_X} \\
\downarrow \\
\nu_{\nu_X} \\
\end{array}
\]

by Leibniz rule, for \( f, g \in m_X \) \( \nu(fg) = f(x)\nu(g) + g(x)\nu(f) = 0 \).

So we factorize \( \overline{\nu} : \frac{m_X}{m_X^2} \to k \)...

Vice versa, if \( \overline{\nu} \in (\frac{m_X}{m_X^2})^* \) then define

\[
\nu : P(X) \to k \\
f \mapsto \overline{\nu}(f(x))
\]

Exercise: check that this is a derivation \( \Box \)

§2.1 Singular points

Result: Nakayama's Lemma

Given \( X \subseteq \mathbb{A}^n \), affine, one can show (Problem 5 in [V]) that

\[ \forall x \in X \quad \dim T_{x,X} \geq \dim x. \]

(Any \( f \in X \) in complete intersection \( \mathbb{C}^n \))

Now study when equality.
Defn: Let $X$ be an algebraic variety and $x \in X$. We say that $X$ is smooth (or non-singular) in $x$ if $\dim X = \dim T_{x,x}$.

We say that $X$ is smooth if $X$ is smooth in all points.

Rem: If $X$ is irreducible and $x \in X$, then $x \in X_{\text{reg}}$ for some ring $O_X$.

Prop (Jacobian criterion)

Let $X \subseteq \mathbb{A}^n$ be an affine variety of dimension $d$, and $I(X) = (f_1, \ldots, f_r)$. Then we have $X$ smooth in $x$ if and only if $\dim \text{Jac}(f_1, \ldots, f_r) = n-d$.

Proof: $T_{x,x} = \ker \text{Jac}$.

Ex: (a) If $f \in \mathbb{K}[x_1, \ldots, x_n]$ is a polynomial without multiple factors and $(0,b) \in V(f)$, then $X$ is singular in $x$ if and only if

$$\frac{\partial f}{\partial x_1}(0,b) = 0 = \frac{\partial f}{\partial x_2}(0,b)$$

i.e., $f = x_2^2 - x_1^3$. Then $(0,c) \in V(f)$ and

$$\frac{\partial f}{\partial x_1} = -3x_1^2, \quad \frac{\partial f}{\partial x_2} = 2x_2$$

vanish in $(0,c)$. 

5) If \( x = (x_1, \ldots, x_n) \) without multiple factors then \( a \in U^n \) is a singular point of \( V(f) \) iff
\[
\begin{align*}
    f(a) &= 0 \\
    \frac{\partial f}{\partial x_i}(a) &= 0 \quad \forall i = 1 \ldots n
\end{align*}
\]

Proof: Let \( x = 1^n \) be a projective variety. and \( I(X) = (f_1, \ldots, f_r) \) with \( f_i \) homogeneous. Let \( x = (x_0 : \ldots : x_n) \in X \) be a point.

Set
\[
A := \left( \frac{\partial f_i}{\partial x_i} \right)_{i=0 \ldots r, \quad i = 0 \ldots n}
\]

Then \( X \) is smooth in \( x \) iff \( \text{rk} A = n - \text{dim} X \).

Proof: wlog \( x_0 = 1 \). Set \( X_b := X \setminus \{x_0 = 0\} \)

then \( I(X_b) = (f_1, \ldots, f_r) \) (cf. exercises)

and \( X \) smooth in \( x \) \( \Leftrightarrow \) \( X_b \) smooth in \( X_b \) \( \Leftrightarrow \) \( \text{rk} B = n - \text{dim} X \)

where
\[
B = \left( \frac{\partial f_i}{\partial x_i}(x) \right)_{i=0 \ldots r, \quad i = 0 \ldots n}
\]

Since \( f_j \) homogeneous we have
\[
\frac{\partial f_j}{\partial x_i}(x) = \frac{\partial f_j}{\partial x_i}(0)
\]

\( \Rightarrow \) \( B \) is obtained from \( A \) by deleting the first column

\( \Rightarrow \) \( \text{rk} B \leq \text{rk} A \).
Now prove equality: \( f_j \) is homogeneous of degree \( d_j \) so by Euler's formula
\[ \text{deg} f_j = \sum_{i=0}^{d_j} x_i \frac{\partial f_j}{\partial x_i} \]

Since \( x_0 = 1 \) and \( f_j(x) = 0 \) we obtain
\[ \frac{\partial f_j}{\partial x_0}(x) = -\sum_{i=1}^{d_j} x_i \frac{\partial f_j}{\partial x_i}(x) \]

\( \Rightarrow \) first column of \( A \) is linear combination of others \( \checkmark \)

§3 Local rings and multiplicity

Prop: Let \( R \) be a local noetherian ring, \( m \in R \) the max ideal and \( k = R/m \). Then \( \dim_k \frac{m}{m^2} \) is equal to the minimal number of generators of \( m \).

Proof: let \( \alpha \) be a gen. of \( m \).

1) \( m = (x_1, \ldots, x_d) \) then \( x_1, \ldots, x_d \) generates \( \frac{m}{m^2} \) over \( k \).

Indeed if \( x \in m \) then \( x = \sum_{i} a_i x_i \), so \( x = \sum_{i} a_i \frac{x_i}{x_0} \cdot \frac{x_i}{x_0} \in \frac{m}{m^2} \)

2) \( m = (x_1, \ldots, x_d) \) and \( x_1, \ldots, x_{d-1} \) is a base of \( \frac{m}{m^2} \) then
\( m = (x_1, \ldots, x_{d-1}) \)

Proof: we can write \( \frac{x_d}{x_0} = \sum_{i=1}^{d-1} a_i x_i \), \( \Rightarrow \) \( x_d - \sum_i a_i x_i \in m^2 \).
bed \ m^2 \text{ is generated by } X_1 X_j \text{ so }

X_c = \sum a_i X_i = \sum b_{i,j} X_i X_j

(=) \quad X_c \left( I - \sum_{l=1}^{c-1} Y_l X_l \right) = \sum_{l=1}^{c-1} a_l X_l + \sum_{l=1}^{c-1} b_{l,j} X_l X_j

\sum_{l=1}^{\infty} e_m

\sum_{l=1}^{\infty} f_m

\text{so invertible in } A. \ \\ \Box

Geometric interpretation. If } X \in x \text{ then } \dim T_{x,x} \text{ is the embedding dimension, so smallest } n \text{ s.t. (locally) } X \subseteq \mathbb{A}^n.

Corollary. } X \text{ is an algebraic variety of dim } \Lambda.

Then } x \in X \text{ is smooth iff } O_{x,x} \text{ is a principal ring.}

Proof: } x \text{ smooth } \iff \dim \frac{m_x}{m_x^2} = 1 \quad \text{Prop } \iff \text{max gen. by } \Lambda \text{ element, } + \text{ Ext. Sheet.}

Plane curves: let } (-) f \in \mathbb{C}[X_1, X_2] \text{ and }

f = \sum_{i,j} f_{i,j} \quad \text{the decom. in homog. polynomials.}

\text{Defn: The multiplicity of } f \text{ in } (0,0) \text{, denoted by } \mu_p(f) \text{ is the smallest } i \text{ s.t. } f_i \neq 0.

(If } f \text{ is smooth, multiple factors are say that } \nu(f) \text{ has multiplicity } \mu_p(f) \text{ in } (0,0)\text{.}
\[
\mu_p(f) = 0 \implies p \in \text{V}(f)
\]
\[
\mu_p(f) = 1 \implies p \in \text{V}(f) \text{ is a smooth pt.}
\]
\[
\mu_p(f) \geq 2 \implies p \in \text{V}(f) \text{ is a singular pt, so } T_v(p), v \leq 2.
\]

**Define analysis:** if \( f \) homogeneous in \( k \) variables then write
\[
\sum_{i=1}^{\ell} \alpha_i \beta_i X_i = f
\]

**Ex:** nice
\[
f = x_1^3 + x_2^3 - x_1 x_2 = 0, \quad f_2 = x_1 x_2
\]
\[
T_1 = \{ x_1 = 0 \} \quad T_2 = \{ x_2 = 0 \}
\]

**cusp**
\[
x_2^2 - x_1^3 = 0, \quad f_2 = x_2^2
\]
\[
T_1 = \{ x_2 = 0 \} \text{ with multiplicity 2}
\]