Exercise sheet 1

**Exercise 1.** Let $R$ be a ring, and let $I \subset R$ be an ideal.

a) Show that $I$ is a prime ideal if and only if $R/I$ is an integral domain.
b) Show that $I$ is a maximal ideal if and only if $R/I$ is a field.

**Exercise 2.** Let $k$ be a field and $n$ a positive integer.

a) For every $t \in T$ let $S_t \subset k^n$ be an affine set. Show that $\bigcap_{t \in T} S_t$ is an affine set.
b) Let $J_1, J_2 \subset k[X_1, \ldots, X_n]$ be ideals. Show that $V(J_1J_2) = V(J_1) \cup V(J_2)$.

In particular a finite union of affine sets is an affine set.

**Exercise 3.** Let $k$ be a field and $n$ a positive integer.

a) If $X \subset k^n$ is an affine set, then we have $V(I(X)) = X$.
b) Let $M \subseteq N$ be affine sets in $k^n$. Then we have $I(N) \subseteq I(M)$.
c) If $J \subset k[X_1, \ldots, X_n]$ is an ideal, then we have $J \subset I(V(J))$.

**Exercise 4.** Let $X$ be a non-empty topological space. Show that the following properties are equivalent :

a) If $X = F \cup G$ with $F$ and $G$ closed, then $X = F$ or $X = G$.
b) If $U, V \subset X$ are open sets such that $U \cap V = \emptyset$, then $U = \emptyset$ or $V = \emptyset$.
c) If $U \subset X$ is a non-empty open set, then $U$ is dense.

**Exercise 5.** Is the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \sin x_1\}$ an affine set?

**Exercise 6.** Let $k$ be an infinite field. We want to determine the function ring $A_i := k[X_1, X_2]/(F_i)$ for

$$F_1 = X_2 - X_1^2$$
$$F_2 = X_1X_2 - 1$$

a) Show that $A_1 \simeq k[T]$ and $A_2 \simeq k[T, T^{-1}]$. Show that $A_1$ and $A_2$ are not isomorphic.
b) Determine $A_3$ for

$$F_3 = X_1^2 + X_2^2 - 1.$$ 

Is $A_3$ isomorphic to $A_1$ or $A_2$? The answer depends on the field $k$. 

1