Exercise sheet 7

Let $k$ be an algebraically closed field.

**Exercice 1.** Suppose that $k$ has characteristic 0. Determine the singular points of the following projective varieties:

$$X = \{ [X_0 : X_1 : X_2] \in \mathbb{P}^2 \mid X_1X_2^4 + X_2X_0^4 + X_1X_0^4 = 0 \}$$

$$C = \{ [X_0 : X_1 : X_2 : X_3] \in \mathbb{P}^3 \mid X_1X_0 - X_2X_3 = X_2^2 - X_1X_3 = X_3^2 - X_2X_0 = 0 \}$$

**Exercice 2.**

a) Let $F \in k[X_0, \ldots, X_3]$ be a homogeneous polynomial of degree 3, and let $X := V(F) \subset \mathbb{P}^3$ be the hypersurface defined by $F$. Suppose that $a = [1 : 0 : 0 : 0]$ and $b = [0 : 1 : 0 : 0]$ are singular points of $X$. Show that the line

$$D = \{ [X_0 : X_1 : X_2 : X_3] \in \mathbb{P}^3 \mid X_2 = X_3 = 0 \}$$

passing through $a$ and $b$ is contained in $X$.

b) Let $F \in k[X_0, \ldots, X_2]$ be a homogeneous polynomial of degree 3, and let $X := V(F) \subset \mathbb{P}^2$ be the hypersurface defined by $F$. Suppose that $a = [1 : 0 : 0]$ and $b = [0 : 1 : 0]$ and $c = [0 : 0 : 1]$ are singular points of $X$. Show that $X$ is the union of three lines.

**Exercice 3.** Suppose that $k = \mathbb{C}$. Let $F \in k[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree 2.

a) Show that up to linear coordinate change we can suppose that $F = X_0^2 + \ldots + X_r^2$ with $0 \leq r \leq n$.

b) Show that $F$ is irreducible if and only if $r \geq 2$. Describe $V(F)$ if $r = 0$ et $r = 1$.

c) Suppose that $r \geq 2$, then we call the hypersurface $Q = V(F) \subset \mathbb{P}^n$ a quadric. Show that the singular locus of $Q$ is a linear subvariety of dimension $n - r - 1$.

**Exercice 4.**

a) Let $A$ be an integral domain that is not a ring, and let $K$ be its fraction field. Show that the following properties are equivalent:

1. $A$ is a local and principal ring.
2. $A$ is local, noetherian and the maximal ideal $m \subset A$ is principal.
3. There exists a $\pi \in A$ which is not zero, irreducible and such that

$$\forall x \in A, x \neq 0 \text{ on a } x = u\pi^n \text{ where } n \in \mathbb{N}_0, u \in A^*.$$
4. There exists a map \( v : K \rightarrow \mathbb{Z} \cup \infty \) such that:
- \( v(0) = \infty \)
- \( \forall x, y \in K \) we have \( v(x \cdot y) = v(x) + v(y) \)
- \( \forall x, y \in K \) we have \( v(x + y) \geq \inf(v(x), v(y)) \)
- \( v(K) \neq \{0, \infty\} \)
- \( A = \{x \in K \mid v(x) \geq 0\} \).

A ring with these properties is called a discrete valuation ring, \( v \) is its valuation and \( \pi \) is a uniformising parameter.

b) Let \( X \) be an algebraic curve and \( x \in X \) a smooth point. Show that the local ring \( \mathcal{O}_{X,x} \) is a discrete valuation ring. Describe the valuation \( v \) and the uniformising parameter \( \pi \).