ON A CONJECTURE OF BELTRAMETTI AND SOMMESE

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Abstract. Let $X$ be a projective manifold of dimension $n$. Beltrametti and Sommese conjectured that if $A$ is an ample divisor such that $K_X + (n-1)A$ is nef, then $K_X + (n-1)A$ has non-zero global sections. We prove a weak version of this conjecture in arbitrary dimension. In dimension three, we prove the stronger non-vanishing conjecture of Ambro, Ionescu and Kawamata and give an application to Seshadri constants.

1. Introduction

1.A. The main result. The aim of this paper is to study the following effective non-vanishing conjecture, due to Beltrametti and Sommese [BS95, Conj. 7.2.7].

1.1. Conjecture. Let $X$ be a projective manifold of dimension $n$, and let $A$ be an ample Cartier divisor such that $K_X + (n-1)A$ is nef. Then we have

$$H^0(X, \mathcal{O}_X(K_X + (n-1)A)) \neq 0.$$ 

By the classification of Fujita and Ionescu [Ion86, Fuj87] the adjoint divisor $K_X + (n-1)A$ is nef unless we are in a very special situation ($X$ is a projective space, quadric etc.), so the conjecture applies to adjoint linear systems on almost every variety. If $X$ is a surface it is an immediate consequence of the Riemann-Roch formula and classical results on surfaces, but in higher dimension the situation is much more complicated. Conjecture 1.1 and its (conjectural) generalisation due to Ambro [Amb99], Ionescu [Cet93] and Kawamata [Kaw00] have been studied by several authors during the last years [Kaw00], [CCZ05], [Xie05], [Fuk06], [Fuk07], [Bro09], [BH08]. We prove a weak version of the Beltrametti-Sommese conjecture in arbitrary dimension:

1.2. Theorem. Let $X$ be a normal, projective variety of dimension $n \geq 2$ with at most rational singularities, and let $A$ be a nef and big Cartier divisor on $X$ such that $K_X + (n-1)A$ is generically nef (cf. Definition 2.6). Then there exists a $j \in \{1, \ldots, n-1\}$ such that

$$H^0(X, \mathcal{O}_X(K_X + jA)) \neq 0.$$ 

In particular if $A$ is effective, then

$$H^0(X, \mathcal{O}_X(K_X + (n-1)A)) \neq 0.$$ 

If $X$ has irrational singularities, the statement still holds unless $(X, A)$ is birationally a scroll (cf. Definition 1.3) over a curve of positive genus.
Note that the conclusion of our theorem is a priori\(^3\) weaker than Conjecture 1.1, but it should be equally useful for applications.

1.B. The technique. Let \(X\) be a projective manifold of dimension \(n\), and let \(A\) be a nef and big Cartier divisor on \(X\). By the Kawamata-Viehweg vanishing theorem one has
\[
\chi(X, \mathcal{O}_X(K_X + tA)) = h^0(X, \mathcal{O}_X(K_X + tA)) \quad \forall \ t \in \mathbb{N},
\]
so the non-vanishing problem reduces to studying the Hilbert polynomial \(\chi(X, \mathcal{O}_X(K_X + tA))\). By Serre duality
\[
\chi(X, \mathcal{O}_X(K_X + tA)) = (-1)^n \chi(X, \mathcal{O}_X(-tA))
\]
is a polynomial of degree \(n\) in \(t\) which can be computed by the Riemann-Roch formula
\[
\chi(X, \mathcal{O}_X(-tA)) = [ch(-tA) \cdot td(T_X)]_n,
\]
where \([ \ ]_n\) denotes the component of degree \(n\) in \(A(X) \otimes \mathbb{Q}\). Using the formulae
\[
ch(-tA) = \sum_{k=0}^n \frac{(-tA)^k}{k!}
\]
for the Chern character and
\[
td(T_X) = 1 - \frac{1}{2} K_X + \frac{1}{12} (K_X^2 + c_2(X)) + \ldots + \chi(X, \mathcal{O}_X)
\]
for the Todd class of \(T_X\), we see that \(\chi(X, \mathcal{O}_X(K_X + tA))\) equals
\[
(1) \quad \frac{A^n}{n!} t^n + \frac{A^{n-1} \cdot K_X}{2(n-1)!} t^{n-1} + \frac{A^{n-2} \cdot (K_X^2 + c_2(X))}{12(n-2)!} t^{n-2} + \ldots + (-1)^n \chi(X, \mathcal{O}_X).
\]
The idea of the proof of Theorem 1.2 is now as follows: we argue by contradiction and suppose that for all \(j \in \{1, \ldots, n-1\}\) we have \(h^0(X, \mathcal{O}_X(K_X + jA)) = 0\). Thus \(1, \ldots, n-1\) are roots of the Hilbert polynomial \(\chi(X, \mathcal{O}_X(K_X + tA))\). It is an undergraduate exercise to translate the assumption into equations involving the coefficients of the Hilbert polynomial above (cf. Lemma 4.1), but it significantly simplifies the problem by reducing it to controlling the characteristic classes \(K_X, c_2(X)\) and \(\chi(X, \mathcal{O}_X)\). Somewhat surprisingly this immediately allows us to deal with the case where \(X\) is rationally connected (so \(\chi(X, \mathcal{O}_X) = 1\)). If \(X\) is a minimal model Miyaoka’s theorem tells us that the second Chern class \(c_2(X)\) is pseudoeffective which is largely sufficient to conclude. More generally if \(X\) is not uniruled we still know that \(\Omega_X\) is generically nef, so a twisted version of Miyaoka’s statement [Fuk05, Thm.2.1], [BH08, Cor.3.13] still allows us to control the second Chern class. The most delicate case is thus when \(X\) is uniruled but not rationally connected. In particular the case of birational scrolls will need some additional effort.

1.3. Definition. Let \(X\) be a normal, projective variety, and let \(A\) be a nef and big Cartier divisor on \(X\). We say that \((X, A)\) is birationally a scroll if there exists a birational morphism \(\mu : X' \rightarrow X\) from a projective manifold \(X'\) and a fibration \(\varphi : X' \rightarrow Y\) onto a projective manifold \(Y\) such that the general fibre \(F\) admits a birational morphism \(\tau : F \rightarrow \mathbb{P}^{n-m}\) and \(\mathcal{O}_F(\mu^* A) \simeq \tau^* \mathcal{O}_{\mathbb{P}^{n-m}}(1)\).

Using the foliated Mori theory due to Miyaoka and Bogomolov-McQuillan we prove the following:

\(^3\)It is an open problem due to Tsuji [MKe04] whether \(h^0(X, \mathcal{O}_X(K_X + jA)) \leq h^0(X, \mathcal{O}_X(K_X + (j + 1)A))\) holds without assuming \(A\) effective.
1.4. Theorem. Let $X$ be a normal, projective variety of dimension $n$. Let $A$ be a nef and big Cartier divisor on $X$. If $(X, A)$ is not birationally a scroll, then $\Omega_X < A >$ is generically nef.

This theorem can be seen as a foliated version of the well-known statement that if $X$ is a projective manifold and $A$ is ample, then $K_X + nA$ is nef unless $X \simeq \mathbb{P}^n$ and $A \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. Note that if $(X, A)$ is birationally a scroll, then $\Omega_X < A >$ is not generically nef even if we assume that $\det(\Omega_X < A >)$ is generically nef. Indeed for $n \geq 2$, set $X := Y \times \mathbb{P}^{n-m}$ where $Y$ is a projective manifold of dimension $1 \leq m \leq n - 1$ with nef canonical divisor. Let $A_Y$ be an ample Cartier divisor on $Y$ and $H$ be the hyperplane divisor on $\mathbb{P}^{n-m}$, then $A := p_Y^* A_Y + p_{\mathbb{P}^{n-m}}^* H$ is ample and $K_X + nA$ is nef. Nevertheless the twisted bundle $\Omega_X < A >$ is not generically nef: this would imply that $\Omega_{X/Y} < A >$ is generically nef, yet even its determinant

$$K_{X/Y} + (n - m)A = (n - m)p_Y^* A_Y - p_{\mathbb{P}^{n-m}}^* H$$

is not generically nef.

1.C. Generalisations and applications. In dimension three, the techniques developed for the proof of Theorem 1.2 give an affirmative answer for the stronger non-vanishing conjecture of Ambro, Ionescu and Kawamata which so far is only known in rather special cases.

1.5. Theorem. Let $X$ be a normal, projective threefold with at most $\mathbb{Q}$-factorial canonical singularities, and let $A$ be a nef and big Cartier divisor on $X$ such that $K_X + A$ is nef. Then we have

$$H^0(X, \mathcal{O}_X(K_X + A)) \neq 0.$$ 

Note that while $K_X$ is only supposed to be $\mathbb{Q}$-Cartier, it is crucial for our proof to suppose that $A$ is Cartier. In fact the statement is false if $A$ is a Weil divisor which is merely $\mathbb{Q}$-Cartier: Iano-Fletcher has constructed an example [IF00, Ex.16.1] of a $\mathbb{Q}$-Fano threefold $X$ of index 30 with terminal singularities such that

$$H^0(X, \mathcal{O}_X(-K_X)) = H^0(X, \mathcal{O}_X(K_X + 2(-K_X))) = 0.$$ 

As A. Broustet pointed out to me, a desingularisation of the threefold in the Iano-Fletcher example gives an example of a smooth projective threefold $X$ and $A$ a big Cartier divisor on $X$ such that $K_X + A$ is pseudoeffective but not effective. Thus our statement is almost optimal. A direct consequence of Theorem 1.5 is the following special case of Lazarsfeld’s conjecture on Seshadri constants (cf. [Bro09, Lemme 4.11]).

1.6. Theorem. Let $X$ be a normal, projective threefold with at most $\mathbb{Q}$-factorial canonical singularities, and let $A$ be a nef and big Cartier divisor on $X$ such that $K_X + A$ is nef and big. Then we have

$$\varepsilon(K_X + A, x) \geq 1$$

for every $x \in X$ sufficiently general.

In particular if the anticanonical divisor of $X$ is nef and $L$ is a nef and big Cartier divisor on $X$, then

$$\varepsilon(L, x) \geq 1$$

for every $x \in X$ sufficiently general.

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2. Notation and basic material

We work over the complex numbers, topological notions always refer to the Zariski topology. For general definitions we refer to Hartshorne’s book [Har77]. We will frequently use standard terminology and results of the minimal model program (MMP) as explained in [KM98] or [Deb01].

A variety is an integral scheme of finite type over \( \mathbb{C} \), a manifold is a smooth variety. A fibration is a proper, surjective morphism \( \varphi : X \to Y \) between normal varieties such that \( \dim X > \dim Y \) and \( \varphi_* \mathcal{O}_X \cong \mathcal{O}_Y \), that is all the fibres are connected. Fibres are always scheme-theoretic fibres. Points are always supposed to be closed.

Let \( X \) be a normal variety. The singular locus of \( X \) has codimension at least two, so we have an isomorphism \( \text{Cl}(X_{\text{nons}}) \to \text{Cl}(X) \). We define the canonical divisor \( K_X \in \text{Cl}(X) \) as the image of \( \det T_X_{\text{nons}} \).

Let \( X \) be a normal, projective variety. For every \( k \in \{0, \ldots, \dim X\} \) we denote by \( A_k(X) \) the group of \( k \)-dimensional cycles modulo rational equivalence, and by \( \text{Pic}(X) \) the group of isomorphism classes of line bundles. We denote by

\[
\text{Pic}(X)^k \times A_k(X) \to \mathbb{Z}; \quad (D_1, \ldots, D_k, [Z]) \mapsto D_1 \cdots D_k \cdot [Z]
\]

the intersection product as defined in [Ful84, Ch.2]. More generally if we consider Cartier divisors and cycles with coefficients in \( \mathbb{Q} \), we get a pairing with values in \( \mathbb{Q} \) which we often abbreviate by

\[
D_1 \cdots D_k \cdot [Z] =: D_1 \cdots D_k \cdot Z.
\]

Suppose now that \( X \) is a normal, projective variety of dimension \( n \) that is smooth in codimension two. Then we have an isomorphism \( A_n-2(X_{\text{nons}}) \to A_{n-2}(X) \), so if \( E \) is a coherent sheaf on \( X \), we define \( c_2(E) \in A_{n-2}(X) \) as the image of \( c_2(E|_{X_{\text{nons}}}) \) under this isomorphism. In particular we define the second Chern class \( c_2(X) \) as the image of \( c_2(T_{X_{\text{nons}}}) \).

We denote by \( N^1(X)_\mathbb{R} \) the vector space of \( \mathbb{R} \)-Cartier divisors modulo numerical equivalence, and by \( N_1(X)_\mathbb{R} \) its dual, the space of 1-cycles modulo numerical equivalence. A divisor class \( \alpha \in N^1(X)_\mathbb{R} \) is pseudoeffective if it is in the closure of the cone of effective divisors in \( N^1(X)_\mathbb{R} \). By [BDPP04] this is equivalent to

\[
\alpha \cdot C \geq 0
\]

for every \( C \) a member of a covering family of curves for \( X \).

Birationally, every projective manifold admits a fibration that separates the rationally connected part and the non-uniruled part: the MRC-fibration or rationally connected quotient:
2.1. Theorem. [Cam92], [GHS03], [KMM92] Let $X$ be a uniruled, projective manifold. Then there exists a projective manifold $X'$, a birational morphism $\mu : X' \to X$ and a fibration $\varphi : X' \to Y$ onto a projective manifold $Y$ such that the general fibre is rationally connected and the variety $Y$ is not uniruled.

2.2. Remarks.

a) We call $Y$ the base of the MRC-fibration. This is a slight abuse of language since the MRC-fibration is only unique up to birational equivalence of fibrations (cf. [Cam04]). Since the dimension of $Y$ does not depend on the birational model, it still makes sense to speak of the dimension of the base of the MRC-fibration.

b) If $X$ is a normal, projective variety, we define the MRC-fibration of $X$ to be the MRC-fibration of some desingularisation $X' \to X$. We say that a normal variety is rationally connected if $X'$ is rationally connected. Note that with this definition a cone over an elliptic curve is not rationally connected (it is merely rationally chain-connected).

c) The MRC-fibration is almost regular, i.e. there exist open dense sets $X_0 \subset X$ and $Y_0 \subset Y$ such that the restriction of the rational map $\varphi : X \dashrightarrow Y$ to $X_0$ gives a regular (proper) fibration $\varphi|_{X_0} : X_0 \to Y_0$. In particular we can see the general $\varphi$-fibre as a submanifold of $X$. Note also that if $Y$ has dimension one, the almost regular map $\varphi$ is regular.

2.3. Definition. [Miy87] Let $X$ be a normal, projective variety. A $\mathbb{Q}$-twisted sheaf $F<\delta>$ is an ordered pair consisting of a coherent sheaf $F$ and a numerical equivalence class $\delta \in N^1(X)_{\mathbb{Q}}$. The $\mathbb{Q}$-twisted sheaf $F<\delta>$ is torsion-free if $F$ is torsion-free. If $A$ is $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ we write $F<A>$ for the twist of $F$ by the numerical class of $A$.

In the situations we are interested in $F$ will either be a torsion-free sheaf on a normal variety or the sheaf of Kähler differentials of a normal variety.

2.4. Definition. Let $X$ be a normal, projective variety, and let $H_1, \ldots, H_{n-1}$ be a collection of ample Cartier divisors. A MR-general curve $C \subset X$ is an intersection $D_1 \cap \ldots \cap D_{n-1}$ for general $D_j \in |m_jH_j|$ where $m_j \gg 0$.

2.5. Remark. The abbreviation MR stands of course for Mehta-Ramanathan, alluding to the well-known fact [MR82] that the Harder-Narasimhan filtration of a torsion-free sheaf commutes with restriction to a MR-general curve.

Let $X$ be a normal, projective variety, and let $F$ be a coherent sheaf that is locally free in codimension one. A MR-general curve $C \subset X$ is an intersection $C \cap D_1 \cap \ldots \cap D_{n-1}$.

2.6. Definition. Let $X$ be a normal, projective variety of dimension $n$, and let $F$ be a coherent sheaf on $X$ that is locally free in codimension one. The $\mathbb{Q}$-twisted sheaf $F<\delta>$ is generically nef if its restriction to every MR-general curve $C$ is a nef $\mathbb{Q}$-vector bundle in the sense of [Laz04, Defn. 6.2.3], i.e.

$$c_1(O_{\mathbb{P}(FC)}(1)) + \pi^*\delta$$
is a nef class in $N^1(\mathbb{P}(\mathcal{F}|_C))_{\mathbb{R}}$, where $\mathcal{O}_{\mathbb{P}(\mathcal{F}|_C)}(1)$ is the tautological line bundle on the projectivised vector bundle $\mathbb{P}(\mathcal{F}|_C)$.

A $\mathbb{Q}$-divisor $D$ on $X$ is generically nef if
$$D \cdot H_1 \cdots H_{n-1} \geq 0$$
for any collection of ample Cartier divisors $H_1, \ldots, H_{n-1}$.

2.7. Remarks. a) An effective $\mathbb{Q}$-divisor $D$ is generically nef.
b) A $\mathbb{Q}$-divisor $D$ on $X$ is generically nef if for $m \gg 0$ sufficiently divisible the reflexive sheaf $\mathcal{O}_X(mD)$ is generically nef.
c) If $D$ is a pseudoeffective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, it is generically nef.

For lack of reference we collect some basic properties of generically nef sheaves. The proof is elementary and left to the reader.

2.8. Lemma.

a) Let $X$ be a normal, projective variety of dimension $n$, and let $\mathcal{F}$ be a coherent sheaf on $X$ that is locally free in codimension one. The $\mathbb{Q}$-twisted sheaf $\mathcal{F}^\langle \delta \rangle$ is generically nef if and only if its bidual $\mathcal{F}^* \langle \delta \rangle$ is generically nef.
b) A $\mathbb{Q}$-divisor $D$ on a normal, projective variety $X$ is generically nef if and only if
$$D \cdot H_1 \cdots H_{n-1} \geq 0$$
for any collection of nef Cartier divisors $H_1, \ldots, H_{n-1}$.
c) Let $\mu : X' \to X$ be a birational morphism between normal varieties, and let $A$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Let $C \subset X$ be a MR-general curve, then
$$(K_{X'} + \mu^* A) \cdot C = (K_X + A) \cdot C.$$
If $K_X + A$ is not generically nef, then $K_{X'} + \mu^* A$ is not generically nef.

Let $X$ be a projective variety that is smooth in codimension two, and let $D$ be a $\mathbb{Q}$-divisor on $X$. Let $S \subset X$ be a surface that is a complete intersection of general very ample divisors. Then $S$ is not contained $\text{Supp}(D)$, so the restriction $D|_S$ is well-defined. Moreover $S$ is smooth, so $D|_S$ is $\mathbb{Q}$-Cartier and the following definition makes sense.

2.9. Definition. Let $X$ be a normal, projective variety of dimension $n$ that is smooth in codimension one and let $D$ be a $\mathbb{Q}$-divisor on $X$. We say that $D$ is nef in codimension one if for every collection $H_1, \ldots, H_{n-2}$ of ample Cartier divisors and $S \subset X$ a complete intersection $D_1 \cap \ldots \cap D_{n-2}$ of general $D_j \in |m_jH_j|$ where $m_j \gg 0$, the restriction $D|_S$ is nef.

2.10. Lemma. (Miyaoka) Let $X$ be a normal, projective variety of dimension $n \geq 2$ that is smooth in codimension two. Let $E$ be a reflexive sheaf over $X$ such that $\det E$ is $\mathbb{Q}$-Cartier, and $\delta$ a numerical equivalence class in $N^1(X)_{\mathbb{Q}}$. If $E^\langle \delta \rangle$ is generically nef and $c_1(E^\langle \delta \rangle)$ is nef in codimension one, then
$$H_1 \cdots H_{n-2} \cdot c_2(E^\langle \delta \rangle) \geq 0,$$
where $H_1, \ldots, H_{n-2}$ is a collection of ample Cartier divisors on $X$.\footnote{A MR-general curve does not meet the image of the exceptional locus, so we can consider it also as a curve in $X'$.}
Recall that the isomorphism $A_{n-2}(X_{\text{non}}) \rightarrow A_{n-2}(X)$ allows to define the second Chern class $c_2(E<\delta>)$.

**Proof.** By linearity of the intersection form it is sufficient to prove that if $S$ is a complete intersection cut out by general elements $D_j \in |m_jH_j|$ for $m_j \gg 0$, then

$$D_1 \cdots D_{n-2} \cdot c_2(E<\delta>) = c_2(E<\delta>|_S) \geq 0.$$ 

Since $X$ is smooth in codimension two, the restriction $E|_S$ is a vector bundle. Moreover $E<\delta>|_S$ is generically nef and $c_1(E<\delta>|_S)$ is nef. We conclude with [LM97, Thm. 8].

2.11. **Corollary.** Let $X$ be a normal, projective variety of dimension $n \geq 2$ that is smooth in codimension two. Let $D$ be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ such that $\Omega_X<\frac{1}{n}D>$ is generically nef and $K_X + D$ is nef in codimension one. Then we have

$$H_1 \cdots H_{n-2} \cdot c_2(X) \geq -H_1 \cdots H_{n-2} \cdot (\frac{n-1}{n}K_X \cdot D + \frac{n-1}{2n}D^2),$$

where $H_1, \ldots, H_{n-2}$ is any collection of nef Cartier divisors on $X$.

We recall that the usual formulas for tensor products of vector bundles extends to $\mathbb{Q}$-vector bundles [Laz04, Ch.6.2, Ch.8.1]: let $X$ be a normal, projective variety that is smooth in codimension two, and let $E$ be a coherent sheaf of rank $r$ over $X$. If $\delta \in N^1(X)_{\mathbb{Q}}$ is a numerical class, then

(2) $$c_1(E<\delta>) = c_1(E) + r\delta$$

(3) $$c_2(E<\delta>) = c_2(E) + (r-1)c_1(E) \cdot \delta + \frac{r(r-1)}{2}\delta^2.$$ 

**Proof of Corollary 2.11.** By the linearity of the intersection form, it is sufficient to show the statement in the case where the Cartier divisors $H_i$ are ample. Moreover by Lemma 2.8,a) the sheaf $\Omega_X<\frac{1}{n}D>$ is generically nef if and only its bidual $\Omega_X^{*n}<\frac{1}{n}D>$ is generically nef. Since $X$ is smooth in codimension two, we have $c_2(\Omega_X) = c_2(\Omega_X^*)$. Therefore Lemma 2.10 applies and yields

$$H_1 \cdots H_{n-2} \cdot c_2(\Omega_X<\frac{1}{n}D>) \geq 0.$$ 

Since by Formula (3)

$$c_2(\Omega_X<\frac{1}{n}D>) = c_2(X) + \frac{n-1}{n}K_X \cdot D + \frac{n-1}{2n}D^2,$$

we get

$$H_1 \cdots H_{n-2} \cdot c_2(X) \geq -H_1 \cdots H_{n-2} \cdot \left(\frac{n-1}{n}K_X \cdot D + \frac{n-1}{2n}D^2\right).$$

2.12. **Lemma.** Let $X$ be a normal, projective variety of dimension $n$, and let $A$ be a Cartier divisor on $X$. Let $\nu : X' \rightarrow X$ be a desingularisation. Then for all $j \in \mathbb{Z}$ we have an inclusion:

$$H^0(X', \mathcal{O}_{X'}(K_{X'} + j\nu^*A)) \subseteq H^0(X, \mathcal{O}_X(K_X + jA)).$$
Proof. Since $A$ is Cartier, the projection formula yields
\[ \nu_* \mathcal{O}_{X'}(K_{X'} + j\nu^* A) \cong \nu_* \mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_X(jA). \]

Note that since $\nu_* \mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_X(jA)$ is torsion-free we have an inclusion
\[ \nu_* \mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_X(jA) \hookrightarrow (\nu_* \mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_X(jA))^\vee. \]

Moreover for any reflexive sheaf $\mathcal{F}$ on a normal variety we have
\[ j_* (\mathcal{F}|_{X_{\text{nons}}}) \cong \mathcal{F}, \]
where $j : X_{\text{nons}} \hookrightarrow X$ is the inclusion. Thus $(\nu_* \mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_X(jA))^\vee$ and $\mathcal{O}_X(K_X + jA)$ are isomorphic since they coincide on $X_{\text{nons}}$ and we get an inclusion
\[ \nu_* \mathcal{O}_{X'}(K_{X'} + j\nu^* A) \hookrightarrow \mathcal{O}_X(K_X + jA). \]

\[ \square \]

2.13. Proposition. Let $X$ be a normal, projective variety of dimension $n$, and let $A$ be a nef and big Cartier divisor on $X$. Then the following holds:

a) There exists $a j \in \{1, \ldots, n+1\}$ such that $H^0(X, \mathcal{O}_X(K_X + jA)) \neq 0$. In particular the divisor $K_X + (n+1)A$ is generically nef.

b) If $(K_X + nA) \cdot A^{n-1} \geq 0$, there exists a $j \in \{1, \ldots, n\}$ such that $H^0(X, \mathcal{O}_X(K_X + jA)) \neq 0$.

c) If $(K_X + nA) \cdot A^{n-1} < 0$, there exists a birational morphism $\tau : X \rightarrow \mathbb{P}^n$ such that $\mathcal{O}_X(A) \cong \tau^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Proof. By Lemma 2.12 statement a) follows from [Laz04, Prop.9.4.23].

b) $\Rightarrow$ c) Let $\nu : X' \rightarrow X$ be a desingularisation. We have $\nu_* K_{X'} = K_X$, so by the projection formula
\[ (K_X + nA) \cdot A^{n-1} = (K_{X'} + n\nu^* A) \cdot \nu^* A^{n-1} \]
Thus the condition lifts to $X'$ and we conclude by Lemma 2.12 and [Fuj89, Thm.2.2]. \[ \square \]

The following basic fact is well-known to experts. For the convenience of the reader we include a proof.

2.14. Lemma. Let $\varphi : X \rightarrow Y$ be a fibration between projective manifolds $X$ and $Y$, and let $A$ nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Suppose that for a general fibre $F$ one has
\[ H^0(F, \mathcal{O}_F(D)) \neq 0, \]
where $D$ is a Cartier divisor on $F$ such that $D \sim_{\mathbb{Q}} K_F + A|_F$. Then $K_{X/Y} + A$ is pseudoeffective.

Proof. It is sufficient to show that $m(K_{X/Y} + A)$ is pseudoeffective for $m \gg 0$ sufficiently divisible so we choose $m \in \mathbb{N}$ such that $mA$ is Cartier and $H^0(F, \mathcal{O}_F(mK_F + mA|_F)) \neq 0$. Hence the direct image sheaf $\varphi_* (\mathcal{O}_X(m(K_{X/Y} + A)))$ is not zero. The Cartier divisor $mA$ is nef and big, so it follows from [Vie95, Ch.2], [Cam04, Thm.4.13], [BP08, Thm.0.2] that $\varphi_* (\mathcal{O}_X(m(K_{X/Y} + A)))$ is weakly positive in the sense of Viehweg. Since $\mathcal{O}_X(m(K_{X/Y} + A))$ has rank one, the canonical morphism
\[ \varphi^* \varphi_* (\mathcal{O}_X(m(K_{X/Y} + A))) \rightarrow \mathcal{O}_X(m(K_{X/Y} + A)) \]
is generically surjective, so $\mathcal{O}_X(m(K_{X/Y} + A))$ is also weakly positive. Thus the divisor $m(K_{X/Y} + A)$ is pseudoeffective. \[ \square \]

If $A$ is a Cartier divisor we can combine Lemma 2.14 and Proposition 2.13 to obtain:
2.15. Proposition. Let $X$ be a projective manifold of dimension $n$. Let $\mu : X' \to X$ and $\varphi : X' \to Y$ be a model of the MRC-fibration (cf. Theorem 2.1), and denote by $m$ the dimension of $Y$. Let $A$ be a nef and big Cartier divisor on $X$. Then
\[ K_{X'/Y} + (n - m + 1)\mu^* A \]
is pseudoeffective. If
\[ K_{X'/Y} + (n - m)\mu^* A \]
is not pseudoeffective, the general $\varphi$-fibre $F$ admits a birational morphism $\tau : F \to \mathbb{P}^{n-m}$ such that $\mathcal{O}_F(A) \simeq \tau^* \mathcal{O}_{\mathbb{P}^{n-m}}(1)$.

In particular $K_X + (n - m + 1)A$ is pseudoeffective. If $K_X + (n - m)A$ is not pseudoeffective, the manifold $F$ admits a birational morphism $\tau : F \to \mathbb{P}^{n-m}$ such that $\mathcal{O}_F(A) \simeq \tau^* \mathcal{O}_{\mathbb{P}^{n-m}}(1)$.

The second statement of the proposition is a consequence of the first and the fundamental result due to Boucksom, Demailly, Păun and Peternell [BDPP04, Cor.0.3] on the pseudoeffectiveness of the canonical bundle of a non-uniruled, projective manifold.

3. The cotangent sheaf of uniruled varieties

Theorem 1.4 will be a consequence of the following statement.

3.1. Theorem. Let $X$ be a normal, projective variety of dimension $n$. Let $A$ be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Then the $\mathbb{Q}$-twisted sheaf $\Omega_X < A >$ is generically nef (cf. Definition 2.6) unless there exists a birational morphism $\mu : X' \to X$ from a projective manifold $X'$ and a fibration $\varphi : X' \to Y$ onto a projective manifold $Y$ of dimension $m < n$ such that the general fibre $F$ is rationally connected and
\[ H^0(F, \mathcal{O}_F(D)) = 0 \]
where $D$ is any Cartier divisor on $F$ such that $D \sim K_F + j\mu^* A|_F$ with $j \in [0, n-m] \cap \mathbb{Q}$.

Let $\varphi : X \to Y$ be a fibration between projective manifolds, and let $\Omega_X \to \Omega_{X/Y} \to 0$ be the canonical map between the sheaves of Kähler differentials. We define the relative tangent sheaf $T_{X/Y}$ to be the saturation of
\[ \Omega_X^{*} = T_X \]
in $T_X$, and det $T_{X/Y}$ the divisor corresponding to its determinant. The main difficulty of the proof is that in general the relative canonical bundle of a fibration does not coincide with the dual of det $T_{X/Y}$. We overcome this difficulty by making an appropriate base change.

Proof of Theorem 3.1. Let us assume that $\Omega_X < A >$ is not generically nef. We fix $L_1, \ldots, L_{n-1}$ ample Cartier divisors on $X$ such that $\Omega_X < A >$ is not generically nef with respect to $L_1, \ldots, L_{n-1}$. Let
\[ C = D_1 \cap \ldots \cap D_{n-1} \]
be a MR-general curve where $D_i \in |m_i L_i|$ general and $m_i \gg 0$ such that $\Omega_X < A >|_C$ is not nef. If $\mathcal{F} < A >$ is a non-zero torsion-free $\mathbb{Q}$-twisted sheaf on $X$, we define the slope
\[ \mu(\mathcal{F} < A >) := \frac{c_1(\mathcal{F} < A >|_C)}{\text{rk} \mathcal{F}}. \]
By Equation (2) one has

$$c_1(F^{<A>|c|}) = c_1(F|c|) + A \cdot C.$$  

By definition the $\mathbb{Q}$-twisted sheaf $F^{<A>}$ is semistable if for every non-zero torsion-free subsheaf $E \subset F$, we have $\mu(E^{<A>}) \leq \mu(F^{<A>})$.

Denote by $T_X := \Omega_X$ the tangent sheaf of $X$, and let

$$0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_r = T_X$$

be the Harder-Narasimhan filtration of $T_X$ with respect to $L_1, \ldots, L_{n-1}$. Then for $i = 1, \ldots, r$, the graded pieces $G_i := F_i/F_{i-1}$ are semistable torsion-free sheaves and if $\mu(G_i)$ denotes the slope, we have a strictly decreasing sequence

$$\mu(G_1) > \mu(G_2) > \ldots > \mu(G_r).$$

Since twisting with a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor does not change the stability properties of a torsion-free sheaf, the Harder-Narasimhan filtration of $T_{X^{<-A>}}$ is

$$0 = F_0^{<-A>} \subseteq F_1^{<-A>} \subseteq \ldots \subseteq F_r^{<-A>} = T_X^{<-A>},$$

with graded pieces $G_i^{<-A>}$ and slopes

$$\mu(G_i^{<-A>}) = \mu(G_i) - A \cdot C.$$\n
We claim that

$$(4) \quad \mu(G_1^{<-A>}) = \mu(F_1^{<-A>}) > 0.$$\n
Otherwise the slopes of all the graded pieces $G_i^{<-A>}$ are non-positive. By the Mehta-Ramanathan theorem [MR82, Thm.6.1] the Harder-Narasimhan filtration commutes with restriction to $C$, so the $\mathbb{Q}$-twisted vector bundles $G_i^{<-A>|C}$ are semistable of non-positive slope, hence antinef. Thus $\Omega_X^{<A>|C}$ is an extension of nef $\mathbb{Q}$-vector bundles, hence nef. This contradicts our hypothesis.

The $\mathbb{Q}$-Cartier divisor $A$ being nef $\mu(F_1^{<-A>}) > 0$ implies $\mu(F_1) > 0$, so $F_1|C$ is ample. We know by standard arguments in stability theory [MP97, p.61ff] that $F_1$ is integrable, moreover the MR-general curve $C$ does not meet the singular locus of the foliation by Remark 2.5. Thus we can apply the Bogomolov-McQuillan theorem [BM01, Thm.0.1], [KSCT07, Thm.1] to see that the closure of a $F_1$-leaf through a generic point of $C$ is algebraic and rationally connected. Since $C$ moves in a covering family the generic $F_1$-leaves are algebraic with rationally connected closure. If $C(X)$ denotes the Chow scheme of $X$, we get a rational map $X \dashrightarrow C(X)$ that sends a general point $x$ to the closure of the unique leaf through $x$. Let $Y$ be a desingularisation of the closure of the image, and let $X'$ be a desingularisation of the universal family over $Y$. By construction the natural map $\mu : X' \to X$ is birational and the general fibres of the fibration $\varphi : X' \to Y$ map onto the closure of general $F_1$-leaves.

By Remark 2.5 the MR-general curve $C$ does not meet the exceptional locus of $\mu$, so we can see it as a curve in $X'$. Denote by $X_C$ the normalisation of the fibre product $X' \times_Y C \subset X' \times C$, and let $p_X : X_C \to X$ the projection on the first factor. The fibration $X' \times_Y C \to C$ admits a natural section

$$C \to X' \times_Y C \subset X' \times C, \quad c \mapsto (c,c),$$

by the universal property of the normalisation we get a section of $p_C : X_C \to C$ which we denote by $s : C \to X_C$. By [KSCT07, Rem.19] the normal variety $X_C$ is smooth in an analytic neighbourhood $U \subset X_C$ of $s(C)$ and

$$T_{X_C/C|U} \simeq (p_X^* \mu^* F_1)|U.$$
In particular by the inequality (4), one has 
\[(\det T_{X_C/C} - (n-m)p_X^*A) \cdot s(C) = (\det F_1 - (n-m)A) \cdot C = (n-m)\mu(F_1 < -A>) > 0.\]

Since \(s(C)\) is a section of the fibration it does not meet any multiple fibre components, so \(-K_{X_C/C}\) and \(\det T_{X_C/C}\) coincide in a neighbourhood of \(s(C)\). Thus 
\[\tag{5} (-K_{X_C/C} - (n-m)p_X^*A) \cdot s(C) = (\det T_{X_C/C} - (n-m)p_X^*A) \cdot s(C) > 0.\]

Since \(X_C\) is smooth in a neighbourhood of \(s(C)\), we can replace \(X_C\) by a desingularisation without changing the inequality (5). We will now argue by contradiction and suppose that there exists a Cartier divisor \(D\) on a general \(\varphi\)-fibre \(F\) such that \(D \sim_Q K_F + j\mu^*A\) for some \(j \in [0, n-m] \cap \mathbb{Q}\) and 
\[H^0(F, \mathcal{O}_F(D)) \neq 0.\]

Since the general \(\mu_C\)-fibre is a general \(\varphi\)-fibre this implies by Lemma 2.14 that \(K_{X_C/C} + jp_X^*\mu^*A\) is pseudoeffective. Since \(s(C)\) is a section, its normal bundle is isomorphic to \(T_{X_C/C|s(C)} \simeq F_1|C\) which is ample. This implies by [Laz04, Cor.8.4.3] that \(E \cdot s(C) \geq 0\) for every effective divisor \(E \subset X_C\), hence 
\[(K_{X_C/C} + (n-m)p_X^*A) \cdot s(C) \geq (K_{X_C/C} + jp_X^*\mu^*A) \cdot s(C) \geq 0.\]

This contradicts the inequality (5). \(\square\)

**Proof of Theorem 1.4.** Suppose that \(\Omega_X < A>\) is not generically nef. Applying Theorem 3.1 yields a birational morphism \(\mu : X' \to X\) from a projective manifold \(X'\) and a fibration \(\varphi : X' \to Y\) onto a projective manifold \(Y\) of dimension \(m\) such that the general fibre \(F\) satisfies 
\[H^0(F, \mathcal{O}_F(K_F + j\mu^*A)) = 0 \quad \forall j \in \{1, ..., n-m\}.\]

It follows from Prop.2.13.b) and Prop.2.13.c) that \((X, A)\) is birationally a scroll. \(\square\)

In Section 4 we use the following technical lemma:

**3.2. Lemma.** In the situation of the proof of Theorem 3.1, suppose that \(A\) is a Cartier divisor. Suppose moreover that \(\Omega_X < -A>\) is not generically nef, i.e. 
\[\mu(\mathcal{G}_1 < -A>) > 0.\]

Denote by \(l \in \mathbb{N}\) the maximal number such that 
\[\mu(\mathcal{G}_i < -A>) > 0 \quad \forall i \in \{1, ..., l\}.\]

Then the following holds:

a) For every \(i \in \{1, ..., l\}\) we have 
\[\mu(\mathcal{F}_i < -\frac{\text{rk}\mathcal{F}_i + 1}{\text{rk}\mathcal{G}_i} A>) \leq 0.\]

b) There exists a sequence of rational numbers \(w_1, ..., w_l\) such that 
\[w_i \in [\text{rk}\mathcal{G}_i, \text{rk}\mathcal{G}_i + 1] \quad \forall i \in \{1, ..., l\}\]

and 
\[\sum_{i=1}^l w_i = \left(\sum_{i=1}^l \text{rk}\mathcal{G}_i\right) + 1\]

and 
\[\mu(\mathcal{G}_i < -\frac{w_i}{\text{rk}\mathcal{G}_i} A>) \leq 0 \quad \forall i \in \{1, ..., l\}.\]
Proof. For statement a) we argue as in the proof of Theorem 3.1: For \( i \in \{1, \ldots, l\} \) the saturated subsheaf \( F_i \subset T_X \) is integrable and we get a birational morphism \( \mu_i : X_i \rightarrow X \) and a fibration \( \varphi_i : X_i \rightarrow Y_i \) such that \( F_i \) corresponds to the relative tangent sheaf of \( \varphi_i \). We argue by contradiction and suppose that
\[
\mu(F_i < -\frac{\text{rk}F_i + 1}{\text{rk}F_i} A>) > 0.
\]
As in the proof of Theorem 3.1 we see that the general \( \varphi_i \)-fibre \( F_i \) satisfies
\[
H^0(F_i, O_F(K_{F_i} + j\mu^* A)) = 0 \quad \forall j \in \{1, \ldots, \text{rk}F_i + 1\}.
\]
This contradicts Proposition 2.13, a).

If \( l = 1 \) the statement b) is an immediate consequence of a): just take \( w_1 = \text{rk}G_1 + 1 \).

Suppose now that \( l > 1 \) and denote by \( C \) the MR-general curve we used in the proof of Theorem 3.1 to compute the slopes. By statement a) we have
\[
[c_1(F_i) - (\text{rk}F_i + 1)A] \cdot C \leq 0 \quad \forall i \in \{1, \ldots, l\}.
\]
Since the sheaves \( G_d \) are the graded pieces of the filtration \( F^* \) this implies that for all \( i \in \{1, \ldots, l\} \) we have
\[
(*) \quad \left[ \sum_{d=1}^{i} c_1(G_{d}) - ((\sum_{d=1}^{i} \text{rk}G_{d}) + 1)A \right] \cdot C \leq 0.
\]
We will now construct a sequence \( w_i \) inductively by using the inequalities (*)).

Start of the induction \( i = 1 \). By hypothesis have
\[
(c_1(G_1) - \text{rk}G_1 A) \cdot C > 0
\]
and since \( G_1 = F_1 \) by (*)
\[
[c_1(G_1) - (\text{rk}G_1 + 1)A] \cdot C \leq 0.
\]
We define \( w_1 \) to be the unique rational number such that
\[
(c_1(G_1) - w_1 A) \cdot C = 0,
\]
i.e. the slope of \( G_1 < -\frac{w_1}{\text{rk}G_1} A > \) equals zero.

Induction step \( i - 1 \rightarrow i \). We have constructed so far \( w_1, \ldots, w_{i-1} \) such that
\[
w_d \in [\text{rk}G_d, \text{rk}G_d + 1] \quad \forall d \in \{1, \ldots, i-1\}
\]
and\(^3\)
\[
(**) \quad \sum_{d=1}^{i-1} w_d \leq \left( \sum_{d=1}^{i-1} \text{rk}G_{d} \right) + 1
\]
and
\[
(c_1(G_d) - w_d A) \cdot C = 0 \quad \forall d \in \{1, \ldots, i-1\}.
\]
Plugging these equalities in the inequality (*) we obtain
\[
\left[ c_1(G_i) - \left( \text{rk}G_i + \left( \sum_{d=1}^{i-1} \text{rk}G_{d} \right) + 1 - \left( \sum_{d=1}^{i-1} w_d \right) \right) A \right] \cdot C \leq 0.
\]
By (**) we have \( (\sum_{d=1}^{i-1} \text{rk}G_{d}) + 1 - (\sum_{d=1}^{i-1} w_d) \geq 0 \) and by hypothesis
\[
(c_1(G_i) - \text{rk}G_i A) \cdot C > 0
\]
\(^3\)For \( i = 2 \) the inequality (**) is just \( w_1 \in [\text{rk}G_1, \text{rk}G_1 + 1] \), for \( i > 2 \) this will be established at the end of the preceding induction step.
If \( i < l \) we define \( w_i \) to be the unique rational number such that
\[
(c_i(G_i) - w_iA) \cdot C = 0,
\]
i.e. the slope of \( G_i < \frac{-w_i}{\text{rk} G_i} \) equals zero. Since we have
\[
w_i \in [\text{rk} G_i, \text{rk} G_i + (\sum_{d=1}^{i-1} \text{rk} G_d) + 1 - (\sum_{d=1}^{i-1} w_d)]
\]
we see immediately that \( (**_{i+1}) \) holds, so the induction can continue.
If \( i = l \) we set
\[
w_l := \text{rk} G_l + (l - 1) \sum_{d=1}^{l-1} \text{rk} G_d + 1 - (\sum_{d=1}^{l-1} w_d),
\]
so we have \( \sum_{i=1}^{l} w_i = (\sum_{i=1}^{l} \text{rk} G_i) + 1. \)

\[\Box\]

4. The Beltrametti-Sommese conjecture

The following lemma is the technical cornerstone of our approach.

4.1. Lemma. Let \( X \) be a projective manifold, and let \( A \) be a Cartier divisor on \( X \). Suppose that \( 1, \ldots, n-1 \) are roots of the Hilbert polynomial \( \chi(X, \mathcal{O}_X(K_X + tA)) \). Then one has
\[
\chi(X, \mathcal{O}_X) + \frac{1}{2} A^{n-1} \cdot (K_X + (n-1)A) = 0
\]
and
\[
A^{n-2} \cdot [2(K_X^2 + c_2(X)) + 6nA \cdot K_X + (n+1)(3n-2)A^2] = 0.
\]

Remark. For \( n = 2 \) the left hand side of Equation (6) and (7) are (multiples of) the Riemann-Roch formula for \( \chi(X, \mathcal{O}_X(K_X + A)) \). This corresponds well with the origin of the Beltrametti-Sommese conjecture [BS95, Ch. 7.2]: the linear system \( K_X + (n-1)A \) should behave as an adjoint linear system on a surface.

Proof. By hypothesis
\[
\chi(X, \mathcal{O}_X(K_X + tA)) = \frac{A^n}{n!} (t - a) \prod_{j=1}^{n-1} (t - j),
\]
where \( a \) is a parameter. Since
\[
\prod_{j=1}^{n-1} (t - j) = t^{n-1} - (\sum_{j=1}^{n-1} j)t^{n-2} + (\sum_{j,k=1}^{n-1} jk)t^{n-3} - \ldots + (-1)^{n-1}(n-1)!,
\]
we have
\[
(t - a) \prod_{j=1}^{n-1} (t - j) = t^n - (a + \sum_{j=1}^{n-1} j)t^{n-1} + (\sum_{j,k=1}^{n-1} jk + a \sum_{j=1}^{n-1} j)t^{n-2} - \ldots + (-1)^{n}a(n-1)!,
\]
Comparing coefficients with Riemann-Roch formula (1), we get

\[
\frac{A^{n-1} \cdot K_X}{2(n-1)!} = -(a + \sum_{j=1}^{n-1} j) \frac{A^n}{n!},
\]

\[
\frac{A^{n-2} \cdot (K_X^2 + c_2(X))}{12(n-2)!} = (\sum_{j,k=1 \atop j<k}^{n-1} jk + a \sum_{j=1}^{n-1} j) \frac{A^n}{n!},
\]

\[
\chi(X, O_X) = \frac{A^n}{n}.
\]

The statement follows by plugging these expressions into the Equations (6) and (7) and using the elementary formula

\[
\sum_{j,k=1 \atop j<k}^{n-1} jk = \frac{1}{24}(n-2)(n-1)n(3n-1).
\]

\[\square\]

**Proof of Theorem 1.2.** We argue by contradiction and suppose that \(H^0(X, O_X(K_X + jA)) = 0\) for all \(j \in \{1, \ldots, n-1\}\). Let \(\nu : X' \to X\) be a resolution of singularities, then by Lemma 2.12 one has

\[
H^0(X', O_{X'}(K_{X'} + j\nu^*A)) = 0 \quad \forall j \in \{1, \ldots, n-1\}.
\]

Since \(\nu^*A\) is nef and big, the Kawamata-Viehweg theorem implies that

\[
\chi(X', O_{X'}(K_{X'} + j\nu^*A)) = h^0(X', O_{X'}(K_{X'} + j\nu^*A)) = 0 \quad \forall j \in \{1, \ldots, n-1\},
\]

in particular Lemma 4.1 applies. Let \(Y\) be the base of the MRC-fibration of \(X'\).

**Case I:** \(\dim Y = 0\).

Since \(A\) is nef and \(K_X + (n-1)A\) is generically nef, one has

(8) \((K_{X'} + (n-1)\nu^*A) \cdot (\nu^*A)^{n-1} = (K_X + (n-1)A) \cdot A^{n-1} \geq 0\).

By Lemma 4.1 we have

\[
\chi(X', O_{X'}) + \frac{1}{2}(K_{X'} + (n-1)\nu^*A) \cdot (\nu^*A)^{n-1} = 0.
\]

Yet \(X'\) is rationally connected, so \(\chi(X', O_{X'}) = 1\). This contradicts the inequality (8).

**Case II:** \(\dim Y = 1\).

Since the base of the MRC-fibration has dimension one, we have a morphism \(\varphi : X' \to Y\) onto a smooth curve of genus at least one (cf. Remark 2.2). By Proposition 2.15 the divisor \(K_{X'} + (n-1)\nu^*A\) is generically nef unless \((X', \nu^*A)\) (and hence \((X, A)\)) is birationally a scroll with base \(Y\).

a) If \((X, A)\) is not birationally a scroll with base \(Y\), denote by \(F'\) a general \(\varphi\)-fibre, then by Proposition 2.13 there exists a \(j \in \{1, \ldots, n-1\}\) such that \(H^0(F', O_{F'}(K_{F'} + j\nu^*A)) \neq 0\). In particular the direct image sheaf \(\varphi_* O_{X'}(K_{X'/Y} + j\nu^*A)\) is not zero and an ample vector bundle by [Vie01, Cor.3.7]. Thus

\[
h^0(X', O_{X'}(K_{X'} + j\nu^*A)) = h^0(Y, \varphi_* O_{X'}(K_{X'/Y} + j\nu^*A)) \geq \chi(Y, O_Y(K_Y) \otimes \varphi_* O_{X'}(K_{X'/Y} + j\nu^*A)) > 0
\]

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by an easy Riemann-Roch computation for vector bundles on curves.

b) If \((X, A)\) is birationally a scroll with base \(Y\), then by assumption \(X\) has rational singularities (cf. the statement of Theorem 1.2). The Albanese morphism \(\alpha : X' \to \text{Alb}(X')\) identifies to the composition of the MRC-fibration \(\varphi : X' \to Y\) and the embedding \(\alpha_Y : Y \to \text{Alb}(Y)\). Since \(X\) has rational singularities the Albanese map of \(X'\) factors through \(\nu\) [BS95, Lemma 2.4.1], so we get a fibration \(\psi : X \to Y\) such that \(\varphi = \psi \circ \nu\).

A general \(\psi\)-fibre \(F\) is a Cartier divisor in \(X\), so

\[
(K_F + (n - 1)A[F]) \cdot A[2]^{n-2} = (K_X + (n - 1)A) \cdot A^{n-2} \geq 0.
\]

In particular by Proposition 2.13.b) there exists a \(j \in \{1, \ldots, n - 1\}\) such that \(H^0(F, \mathcal{O}_F(K_F + jA)) \neq 0\). Since \(\nu_*\mathcal{O}_X(K_{X'}) \simeq \mathcal{O}_X(K_X)\) this shows that the direct image sheaf

\[
\varphi_*\mathcal{O}_{X'}(K_{X'}/Y + j\nu^*A) \simeq \psi_*\mathcal{O}_X(K_{X}/Y + jA)
\]

is not zero and we conclude as in a).

The following example shows why our strategy of proof does not apply if \((X, A)\) birationally a scroll and \(X\) has irrational singularities.

4.2. Example. Let \(C \subset \mathbb{P}^2\) be a smooth curve of degree three, and set \(\mathcal{O}_C(1)\) for the restriction of the hyperplane divisor to \(C\). Denote by \(A'\) the tautological divisor on the projectivised bundle \(\varphi : S' := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(1)) \to C\). Then \(\mathcal{O}_S(A')\) is globally generated and induces a birational map \(\nu : S' \to S \subset \mathbb{P}^3\) that contracts the section corresponding to the quotient bundle \(\mathcal{O}_C \oplus \mathcal{O}_C(1) \to \mathcal{O}_C\). The surface \(S\) has degree three and is of course the cone over the elliptic curve \(C\). Thus \(S\) is normal, Gorenstein and \(K_S = -H|_S\), where \(H\) is the hyperplane divisor. The Cartier divisor \(A := H\) is ample, the adjoint bundle \(\mathcal{O}_S(K_S + A)\) is trivial, so nef and

\[
H^0(S, \mathcal{O}_S(K_S + A)) = \mathbb{C}.
\]

It is not possible to prove the existence of this global section by looking only at the nonsingular surface \(S'\): the divisor \(K_{S'} + \nu^*A = K_{S'} + A'\) is not generically nef, its restriction to a \(\varphi\)-fibre is \(\mathcal{O}_P(1)\).

Case III: \(\dim Y = 2\).

In order to simplify the notation we assume without loss of generality that \(X\) is smooth, so \(X' = X\). Note that by Proposition 2.15 the divisor \(K_X + (n - 1)A\) is pseudoeffective. By Lemma 4.1 one has

\[
\chi(X, \mathcal{O}_X) + \frac{1}{2}(K_X + (n - 1)A) \cdot A^{n-1} = 0.
\]

Since \(K_X + (n - 1)A\) is pseudoeffective we get a contradiction if \(\chi(X, \mathcal{O}_X) > 0\).

Suppose now that \(\chi(X, \mathcal{O}_X) \leq 0\). Since there are no holomorphic forms on a rationally connected variety and the general fibre of the MRC-fibration has dimension \(n - 2\), we see that

\[
h^k(X, \mathcal{O}_X) = h^0(X, \Omega_X^k) = 0 \quad \forall \ k \geq 3.
\]

Thus \(\chi(X, \mathcal{O}_X) \leq 0\) implies that \(h^1(X, \mathcal{O}_X) \neq 0\) and we have a non-trivial Albanese morphism \(\alpha : X \to \text{Alb}(X)\). We claim that there exists a \(j \in \{1, \ldots, n - 1\}\) such that the direct image sheaf \(\alpha_*\mathcal{O}_X(K_X + jA)\) is not zero: indeed if \(F\) is a general non-empty fibre of \(\alpha\), then by Proposition 2.13 there exists a \(j \in \{1, \ldots, n - 1\}\) such that

\[
H^0(F, \mathcal{O}_F(K_F + jA)) \neq 0.
\]
We now argue as in [Xie09]: let $P \in \text{Pic}^0(\text{Alb}(X))$ be a numerically trivial Cartier divisor, then $jA + \alpha^*P$ is nef and big. Using the relative Kawamata-Viehweg theorem and the Leray spectral sequence one obtains

$$H^i(\text{Alb}(X), \alpha_*\mathcal{O}_X(K_X + jA) \otimes \mathcal{O}_{\text{Alb}(X)}(P)) = 0 \quad \forall i > 0.$$ 

Therefore [Muk81, Cor.2.4] implies that

$$H^0(X, \mathcal{O}_X(K_X + jA + \alpha^*P)) \simeq H^0(\text{Alb}(X), \alpha_*\mathcal{O}_X(K_X + jA) \otimes \mathcal{O}_{\text{Alb}(X)}(P)) \neq 0$$

for some $P \in \text{Pic}^0(\text{Alb}(X))$. In particular

$$\chi(X, \mathcal{O}_X(K_X + jA + P)) \neq 0.$$ 

Since tensoring with a numerically trivial Cartier divisor does not change the Euler characteristic, we get a contradiction to $\chi(X, \mathcal{O}_X(K_X + jA)) = 0$.

Case IV: $\dim Y \geq 3$.

The following lemma is due to Fujita [Fuj87, Lemma 2.5] in the case where $A$ is an ample Cartier divisor and $X$ is Gorenstein and to Andreatta in the log-terminal setting [And95, Thm.2.1].

4.3. Lemma. Let $X$ be a normal, projective variety of dimension $n$ with at most log-terminal singularities. Let $\mu : X \to X'$ be an elementary contraction of birational type contracting a $K_X$-negative extremal ray $\Gamma$. Let $\mu^{-1}(y)$ be a fibre of dimension $r > 0$.

If $A$ is a nef and big Cartier divisor on $X$ such that $A \cdot \Gamma > 0$, then

$$(K_X + rA) \cdot \Gamma > 0.$$ 

In order to simplify the notation we assume without loss of generality that $X$ is smooth, so $X' = X$. Note that by Proposition 2.15 the divisor $K_X + (n - 2)A$ is pseudoeffective, in particular $K_X + (n - 1)A$ is big.

Step 1. Reduction to the case where $K_X + (n - 1)A$ is nef and big. Our goal is to prove that there exists a birational map $\psi : X \rightarrow X_{\min}$ onto a projective variety $X_{\min}$ with at most terminal singularities and a nef and big Cartier divisor $A_{\min}$ on $X_{\min}$ such that $K_{X_{\min}} + (n - 1)A_{\min}$ is nef and

$$H^0(X_{\min}, \mathcal{O}_{X_{\min}}(K_{X_{\min}} + jA_{\min})) \simeq H^0(X, \mathcal{O}_X(K_X + jA)) \quad \forall j \in \{1, \ldots, n - 1\}.$$ 

We will construct $X_{\min}$ by using an appropriate MMP: since $A$ is nef and big, there exists an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $D \sim_{\mathbb{Q}} (n - 1)A$ and the pair $(X, D)$ is klt. Since $K_X + D$ is pseudoeffective and $D$ is big we know by [BCHM06, Thm.1.2] that the pair $(X, D)$ has a log-minimal model $(X_{\min}, D_{\min})$, i.e. we can run a $K_X + D$-MMP with scaling

$$(X_0, D_0) := (X, D) \xrightarrow{\mu_0} (X_1, D_1) \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_s} (X_s, D_s) =: (X_{\min}, D_{\min}).$$ 

We claim that if $\mu_i : (X_i, D_i) \dashrightarrow (X_{i+1}, D_{i+1})$ is an elementary contraction contracting an extremal ray $\Gamma_i$ in this MMP, then $D_i \cdot \Gamma_i = 0$. Moreover one has $D_{i+1} \sim_{\mathbb{Q}} (n - 1)A_{i+1}$ with $A_{i+1}$ a nef and big Cartier divisor such that $A_i = \mu_i^*A_{i+1}$ and

$$H^0(X_i, \mathcal{O}_{X_i}(K_{X_i} + jA_i)) \simeq H^0(X_{i+1}, \mathcal{O}_{X_{i+1}}(K_{X_{i+1}} + jA_{i+1})) \quad \forall j \in \{1, \ldots, n - 1\}.$$ 

In particular the $K_X + D$-MMP is a $K_X$-MMP, so $X_{\min}$ has terminal singularities. Hence if we set $A_{\min} := A_{s}$, then $K_{X_{\min}} + (n - 1)A_{\min}$ is nef and our non-vanishing problem descends to $X_{\min}$. 

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Proof of the claim. Since the contraction $\mu_i$ is $K_{X_i} + D_i$-negative, we have

$$(K_{X_i} + (n-1)A_i) \cdot \Gamma_i = (K_{X_i} + D_i) \cdot \Gamma_i < 0.$$  

Since $\mu_i$ is birational Lemma 4.3 shows that $A_i \cdot \Gamma_i = 0$.

a) If the contraction is divisorial, then $\mu_i$ is a morphism and $K_{X_i} = \mu_i^* K_{X_{i+1}} + E_i$ with $E_i$ an effective $\mathbb{Q}$-divisor. Since $A_i \cdot \Gamma_i = 0$ there exists a nef and big Cartier divisor $A_{i+1}$ on $X_{i+1}$ such that $A_i = \mu_i^* A_{i+1}$. Thus we have $\Delta_{i+1} \sim (n-1)A_{i+1}$ and

$$H^0(X_i, \mathcal{O}_{X_i}(K_{X_i} + jA_i)) \simeq H^0(X_{i+1}, \mathcal{O}_{X_{i+1}}(K_{X_{i+1}} + jA_{i+1})) \quad \forall \ j \in \{1, \ldots, n-1\}.$$  

b) If the contraction is small, denote by $\nu : X_i \to X'$ and $\nu_+ : X_{i+1} \to X'$ the birational morphisms defining the flip. Since $A_i \cdot \Gamma_i = 0$ there exists a nef and big Cartier divisor $A'$ on $X'$ such that $A_i = \nu^* A'$. Thus $A_{i+1} := \nu_+^* A'$ is a nef and big Cartier divisor such that $\Delta_{i+1} \sim (n-1)A_{i+1}$. Since $X_i$ and $X_{i+1}$ are isomorphic in codimension one, we have

$$H^0(X_i, \mathcal{O}_{X_i}(K_{X_i} + jA_i)) \simeq H^0(X_{i+1}, \mathcal{O}_{X_{i+1}}(K_{X_{i+1}} + jA_{i+1})) \quad \forall \ j \in \{1, \ldots, n-1\}.$$  

Step 2. The computation. We know that $K_{X_{\min}} + (n-2)A_{\min}$ is pseudoeffective and by the first step $K_{X_{\min}} + (n-1)A_{\min}$ is nef and big. The goal of this step is to show that

$$A_{\min}^{n-2} \cdot [2(K_{X_{\min}}^2 + c_2(X_{\min})) + 6n A_{\min} \cdot K_{X_{\min}} + (n+1)(3n-2)A_{\min}^2]$$  

is positive. Note first that

$$A_{\min}^{n-2} \cdot [2(K_{X_{\min}}^2 + c_2(X_{\min})) + 6n A_{\min} \cdot K_{X_{\min}} + (n+1)(3n-2)A_{\min}^2] = 2A_{\min}^{n-2} \cdot (K_{X_{\min}} + (n-1)A_{\min}) \cdot (K_{X_{\min}} + (n+2)A_{\min})$$

$$+ A_{\min}^{n-2} \cdot [(2n-2)K_{X_{\min}} \cdot A_{\min} + (n^2 - n + 2)A_{\min}^2 + 2c_2(X_{\min})].$$  

Since $K_{X_{\min}} + (n-1)A_{\min}$ is nef and big, the first term is positive. Thus we are left to show that

$$A_{\min}^{n-2} \cdot [(2n-2)K_{X_{\min}} \cdot A_{\min} + (n^2 - n + 2)A_{\min}^2 + 2c_2(X_{\min})] \geq 0.$$

1st case. $(X_{\min}, A_{\min})$ is not birationally a scroll. Then $\Omega_{X_{\min}} \langle A_{\min} \rangle$ is generically nef by Theorem 1.4. Since $K_{X_{\min}} + (n-1)A_{\min}$ is nef, det $\Omega_{X_{\min}} \langle A_{\min} \rangle = K_{X_{\min}} + nA_{\min}$ is nef. Since $X_{\min}$ is smooth in codimension two we know by Corollary 2.11 that

$$A_{\min}^{n-2} \cdot c_2(X_{\min}) \geq -(n-1)K_{X_{\min}} \cdot A_{\min} + \frac{(n-1)n}{2} A_{\min}^2.$$

Therefore

$$A_{\min}^{n-2} \cdot [(2n-2)K_{X_{\min}} \cdot A_{\min} + (n^2 - n + 2)A_{\min}^2 + 2c_2(X_{\min})] \geq A_{\min}^{n-2} \cdot [(n^2 - n + 2)A_{\min}^2 - (n-1)nA_{\min}^2] = 2A_{\min}^n \geq 0.$$

2nd case. $(X_{\min}, A_{\min})$ is birationally a scroll.

Since $A_{\min}$ is a limit of ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, the problem reduces to showing that if $S$ is a surface cut out by general divisors $D_j \in |m_j H_j|$ where the $H_j$ are ample Cartier divisors and $m_j \gg 0$, then one has

$$[S] \cdot [(2n-2)K_{X_{\min}} \cdot A_{\min} + (n^2 - n + 2)A_{\min}^2 + 2c_2(X_{\min})] \geq 0.$$

Note that since $X_{\min}$ is smooth in codimension two, the surface $S$ is smooth. The main difficulty is to estimate $[S] \cdot c_2(X_{\min})$ which we will do now.
Denote by $T_{X_{\text{min}}} := \Omega_{X_{\text{min}}}$ the tangent sheaf of $X_{\text{min}}$. Fix $H_1, \ldots, H_{n-1}$ ample Cartier divisors, and let

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_r = T_{X_{\text{min}}}$$

be the Harder-Narasimhan filtration of $T_{X_{\text{min}}}$ with respect to $H_1, \ldots, H_{n-1}$. Then for $i \in \{1, \ldots, r\}$, the graded pieces $\mathcal{G}_i := \mathcal{F}_i / \mathcal{F}_{i-1}$ are semistable torsion-free sheaves and if $\mu(\mathcal{G}_i)$ denotes the slope, we have a strictly decreasing sequence

$$\mu(\mathcal{G}_1) > \mu(\mathcal{G}_2) > \ldots > \mu(\mathcal{G}_r).$$

Set $d_i := \text{rk} \mathcal{G}_i$, and let $l \in \mathbb{N}$ be as in Lemma 3.2, then

$$\mu(\mathcal{G}_i < -A_{\text{min}}>) \leq 0 \quad \forall \ i \geq l + 1.$$

Moreover by Lemma 3.2.b) there exists a sequence of rational numbers $w_1, \ldots, w_l$ such that

$$w_i \in [d_i, d_i + 1] \quad \forall \ i \in \{1, \ldots, l\}$$

and

$$\sum_{i=1}^{l} w_i = (\sum_{i=1}^{l} d_i) + 1$$

and

$$\mu(\mathcal{G}_i < -\frac{w_i}{d_i} \cdot A_{\text{min}}>) \leq 0 \quad \forall \ i \in \{1, \ldots, l\}.$$

Note furthermore that $\Omega_{X_{\text{min}}}$ contains a generically nef subsheaf of rank at least three (the pull-back of the cotangent sheaf of the base of the MRC-fibration). Thus there exists a $k \in \{l + 1, \ldots, r\}$ such that

$$\mu(\mathcal{G}_i) \leq 0 \quad \forall \ i \geq k$$

and $\sum_{i=k}^{r} d_i \geq 3$. For $i \in \{1, \ldots, r\}$ we set

$$V_i := \mathcal{G}_i|_S.$$ 

Then the $V_j$ are locally free sheaves on the surface $S$. Since $S$ is a smooth surface (so every ideal sheaf has a locally free resolution of length at most one), we have by [Uta92, Lemma 10.9]

$$c_2(\mathcal{G}_i|_S) \geq c_2(\mathcal{G}_i|_S)^* = c_2(\mathcal{G}_i|_S)^* \quad \forall \ i \in \{1, \ldots, r\}.$$ 

Moreover we have $\mathcal{G}_i^*|_S \simeq (\mathcal{G}_i|_S)^*$, so we obtain

$$[S] \cdot c_2(X_{\text{min}}) = c_2(T_{X_{\text{min}}}|_S) = c_2(\oplus_{i=1}^{r} \mathcal{G}_i|_S) \geq c_2(\oplus_{i=1}^{r} V_i).$$

Our goal is to estimate $\text{c}_2(\oplus_{i=1}^{r} V_i)$ by applying Lemma 2.10 to a sufficiently positive $\mathbb{Q}$-twist. For $i \in \{l + 1, \ldots, k - 1\}$ we set

$$w_i := d_i$$

and for $i \in \{k, \ldots, r\}$ we set

$$w_i := 0.$$

With this notations the slope estimates imply that for all $i \in \{1, \ldots, r\}$ the twisted vector bundle $V_i < \frac{w_i}{d_i} A_{\text{min}} >$ is generically nef\footnote{In order to simplify the notation we denote by $A_{\text{min}}$ the restriction of $A_{\text{min}}$ to $S$.}. By Formula (2) we have

$$c_1(V_i < \frac{w_i}{d_i} A_{\text{min}} >) = c_1(V_i) + w_i A_{\text{min}} \quad \forall \ i \in \{1, \ldots, r\}.$$ 

Since $V_i < \frac{w_i}{d_i} A_{\text{min}} >$ is generically nef and $A_{\text{min}}$ is nef, this implies

$$c_1(V_i) \cdot A_{\text{min}} \geq -w_i A_{\text{min}}^2 \quad \forall \ i \in \{1, \ldots, r\}.$$
Since $\sum_{i=1}^{l} w_i = (\sum_{i=1}^{l} d_i) + 1$ and $\sum_{i=l+1}^{k-1} w_i = \sum_{i=l+1}^{k-1} d_i$, we have
\[
\sum_{i=1}^{r} w_i = \sum_{i=1}^{k-1} d_i + 1 = n - (\sum_{i=k}^{r} d_i) + 1 \leq n - 2.
\]

Thus if we set
\[
c := \frac{n - 1}{\sum_{i=1}^{r} w_i},
\]
we have $c \geq 1$ and $\sum_{i=1}^{r} cw_i = n - 1$. Since $K_{X_{min}}|S = \sum_{i=1}^{r} c_1(V_i)$ the twisted vector bundle
\[
\bigoplus_{i=1}^{r} V_i <\frac{cw_i}{d_i} A_{min}>
\]
is generically nef with nef determinant $(K_{X_{min}} + (n - 1)A_{min})|S$. Thus its second Chern class is non-negative by Lemma 2.10, so by Lemma 4.4 below we have
\[
0 \leq c_2(\bigoplus_{i=1}^{r} V_i <\frac{cw_i}{d_i} A_{min}>)
\]
\[
= c_2(\bigoplus_{i=1}^{r} V_i) + \frac{1}{2} \left( \sum_{i=1}^{r} (cw_i)^2 - \sum_{i=1}^{r} (\frac{cw_i}{d_i})^2 \right) A_{min}^2 + \sum_{i=1}^{r} \left( \sum_{j=1}^{r} cw_j - \frac{cw_i}{d_i} \right) c_1(V_i) \cdot A_{min}.
\]

Since $\sum_{j=1}^{r} cw_j = n - 1$ and $K_{X_{min}}|S = \sum_{i=1}^{r} c_1(V_i)$ we have
\[
\sum_{i=1}^{r} \left( \sum_{j=1}^{r} cw_j - \frac{cw_i}{d_i} \right) c_1(V_i) \cdot A_{min} = (n - 1)K_{X_{min}}|S \cdot A_{min} - \sum_{i=1}^{r} \frac{cw_i}{d_i} c_1(V_i) \cdot A_{min}.
\]

By inequality (10) this is less or equal than
\[
(n - 1)K_{X_{min}}|S \cdot A_{min} + \sum_{i=1}^{r} \frac{cw_i^2}{d_i} A_{min}^2.
\]

Thus we get a lower bound for the second Chern class:
\[
(11) \quad c_2(\bigoplus_{i=1}^{r} V_i) \geq -(n - 1)K_{X_{min}}|S \cdot A_{min} - \frac{1}{2} \left( (n - 1)^2 + (2c - c^2) \sum_{i=1}^{r} \frac{w_i^2}{d_i} \right) A_{min}^2.
\]

This immediately implies that $|S| \cdot [(2n - 2)K_{X_{min}} \cdot A_{min} + (n^2 - n + 2)A_{min}^2 + 2c_2(X_{min})]$ is greater or equal than
\[
\left[ (n^2 - n + 2) - (n - 1)^2 - (2c - c^2) \sum_{i=1}^{r} \frac{w_i^2}{d_i} \right] |S| \cdot A_{min}^2 = \left[ n + 1 - (2c - c^2) \sum_{i=1}^{r} \frac{w_i^2}{d_i} \right] |S| \cdot A_{min}^2.
\]

We have $2c - c^2 \leq 1$ so we are finished if we show that
\[
\sum_{i=1}^{r} \frac{w_i^2}{d_i} \leq n + 1.
\]

Recall now that by Equation (9) we have $\sum_{i=1}^{l} (w_i - d_i) = 1$, so
\[
\sum_{i=1}^{l} \frac{w_i^2}{d_i} = \sum_{i=1}^{l} \frac{d_i^2 + 2(w_i - d_i)d_i + (w_i - d_i)^2}{d_i} = \sum_{i=1}^{l} d_i + 2 + \sum_{i=1}^{l} \frac{(w_i - d_i)^2}{d_i}.
\]

Since $(w_i - d_i) \in [0, 1]$ and $d_i \geq 1$ we have moreover
\[
\sum_{i=1}^{l} \frac{(w_i - d_i)^2}{d_i} \leq \sum_{i=1}^{l} (w_i - d_i) = 1.
\]
Since \( w_i = d_i \) for \( i \in \{1, \ldots, k - 1\} \) and \( w_i = 0 \) for \( i \in \{k, \ldots, r\} \) we get
\[
\sum_{i=1}^{r} \frac{w_i^2}{d_i} \leq \sum_{i=1}^{k-1} d_i + 3 = n - \sum_{i=k}^{r} d_i + 3.
\]
Since \( \sum_{i=k}^{r} d_i \geq 3 \) this finishes the proof of this step.

**Step 3. The conclusion.** Let \( \mu : X'_{\min} \rightarrow X_{\min} \) be a desingularisation of \( X_{\min} \). Since \( X_{\min} \) is smooth in codimension two one has
\[
(\mu^*A_{\min})^{n-2} \cdot \left[ 2(K_{X'_{\min}}^2 + c_2(X'_{\min})) + 6n^{1/2}A_{\min} \cdot K_{X'_{\min}} + (n + 1)(3n - 2)(\mu^*A_{\min})^2 \right]
\]
which is positive by Step 2. Since terminal singularities are rational, we have
\[
\chi(X'_{\min}, \mathcal{O}_{X'_{\min}}(K_{X'_{\min}} + j\mu^*A_{\min})) = \chi(X_{\min}, \mathcal{O}_{X_{\min}}(K_{X_{\min}} + jA_{\min})) = 0
\]
for all \( j = 1, \ldots, n - 1 \). Thus by Lemma 4.1 the Equation (7) holds, a contradiction to our computation. \( \square \)

**4.4. Lemma.** Let \( S \) be a projective manifold. Let \( V_1, \ldots, V_r \) be vector bundles on \( S \), and let \( A \) be a Cartier divisor class on \( S \). Set \( d_i := \text{rk}V_i \), and let \( \alpha_i \in \mathbb{Q} \) for \( i \in \{1, \ldots, r\} \). Then \( c_2(\oplus_{i=1}^{r} V_i < \frac{\alpha_i}{d_i} A>) \) is equal to
\[
c_2(\oplus_{i=1}^{r} V_i) + \frac{1}{2} \left( \sum_{i=1}^{r} \alpha_i^2 - \sum_{i=1}^{r} \frac{\alpha_i^2}{d_i} \right) A^2 + \sum_{i=1}^{r} \sum_{j=1}^{r} \left( \sum_{i \neq j}^{r} \frac{\alpha_i}{d_i} \right) c_1(V_i) \cdot A.
\]

Proof. Note first that by Formula (3) one has
\[
(12) \quad c_2(V_i < \frac{\alpha_i}{d_i} A>) = c_2(V_i) + \frac{1}{2} \left( \alpha_i^2 - \frac{\alpha_i^2}{d_i} \right) A^2 + (\alpha_i - \frac{\alpha_i}{d_i}) c_1(V_i) \cdot A.
\]
Recall also that for a direct sum of (\( \mathbb{Q} \)-twisted) vector bundles \( \oplus_{i=1}^{r} \mathcal{F}_i \) one has
\[
c_2(\oplus_{i=1}^{r} \mathcal{F}_i) = \sum_{i=1}^{r} c_2(\mathcal{F}_i) + \sum_{i<j} c_1(\mathcal{F}_i) \cdot c_1(\mathcal{F}_j).
\]
Thus
\[
c_2(\oplus_{i=1}^{r} V_i < \frac{\alpha_i}{d_i} A>) = \sum_{i=1}^{r} c_2(V_i < \frac{\alpha_i}{d_i} A>) + \sum_{i<j} c_1(V_i < \frac{\alpha_i}{d_i} A>) \cdot c_1(V_j < \frac{\alpha_j}{d_j} A>)
\]
which by (12) and Formula (2) is equal to
\[
\sum_{i=1}^{r} \left( c_2(V_i) + \frac{1}{2} \left( \alpha_i^2 - \frac{\alpha_i^2}{d_i} \right) A^2 + (\alpha_i - \frac{\alpha_i}{d_i}) c_1(V_i) \cdot A \right) + \sum_{i<j} (c_1(V_i) + \alpha_i A) \cdot (c_1(V_j) + \alpha_j A)
\]
\[
= c_2(\oplus_{i=1}^{r} V_i) + \frac{1}{2} \left( \sum_{i=1}^{r} \alpha_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j - \sum_{i=1}^{r} \frac{\alpha_i^2}{d_i} \right) A^2 + \sum_{i=1}^{r} (\alpha_i - \frac{\alpha_i}{d_i}) c_1(V_i) \cdot A + \sum_{i<j} (\alpha_i c_1(V_i) \cdot A + \alpha_i c_1(V_j) \cdot A).
\]
By the binomial formula the coefficient for \( A^2 \) equals \( \sum_{i=1}^{r} \alpha_i^2 - \sum_{i=1}^{r} \frac{\alpha_i^2}{d_i} \), so we are left to show that
\[
\sum_{i=1}^{r} \alpha_i c_1(V_i) \cdot A + \sum_{i<j} (\alpha_j c_1(V_i) \cdot A + \alpha_i c_1(V_j) \cdot A) = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_j c_1(V_i) \cdot A.
\]
This is elementary by induction on \( r \). \( \square \)
5. The Ambro-Ionescu-Kawamata conjecture

Proof of Theorem 1.5. By [Kaw00, Thm.3.1] we can suppose that $K_X + A$ is nef and big.

Step 1. Terminalisation. By [KM98, Thm.6.23] there exists a terminalisation of $X$, i.e a birational map $\mu : X' \to X$ from a threefold with at most terminal singularities such that $K_{X'} = \mu^* K_X$. Thus $A' := \mu^* A$ is a nef and big Cartier divisor such that $K_{X'} + A'$ is nef and big, moreover we have

$$H^0(X, \mathcal{O}_X(K_X + A)) = H^0(X', \mathcal{O}_{X'}(K_{X'} + A')).$$

Hence the non-vanishing problem lifts to $X'$, in order to simplify the notation we suppose without loss of generality that $X$ has at most terminal singularities.

Step 2. The computation. We claim that the twisted cotangent sheaf $\Omega_X < \frac{2}{3} A >$ is generically nef. Assuming this for the time being, let us show how to conclude. By [Kaw86, p.541], we have

$$\chi(X, \mathcal{O}_X) \geq -\frac{1}{24} K_X \cdot c_2(X).$$

By the Riemann-Roch formula for threefolds with terminal singularities [Rei87, p.413]

$$\chi(X, \mathcal{O}_X(K_X + A)) \geq \frac{1}{12} (K_X + A) \cdot A \cdot (K_X + 2A) + \frac{1}{24} (K_X + 2A) \cdot c_2(X).$$

Since $\Omega_X < \frac{2}{3} A >$ is generically nef and $K_X + 2A$ is nef, we have by Corollary 2.11

$$(K_X + 2A) \cdot c_2(X) \geq -(K_X + 2A) \cdot (\frac{4}{3} K_X \cdot A + \frac{4}{3} A^2) = \frac{4}{3} (K_X + 2A) \cdot (K_X + A) \cdot A.$$

Hence

$$\frac{1}{12} (K_X + A) \cdot A \cdot (K_X + 2A) + \frac{1}{24} (K_X + 2A) \cdot c_2(X) \geq \frac{1}{24} (K_X + 2A) \cdot (K_X + A) \cdot \frac{2}{3} A.$$

Since the three divisors $K_X + A, A$ and $K_X + 2A$ are nef and big this intersection product is strictly positive. Thus by Kawamata-Viehweg vanishing

$$\chi(X, \mathcal{O}_X(K_X + A)) = \chi(X, \mathcal{O}_X(K_X + A)) > 0.$$
Since $K_X' \cdot F = -2$ and $A$ is Cartier, this implies that $\mu^* A \cdot F \geq 3$. Hence $K_F + \frac{2}{3} A|_F$ is $\mathbb{Q}$-linear equivalent to an effective divisor, a contradiction to (13). □

REFERENCES


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