**Exercise sheet 3**

**Exercise 1.** Let $X$ be a real manifold given by some atlas $(U_i, \varphi_i : U_i \to V_i \subset \mathbb{R}^n)_{i \in I}$. The atlas is oriented if the coordinates changes $\varphi_i \circ \varphi_j^{-1}$ have positive determinant, i.e.

$$\det(\text{Jac}_{\varphi_i \circ \varphi_j^{-1}}(x)) > 0 \quad \forall x \in \varphi_j(U_i \cap U_j).$$

Let $X$ be a complex curve given by some atlas $(U_i, \varphi_i : U_i \to V_i \subset \mathbb{C})_{i \in I}$. Show that the corresponding real atlas is oriented. Generalise to complex manifolds $X$ of arbitrary dimension.

**Exercise 2.** Let $X$ be a complex curve.

a) Let $\omega \in H^0(X, \Omega^1_X)$ be a holomorphic one-form. Show that $\omega$ is $d$-closed.

b) Let $\omega \in C^\infty(X, \Omega^{1,0}_X)$ be a differential form of type $(1,0)$ that is $d$-closed. Show that $\omega$ is holomorphic.

c) Deduce that $H^2(P^1, \mathbb{C}) \neq 0$ and $H^{1,1}(P^1) \neq 0$.

**Exercise 3.** Let $\omega$ be the $(1,1)$-form on $P^1$ that is given locally in the standard open sets $U_0$ (resp. $U_1$) by $\omega = \frac{4}{1 + |z|^2} dz \wedge d\bar{z}$ (resp. $\omega = \frac{4}{1 + |w|^2} dw \wedge d\bar{w}$).

a) Show that $\omega$ is well-defined, i.e. the local descriptions coincide on $U_0 \cap U_1$ (recall that $w = \frac{1}{z}$).

b) Show that $\omega$ is $d$-closed. Show that $\int_{U_0 \cap U_1} \omega = 2\pi$ (use that $idz \wedge d\bar{z} = 2 dx \wedge dy$).

c) Deduce that $H^2(P^1, \mathbb{C}) \neq 0$ and $H^{1,1}(P^1) \neq 0$.

**Exercise 4.** Prove the generalised form of Cauchy’s formula: let $U \subset \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a differentiable complex-valued function. Let $\overline{D} \subset U$ be the closure of a disc contained in $U$. Then for every $w \in D$, we have

$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w} dz - \int_{\overline{D}} \frac{i}{2\pi(z-w)} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}.$$

**Exercise 5.** a) Let $f : X \to Y$ be a differentiable map between real manifolds. Show that the pull-back

$$f^* : C^\infty(Y, \Omega^k_{Y,C}) \to C^\infty(X, \Omega^k_{X,C})$$

induces a linear map between the de Rham cohomology groups $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$.

b) Let now $f : X \to Y$ be a holomorphic map between complex manifolds. Show that the pull-back

$$f^* : C^\infty(Y, \Omega^k_{Y,C}) \to C^\infty(X, \Omega^k_{X,C})$$
maps forms of type \((p, q)\) to forms of type \((p, q)\). Hint: prove the claim for forms of type \((1, 0)\) and \((0, 1)\), then generalise.

Show that for all \(p, q \in \mathbb{N}\) the pull-back induces a linear map between the Dolbeault cohomology groups \(f^*: H^{p,q}(Y) \to H^{p,q}(X)\).

**Exercise 6.** (Bonus exercise, probably no time for discussion in the exercise class.) We consider the differentiable 1-form on \(\mathbb{R}^2 \setminus 0\) given by
\[
\alpha = \frac{1}{x^2 + y^2}(-ydx + xdy).
\]
a) Show that \(\alpha\) is \(d\)-closed.

b) Let \(i: S^1 \hookrightarrow \mathbb{R}^2 \setminus 0\) be the inclusion of the unit circle. Show that
\[
\int_{S^1} i^*\alpha = 2\pi.
\]
Deduce that \(\alpha\) and \(i^*\alpha\) are not \(d\)-exact.

c) Let \(\omega\) be a differentiable 1-form on \(S^1\). For every \(x \in S^1\) we define
\[
f(x) := \int_1^x \omega,
\]
where we integrate over the segment of the circle from 1 to \(x\) in counterclockwise direction. Show that \(\lim_{x \to 1} f(x) = 0\) if \(\int_{S^1} \omega = 0\). Deduce that \(\omega\) is \(d\)-exact if and only if \(\int_{S^1} \omega = 0\).

d) Show that \(H^1(S^1, \mathbb{R}) \cong \mathbb{R}\).