Exercise sheet 4

Exercise 1. For this exercise you can admit the following result\(^1\): let \( g : \mathbb{C} \to \mathbb{C} \) be a differentiable function. Then there exists a differentiable function \( f : \mathbb{C} \to \mathbb{C} \) such that \( g = \frac{\partial f}{\partial \bar{z}} \).

a) Show that \( H^0_{\mathbb{C}}(\mathbb{C}) = 0 \) and \( H^1_{\mathbb{C}}(\mathbb{C}) = 0 \).

b) Show that \( H^1_{\mathbb{C}}(\mathbb{P}^1) \cong \mathbb{C} \). Hint: let \( \mathcal{U} \) be the open cover by the standard open sets \( U_0 \) and \( U_1 \). Show that \( H^1_{\mathbb{C}}(\mathbb{P}^1) \cong \check{H}^1(\mathcal{U}, \Omega_{\mathbb{P}^1}) \) and \( \check{H}^1(\mathcal{U}, \Omega_{\mathbb{P}^1}) \) is generated by \( \frac{1}{2}dz \).

Exercise 2. Show that \( \check{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0 \) in two different ways:

a) By using the Riemann-Roch theorem.

b) By a computation via Čech cohomology (use the open cover \( \mathcal{U} \) by the standard open sets \( U_0 \) and \( U_1 \)).

c) Use a computation in Čech cohomology to show that \( \check{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0 \).

Exercise 3. Let \( X = \mathbb{C}/\Lambda \) be a one-dimensional torus, so \( \Omega_X \cong \mathcal{O}_X \) (as proven in the lecture)

a) Show that \( H^1_{\mathbb{C}}(X) \cong \mathbb{C} \), so \( X \) is a compact curve of genus one.

b) Let \( D = x \) be an effective divisor, where \( x \in X \) is an arbitrary point, and denote by \( \mathcal{O}_X(D) \) the associated sheaf. Show that \( h^0(X, \mathcal{O}_X(D)) = 1 \).

c) Deduce that \( h^1(X, \mathcal{O}_X(D)) = 0 \).

Exercise 4.

a) Let \( X \) be a topological space, and \( \mathcal{F} \) a sheaf of abelian groups on \( X \). Let \( \mathcal{U} = (U_i)_{i \in I} \) be an open cover. In the lecture we defined the maps

\[
\delta_1 = C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \quad \delta_2 = C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F}).
\]

Show that \( \delta_2 \circ \delta_1 = 0 \).

b) Let \( X \) be a complex manifold and \( Y \subset X \) a hypersurface. Show that there exists a holomorphic line bundle \( L \to X \) and a global section \( s : X \to L \) such that \( Y = \{ x \in X \mid s(x) = 0 \} \).

Exercise 5. (Picard group)

Let \( X \) be a complex manifold, and let \( \pi : L \to X \) be a holomorphic line bundle, that is a holomorphic vector bundle of rank one. Let \( (U_\alpha)_{\alpha \in A} \) be an open covering of \( X \) such that \( L|_{U_\alpha} \) is trivial. The transition functions of \( L \) are thus holomorphic maps

\[
g_{\alpha \beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*.\]

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\(^1\)You can find the proof in Forster, Riemann surfaces, §13.
a) Suppose that there exist holomorphic functions $s_\alpha : U_\alpha \to \mathbb{C}^*$ such that

$$g_{\alpha\beta} = \frac{s_\beta}{s_\alpha}$$

on $U_\alpha \cap U_\beta$. Show that there exists a global section $s : X \to L$ such that $s(x) \neq 0$ for all $x \in X$. Deduce that $L$ is isomorphic to the trivial line bundle $X$.

b) Show that the isomorphism classes of holomorphic line bundles have a natural group structure given by the tensor product. We call this group the Picard group $\text{Pic}(X)$.

c) Show that the Picard group is isomorphic to the Čech cohomology group $\check{H}^1(X, \mathcal{O}_X^*)^2$.

Hint: start by showing that a Čech cocycle defines transition functions of a line bundle.

\footnote{We compute the Čech cohomology for some sufficiently fine cover of $X$ by polydiscs.}