§1 Holomorphic functions in several variables

Literature: Gunning, Koebe and Karp

Defn. Let $U \subseteq \mathbb{C}^n$ be an open set, denote by $z_1, \ldots, z_n$ the coordinates.

A differentiable $(\geq C^\infty)$ function $f: U \rightarrow \mathbb{C}$ is holomorphic in $\alpha=(a_1, \ldots, a_n) \in U$ if

\[ \forall j=1 \ldots n \quad z_j \mapsto f(a_1, \ldots, a_{j-1}, z_j, a_{j+1}, \ldots, a_n) \]

is holomorphic in $a_j$.

Rem. Equivalent definitions:

* if differentiable and $\frac{\partial f}{\partial z_j}(a) = \frac{1}{2} \left( \frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right)(a) = 0 \quad \forall j=1 \ldots n$
* if differentiable and $df|_a: \mathbb{C}^n \rightarrow \mathbb{C}$ is $\mathbb{C}$-linear.

Defn. Let $a \in \mathbb{C}^n$ be a point, $R \in (\mathbb{R}^*)^n$. The polydisc around $a$ with multiradii $R$ is the open set

\[ D(a, R) := \{ z \in \mathbb{C}^n \mid |z_j-a_j| < R_j \quad \forall j=1 \ldots n \} \]

If $a=0$, $R=(1, \ldots, 1)$ we call $D^n = D(0, R)$ the unit disc.

The set $P(a, R) := \{ z \in \mathbb{C}^n \mid |z_j-a_j| = R_j \quad \forall j=1 \ldots n \}$

is called the distinguished boundary of $D(a, R)$. 
Rem: For \( n \geq 2 \), the distinguished boundary \( \partial \Omega(0, R) \) is not the topological boundary of \( D(0, R) \).

Cauchy's Thm: \( U \subset \mathbb{C}^n \) open, \( f: U \to \mathbb{C} \) differentiable.

\( f \) is holomorphic on \( U \) iff

i) \( \forall z \in \Omega \exists \Omega(z) \) polydisc \( \text{st.} \)

\[
f(z) = \frac{1}{2\pi i} \oint_{\Gamma(z)} \frac{f(z)}{(z-w_1)\cdots(z-w_n)} \, dw_1 \cdots dw_n.
\]

ii) For every polydisc \( D(0, R) \subset U \) and \( \omega \in D(0, R) \) we have

\[
f(z) = (\frac{A}{2\pi i})^n \oint_{\Gamma(0, R)} \frac{f(z)}{(z-w_1)\cdots(z-w_n)} \, dw_1 \cdots dw_n.
\]

Corollary (exercise sheet 1)

Let \( D(0, R) \) be a polydisc and \( f: D(0, R) \to \mathbb{C} \) a holomorphic function.

If \( f \) has a local maximum in some point \( \omega \in D(0, R) \) then \( f \) is constant.

Def: \( U \subset \mathbb{C}^n \) open, \( f: U \to \mathbb{C}^n \) differentiable map is holomorphic in \( U \) if

\[
f_{11}, \ldots, f_{nn} \text{ holom. in } U.
\]
Def.: A holom. map \( f: U \to \mathbb{C}^n \) is biholomorphic on \( U \) if its image \( V = f(U) \) is bijective and \( f^{-1}: V \to U \) is holomorphic.

Def.: Let \( f: U \to \mathbb{C}^m \) be a holom. map. The Jacobian of \( f \) at a point \( a \in U \) is the matrix
\[
J_f(a) := \left( \frac{\partial f_k}{\partial z_j}(a) \right)_{1 \leq k \leq m, 1 \leq j \leq n}
\]

Rank theorem: Let \( f: U \to \mathbb{C}^m \) be a holom. map, suppose that for some \( a \in U \) the Jacobian has rank \( \leq k \) in a neighborhood of \( a \).

Then \( \exists \, V = U \) open, \( \Omega \subset \mathbb{C}^m \) open and \( f(a) \) biholomorphic maps \( \phi \) and \( \psi \) such that
\[
\begin{align*}
\phi: & \quad \mathbb{C}^k \to U \\
\psi: & \quad \mathbb{C}^m \to V
\end{align*}
\]

\[
( z_1, \ldots, z_n ) \mapsto ( z_1, \ldots, z_k, \zeta, \ldots, \zeta )
\]
§ 2 Analytic sets

Defn: Let $V \subset \mathbb{C}^n$ be an open set, $f_1, \ldots, f_m : V \to \mathbb{C}$ holom.

The vanishing set (or zero set) of $f_1, \ldots, f_m$ is

$$Z(f_1, \ldots, f_m) = \{ z \in V \mid f_1(z) = \ldots = f_m(z) = 0 \}$$

Example: 1) $f : \mathbb{C}^2 \to \mathbb{C}$,

$$f(z_1, z_2) \mapsto z_1^2 - z_2^3$$

$$Z(f) = \{ \}$$

2) $f_1 : \mathbb{C}^3 \to \mathbb{C}$,

$$f_1(z_1, z_2, z_3) \mapsto z_3^2 - z_2$$

$f_2 : \mathbb{C}^3 \to \mathbb{C}$,

$$f_2(z_1, z_2, z_3) \mapsto z_1 z_2$$

Defn: Let $V \subset \mathbb{C}^n$ open. A subset $A \subset V$ is analytic if

$$\forall a \in V \exists \forall \epsilon > 0 \text{ open and holom. \phi}_a : \mathbb{C}^n \to \mathbb{C}$$

such that $A \cap \mathbb{V} = Z(f_1, \ldots, f_m)$.

Rem: An analytic subset of $V$ is closed in $V$. 
How to measure the "size" of an analytic set?

Ex 1) Looks like a curve, \(\mathcal{E} \times \mathbb{L} = \text{an} \times \text{line and plane} \)

Defn: \( V \subset \mathbb{C}^n \text{ open}, \ A \subset V \) an analytic set. Fix \( a \in A \).

- We say that \( A \) has codimension \( s \) at \( a \) (written \( s = \text{codim}_a A \)) if
  i) \( \exists s \)-dimensional affine space \( \Pi \subset \mathbb{C}^n \) s.t. \( a \) is an isolated point of \( \Pi \cap A \).

ii) there is no \( s+1 \)-dimensional space with this property.

- If the codimension of \( A \) is equal to \( s \) (\( \text{at all } a \in A \)) we say that \( A \) has dimension \( n-s \).

Example: \( \mathbb{Z}(z^2 - z^3) \subset \mathbb{C}^2 \) has dimension 1.

Exercise! In \((96)\) take \( n = \{ x_2 = 0 \} \)

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Lemma \( V \subset \mathbb{C}^n \text{ open}, f: X \to \mathbb{C} \text{ holomorphic. Then } \forall \ a \in \mathbb{Z}(f) \text{ we have } \text{codim}_a \mathbb{Z}(f) = 1 \).

Proof: i) Take a line \( \Pi \subset \mathbb{C}^n \) s.t. \( a \in \Pi \), but \( \Pi \not\subset \mathbb{Z}(f) \) (existence).

Then \( \Pi \cap \mathbb{Z}(f) = \mathbb{Z}(f|_{\Pi}) \).

If \( a \) is not an isolated point of \( \mathbb{Z}(f|_{\Pi}) \), then \( f|_{\Pi} \equiv 0 \) by identity principle. \( \Rightarrow \) \( \Pi \not\subset \mathbb{Z}(f) \).
Hartogs's thm: \( V \subset \mathbb{C}^n \) open, \( A \subset V \) analytic set.

Suppose that \( A \) has co-dimension \( \geq 2 \) in every point \( a \in A \).

\[ \text{Let } f: V \setminus A \to \mathbb{C} \text{ be holomorphic. Then} \]
\[ \exists \tilde{f}: V \to \mathbb{C} \text{ holom. s.t. } f = \tilde{f} \big|_{V \setminus A}. \]

Rem: no hypothesis on boundedness as in Riemann's rem. sing. thm.

Corollary: \( \text{Let } D^n \text{ be unit disc, } f: D^n \to \mathbb{C} \text{ holom, and } n \geq 2 \)

Suppose that \( f(z) \neq 0 \ \forall z \neq 0 \). Then \( f(0) \neq 0 \).

Proof: \( \frac{1}{f} \) is holom. on \( D^n \setminus 0 \) and extends by Hartogs.

\( \star \)

ii) Suppose \( P \subset \mathbb{C}^n \) is linear subspace of dim \( \geq 2 \).

Then \( P \cap \mathbb{Z}(f) = \mathbb{Z}(f|_P) \) so if \( a \) is an isolated point, \( P \cap \mathbb{Z}(f) = \mathbb{Z}(f|_P) \) so if \( a \) is an isolated point, \( f|_P \) has an isolated zero \( \in P \) to corollary \( \star \)

Fact (not trivial!)

1) \( \text{Let } A = \mathbb{Z}(f_1, \ldots, f_m) \subset \mathbb{C}^n \) be an analytic set, \( f_i \) not constant.

then \( \forall a \in A \) \( \text{codim}_a A \leq m. \)

2) \( \text{If } A \subset V \subset \mathbb{C}^n \) has \( \text{codim} = 1 \) in every point \( a \in A \), then (locally) \( A = \mathbb{Z}(f) \) for some \( f: V \to \mathbb{C} \) holom.
Rem: In general it is not possible to find $f_1, \ldots, f_m$ s.t.
$$\text{codim}_\alpha A = m.$$ (tough exercises)

However

Prop: Let $A = Z(f_1, \ldots, f_m) \subset \mathbb{C}^n$ be an analytic set.
Let $a \in A$ be a point s.t. $\text{rk} J_a (f_1, \ldots, f_m) = m$.
Then $\text{codim}_\alpha A = m$; in fact (up to replacing $V$ by a smaller open set) we have
$$A \sim D^{n-s}.$$

Proof: Note first that $\text{rk} J_z (f_1, \ldots, f_m) = m$ at $z$ in a neighborhood of $a$.

By the rank theorem up to coordinate change we have
$$f_1, \ldots, f_m: X \to \mathbb{C}^m$$
$$(x_1, \ldots, x_n) \mapsto (z_1, \ldots, z_m)$$

In particular $\mathbb{C} (f_1, \ldots, f_m) \cong \{ z \in \mathbb{C} \mid z_1 = \ldots = z_m = 0 \}$, so $A \sim D^{n-s}$.

For codim. statement up to permuting $z_i$ we have
$$\text{rk} \left( \frac{\partial f_k}{\partial z_j} \right)_{1 \leq k \leq m, 1 \leq j \leq m} = m.$$

Then $A \cap \{ z_{m+1} = \ldots = z_n = 0 \}$ has a as an isolated point.
(Observe that $f_1, \ldots, f_m, z_{m+1}, \ldots, z_n$ is not biholom.)
§3 Complex manifolds

Last section: analytic sets \( A \subset V \subset \mathbb{C}^n \)

holom open

Easiest case \( A = V \subset \mathbb{C}^n \) open.

Now: glue these objects to obtain more interesting objects.

Defn: A complex manifold of dimension \( n \) is

- a connected topological space \( X \) that is Hausdorff.
- s.t. there exists an open covering \( (U_i)_{i \in I} \) \( X \) is manifolds

and homeomorphisms

\[ \varphi_i : U_i \to V_i \subset \mathbb{C}^n \text{ open} \]

s.t. \( \forall i, j \in I \)

\[ \varphi_i \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \]

is biholomorphic.

We call \( (U_i, \varphi_i) \) an atlas.

Defn: The complex manifold \( X \) is compact if the top space is compact.
Rem: • Generalisation: complex space, where
\[ \phi_i: U_i \rightarrow V_i \text{ is an analytic set} \]
(see exercises)

• Variant: real manifold, where
\[ \phi_i: U_i \rightarrow V_i \text{ is } \mathbb{R}^n \]
and \( \phi_i \circ \phi_j^{-1} \) is a diffeomorphism.

Easy fact: A complex manifold of dimension \( n \) is always a real manifold of (real) dimension \( 2n \).
However: \( X_1 \not\approx X_2 \) \( \Rightarrow \) \( X_1 \not\sim X_2 \)
(see exercises)

Examples:

1) \( \Lambda = \mathbb{C}^n \) lattice of rank \( 2n \), i.e.
\[ \Lambda = \bigoplus_{i=1}^{2n} \mathbb{Z}w_i \]
\[ \text{s.t. } w_1, \ldots, w_{2n} \text{ is an } \mathbb{R}-basis of } \mathbb{C}^n. \]

\[ \Rightarrow \text{ } X := \mathbb{C}^n/\Lambda \text{ is a compact complex manifold, called a } \]
\[ n \text{-dimensional torus.} \]

Proof for \( n = 1 \)

Take \( V_i = \mathbb{C} \) s.t. \( \pi: \mathbb{C} \rightarrow \mathbb{C}/\Lambda \) is bijective onto its image.

Set \( U_i = \pi(V_i) \) and \( \phi_i: U_i \rightarrow V_i \)
\[ \pi|_{U_i}^{-1} \]

then \( \phi_i \circ \phi_j^{-1} \) satisfies that \( \phi_i \circ \phi_j^{-1}(z) \in \Lambda \)
(two points mapping to same class differ by elem. of \( \Lambda \)).
2) Projective spaces: \( V \) a \( n+1 \)-dimensional \( \mathbb{C} \) vector space

\[
\text{IP}(V) := \{ \text{sub vector spaces of } V \text{ of dim } L \} \]

\( \pi: V \setminus \{ 0 \} \rightarrow \text{IP}(V), \quad v \mapsto [v] \quad \text{class of line through } cv. \)

is surjective and \( \pi^{-1}(U) = U \setminus \{0\} \) (so \( \text{IP}(V) \cong \mathbb{P}^n \)).

Affine charts: Fix coordinates \( v_0, \ldots, v_n \) on \( V = \mathbb{C}^{n+1} \).

\( \sim \) homogeneous coordinates \( [v] = [v_0 : \ldots : v_n] \)

Set \( U_i := \{ v \in \text{IP}(V) \mid v_i \neq 0 \} \) standard open set.

and \( \varphi_i: U_i \rightarrow \mathbb{A}^n, \quad [v] \mapsto \left( \frac{v_0}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i} \right) \)

\( \sim \) \( \varphi_j(\varphi_i(U_i \cap U_j)) = \{ (z_1, \ldots, z_n) \in \mathbb{A}^n \mid z_i \neq 0 \} \)

and \( \varphi_i \circ \varphi_j^{-1} \) is given by

\[
(z_1, \ldots, z_n) \quad \mapsto \quad \left( z_1, \ldots, \frac{z_i}{z_j}, \frac{z_j}{z_i}, \ldots, \frac{z_n}{z_i} \right)
\]

Note: \( \text{IP}(V) \) compact since \( \mathbb{V} \setminus \{ 0 \} \rightarrow \text{IP}(V) \) is compact.
3) Hopf manifolds: fix $\lambda \in \mathbb{R}$, $0 < \lambda < 1$

$\mathbb{C}^n$ acts on $\mathbb{C}^n \setminus 0$ by $\mathbb{Z} \times \mathbb{C}^n \setminus 0 \to \mathbb{C}^n \setminus 0$

$((m, z)) \mapsto \lambda^m z$

$H_\lambda = \mathbb{C}^n \setminus 0 / \mathbb{Z}$ is a compact complex manifold.

**Defn:** Let $X^n$ and $Y^m$ be complex manifolds

$(\psi_i : U_i \to V_i)_{i \in I}$ and $(\phi_j : M_j \to N_j)_{j \in J}$ the atlases of $X$ and $Y$.

A continuous map $f : X \to Y$ is holomorphic if

$$
\forall i, j \in I \times J \quad \forall \psi_i : V_i \to \mathbb{C}^n \quad \phi_j : M_j \to \mathbb{C}^m \quad \text{the composition } f \circ \psi_i^{-1} : V_i \to M_j \text{ is holomorphic.}
$$

Examples:

1) $X$ is a complex manifold, $f : X \to \mathbb{C}$ constant.

Interesting fact (exercise: show it!): If $X$ is a compact complex manifold and $f : X \to \mathbb{C}$ holomorphic, then $f$ is constant.

Consequence: can't use holomorphic functions to study $X$.  

2) $H = \mathbb{C}^{n}/\mathbb{Z}$ Hopf manifold

\[
\mathbb{P}^{n-1} = \mathbb{C}^{n}/\mathbb{C}^{*} \quad \mathbb{C}^{n}/\mathbb{C}^{*} \xrightarrow{\text{Id}} \mathbb{C}^{n}/\mathbb{C}^{*} \\
\downarrow \quad f \quad \downarrow \\
H \xrightarrow{f} \mathbb{P}^{n-1} \quad \text{Hopf fibration.}
\]

Exercise: $\forall y \in \mathbb{P}^{n-1}$ $f^{-1}(y)$ is a 1-dimensional torus.

Definition: Let $X$ be a complex manifold. A submanifold of codim $k$ is a connected closed set $Y \subseteq X$.

- s.t. $\forall x \in X \subseteq \supset Y$ open and $f : U \rightarrow \mathbb{D}^{k}$ holom.
- s.t. $U \cap Y = f^{-1}(0)$

Example: Projective manifolds

Fix $f_{j}$ homogeneous polynomials of degree $d_{j}$ in $n+1$ variables.

Set $X := \{ x \in \mathbb{P}^{n} \mid f_{1}(x) = \ldots = f_{d}(x) = 0 \}$

NB: $f_{j}(x)$ is not a well-defined complex number ($f_{j}$ not a function)

However, $f_{j}(x) = 0$ well-defined since:

\[
f_{j}(\lambda x_{0}, \ldots, \lambda x_{n}) = \lambda^{d_{j}} f(x_{0}, \ldots, x_{n})
\]

In general $X$ is not a complex manifold (only a complex space)
Prop/Exrc: Consider \( f_1, \ldots, f_k \) as holomorphic functions on \( \mathbb{C}^{n+1} \).

If \(337,142) \( \forall z \in \mathbb{C}^{n+1}, \) \( \text{and } \exists \mathcal{L} \text{ such that } f(z) = 0 \)
then \( X \) is a complex manifold of co-dimension \( s \).

Ex: \( f = x_0^d + x_1^d + x_2^d \)
then \( X = \{ x \in \mathbb{R}^2 \mid f(x) = 0 \} \) is a proper submanifold.

Fun fact: if \( d = 3 \) then \( X \cong \mathbb{C}/\Lambda \)

Defn: Let \( X \) be a complex curve. A meromorphic function on \( X \)
is a \( \Phi \)-form \( f : X_0 \to \mathbb{C} \) where
i) \( X \setminus X_0 \) has only isolated points
ii) \( \forall z \in X \setminus X_0 \) \( \exists \mathcal{U} \subset X \text{ open and } h : \mathcal{U} \to \mathbb{C} \text{ holom.} \)
\( h(z) = \frac{\Phi(z)}{\Phi_0(z)} \)

Rem: If \( \lim_{z \to z_0} |f(z)| = \infty \) we call \( z_0 \) a pole of \( f \).

Generalise definition to arbitrary dimension by setting \( X \setminus X_0 \) is an analytic hypersurface.
Prop: Let $X$ be a complex curve and $f: X \to \mathbb{C}$ merom. For every pole $z_0 \in X$ we set $f(z_0) = \infty$. Then $f: X \to \mathbb{P}^1$ is a holomorphic map.

Proof: Only have to prove that $f$ is holomorphic near poles $z_0$.

By defn we have loc. $f(z) = \frac{g(z)}{h(z)}$ and since $\lim_{z \to z_0} |f(z)| = \infty$ we have $\frac{g(z)}{h(z)} \neq 0$.

Thus $f: (X \setminus \{z_0\} \to \mathbb{P}^1 \text{ homog. coord.}$

$z \mapsto [1 : \frac{g(z)}{h(z)}] = [h(z) : g(z)]$

$\Rightarrow f: X \to \mathbb{P}^1$ is given by $z \mapsto [h(z) : g(z)]$

and holomorphic in $z_0$ (use chart $\frac{x_0}{x_1} \to z$).

Fact: Let $X$ be a compact complex curve.

Then for every $z_0 \in X$ there exists $f: X \setminus \{z_0\} \to \mathbb{C}$ merom. s.t. $z_0$ is a pole of $f$.

In particular $X$ has "lots of" meromorphic functions.

NB: Not true in higher dimension.
Then: let \( f : X \to Y \) be a holomorphic map between complex curves. Fix \( z_0 \in X \). Then there exist local charts

\[
\varphi : U \to \mathbb{C} \quad \psi : V \to \mathbb{C}
\]

such that \( \psi \circ f \circ \varphi^{-1} : V \to \tilde{V} \) is given by

\[
2 \mapsto z^k
\]

We call \( k \) the multiplicity of \( f \) in \( z_0 \).

Proof: up to translating and restricting we can suppose \( \varphi(z_0) = 0 \), \( \varphi(f(z_0)) = 0 \).

\[
U \cong U_0 \quad \tilde{V} \cong V_0
\]

\[\Rightarrow \text{ get } \psi \circ f \circ \varphi^{-1} : U_0 \to \tilde{V}_0 \text{ holom. s.t. } \hat{\varphi}(0) = 0.\]

Then \( h(z) \neq 0 \) so \( h(\hat{\varphi}(z)) = \hat{g}(z) \) with \( g \) holom.

and \( z \mapsto z^k \) is holomorphic near 0.

Cor: In situation above, let \( w \in \tilde{V} \) be an arbitrary point. Then

\((\psi \circ f \circ \varphi^{-1})'(w) \) consists of \( k \) points if we count multiplicities.

Then: let \( f : X \to Y \) be a proper holomorphic map between complex curves. Fix \( w \in Y \).

Then for every \( w \in Y \), the preimage \( f^{-1}(w) \) consists of \( n \) points (in index of \( w \)) if we count multiplicities.