How to compute cohomology if $X$ not a polydisc?

Čech cohomology (only for $q = 1$, higher $q$ in my notes)

Let $X$ be a topd. space, $F$ a sheaf of abelian groups on $X$

\( \pi \)

Let \((U_i)_{i \in I} \) be an open covering.

We define

\[
C^0(M, F) = \prod_{i \in I} F(U_i)
\]

\[
C^1(M, F) = \prod_{i_0 < i_1} F(U_{i_0} \cap U_{i_1})
\]

\[
C^2(M, F) = \prod_{i_0 < i_1 < i_2} F(U_{i_0} \cap U_{i_1} \cap U_{i_2})
\]

and maps

\[
\partial : C^0(M, F) \to C^1(M, F)
\]

\[
\alpha = (\alpha_i)_{i \in I} \mapsto \partial \alpha,
\]

where

\[
(\partial \alpha)_{i_0 < i_1} = \alpha_{i_1} - \alpha_{i_0}
\]

\[
\delta : C^1(M, F) \to C^2(M, F)
\]

\[
\alpha = (\alpha_{i_0 i_1})_{i_0 < i_1} \mapsto \delta \alpha
\]

where

\[
(\delta \alpha)_{i_0 < i_1 < i_2} = \alpha_{i_0 i_2} - \alpha_{i_0 i_1} + \alpha_{i_1 i_2}
\]

Exercise: \( \delta \circ \partial = 0 \).
Def: The first Čech cohomology group of $\tilde{X}$ (with respect to the covering $U$) is

$$\check{H}^1(U,\tilde{X}) := \{ \alpha \in C^1(U,\tilde{X}) \mid d\alpha = 0 \}$$

Čech cochains

Example: $X$ complex manifold, $\tilde{X} = \mathbb{C}^*$, holom. funct. that do not vanish.

$$\alpha \in C^1(U,\mathbb{C}^*)$$
and

$$(\alpha_{i_0i_1})_{i_0 < i_1} \in G^\times_x(U_{i_0}\cap U_{i_1})$$

What does $d\alpha = 1$ mean?

$$\alpha = (d\alpha)_{i_0 < i_1} = \alpha_{i_0i_1} \cdot \alpha_{i_0i_2} \cdot \alpha_{i_1i_2} \bigg|_{U_{i_0} \cap U_{i_1} \cap U_{i_2}}$$

$$= \alpha_{i_0i_2}^* \cdot \alpha_{i_0i_1} \cdot \alpha_{i_1i_2} = \alpha_{i_0i_1} \cdot \alpha_{i_1i_2}$$

Cocycle relation

Then we get a map $\check{H}^1(U,\mathbb{C}^*) \to \text{Pic} X$ (see ex. sheet)

(Isomorphism if $U$ is aff. fine)

Then (comparison thm of de Rham–Weil + Leray's thm)

Let $X$ be a manifold and $U = (U_i)_{i \in I}$ be a covering
such that $H^p(U_i) = 0 \forall i \in I$. Then we have an isomorphism

$$\check{H}^p(U,\mathbb{C}^*) \cong H^p_{\text{dR}}(X)$$

Čech cohomology of $\mathbb{C}^*$ Dolbeault cohom.
"Proof": We construct a map \( H^p_x(X) \to H^q(U, \Omega^p_x) \).

A class in \( H^p_x(X) \) is represented by
\[
\alpha \in C^r_x(X, \Omega^p_x)\quad \text{s.t.} \quad d\alpha = 0.
\]

In particular, \( \overline{\partial}\alpha |_{U_c} = 0 \) \( \Rightarrow \exists \beta \in C^r(U_c, \Omega^p_x) \)
\[
\text{s.t.} \quad \alpha = \overline{\partial}\beta.
\]

On \( U_{i_0} \cap U_{i_1} \) we have \( \overline{\partial}\beta_{i_0} - \overline{\partial}\beta_{i_1} = 0 \)
\[
\Rightarrow (\beta_{i_0} - \beta_{i_1})|_{U_{i_0} \cap U_{i_1}} \quad \text{is holom. so \ element of} \quad \Omega^p_x(U_{i_0} \cap U_{i_1})
\]
\[
\Rightarrow \gamma = (\gamma_{i_0i_1}) \in C^r(U, \mathfrak{F}).
\]

Check the cocycle relation: on \( U_{i_0} \cap U_{i_1} \cap U_{i_2} \) we have
\[
\gamma_{i_1i_2} - \gamma_{i_0i_2} + \gamma_{i_0i_1} = \beta_{i_1} - \beta_{i_2} - (\beta_{i_0} - \beta_{i_2}) + \beta_{i_0} - \beta_{i_1} = 0
\]
\[
\Rightarrow \text{get} \ [\gamma] \in H^q(U, \Omega^p_x)
\]

Rem: By Dolbeault's lemma a covering \( U \) by polydiscs satisfies the condition of the theorem.

Rem: Analogous statements for \( H^q_x(X) \) \( q \) arbitrary
\[
H^q(U, \mathfrak{F}) \cong H^q_{\text{dR}}(X, \mathfrak{F})
\]

sheaf of \( \mathcal{C}_\mathbb{C} \) constant forms.
Example: We have \( H^q(M) = H^q(M, \Omega) = 0 \).

Proof: Standard covering \( c: U \rightarrow \mathbb{P}^1 \),

\[
E \quad \text{coord. change } z \mapsto \frac{1}{z} = \omega
\]

We have shown in exercises that \( H^0(M) = 0 \), \( \Rightarrow \) Thm. applies.

For this covering a cocycle \( \alpha_{i_0-i_1} \in C^1(U, \Theta) \) is simply

a holom. form \( f_{i_0} : U_0 \rightarrow C \), \( f_{i_1} : U_1 \rightarrow C \) holom. s.t. \( f_{i_1} = f_i - f_0 \).

\( f_{i_0} \) is given by Laurent series \( \sum_{n=0}^{\infty} c_n z^n \).

We set \( f_0 : U_0 \rightarrow C \), \( f_1 : U_1 \rightarrow C \) where

\[
f_0 = \sum_{n=0}^{\infty} c_n z^n \quad f_1 = \sum_{n=0}^{\infty} c_n \omega^n
\]

since \( \omega = \frac{1}{z} \) we then have the desired property. \( \square \)

Defn: Let \( X \) be a compact curve. We call \( \dim H^0(M) \)
the genus \( g(X) \).

Rem: Example shows \( X \cong \mathbb{P}^1 \implies g(X) = 0 \).

In \( \S 8 \) we show \( g(X) = 0 \implies X \cong \mathbb{P}^1 \).
§ 7 Hypersurfaces and line bundles

Intuition: A hypersurface is locally defined by a holomorphic, non-constant function. However, \( z = 0 \) and \( z^2 = 0 \) define the same set in \( \mathbb{C}^n \).

We need a precise notion of "defined by".

Definition: Let \( X \) be a complex manifold. A (reduced) complex subspace \( Y \) of \( X \) is a closed set \( Y \subseteq X \) s.t. for every open set \( U \supseteq Y \) in \( X \), the intersection \( U \cap Y \) is an analytic set.

We say that \( Y \) is a hypersurface if we can find an open covering \( (U_i)_{i \in I} \) of \( X \) by polydiscs s.t. \( \forall i \in I \), the ideal

\[
\mathfrak{I}(Y \cap U_i) = \{ f: U_i \to \mathbb{C} \text{ holom} \mid f(z) = 0 \ \forall z \in U_i \cap Y \}
\]

\( \mathcal{O}(U_i) = \text{holomorphic functions on } U_i \)

is principal, i.e., generated by a unique function \( f_i \).

We say that \( f_i \) defines \( Y \) on the open set \( U_i \).

Lemma: \( Y \) hypersurface \( \iff \) \( \dim_X Y = n \) \( \forall x \in Y \)

not trivial.

Proposition: Let \( X \) be a complex manifold and \( L \to X \) a holomorphic line bundle.

Let \( s: X \to L \) be a non-zero section, and set

\[
Y = \{ x \in X \mid s(x) = 0 \}
\]

Then \( Y \) is a hypersurface.
Proof: let \((U_i)_{i \in I}\) be a covering by polydiscs \(U_i\) that trivializes \(L\), i.e. \(L \mid u_i = u_i \times e\).

Then \(o \mid u_i : u_i \rightarrow L \mid u_i \cong u_i \times e \xrightarrow{p_2} e\) is holomorphic and \(Y \cap u_i = \mathbb{T} \circ (u_i) \quad \text{or} \quad Y \cap u_i\) is empty,

\(\Rightarrow Y\) is hypersurface \(\Box\)

Prop: let \(X\) be a complex manifold and \(Y \subset X\) a hypersurface.

Then \(\exists \mathbb{T} \rightarrow X\) line bundle and section \(s : X \rightarrow \mathbb{T}\)

s.t. \(Y = \mathbb{T}(s)\).

Proof: cover \(X\) by polydiscs \(U_i\) s.t. \(Y \cap u_i\) = \((f_i)\).

On \(U_i \cap U_j\) we have \((f_i \mid _{U_i \cap U_j}) = (f_j \mid _{U_i \cap U_j})\).

On \(U_i \cap U_j\) we can write

\[ f_i = g_{ij} f_j \quad \text{and} \quad f_j = g_{ij} f_i \]

and is holomorphic.

Moreover we have on \(U_i \cap U_k \cap U_x\) cyclic

\[ g_{ik} \cdot \frac{1}{g_{ik}} \cdot g_{ij} = f_i \cdot (f_k \cdot f_i)^{-1} \cdot f_j = 1 \quad \text{so} \quad (g_{ij})_{i \neq j}\]

\[ E_x: \quad x = 1p^4, \quad y = \{u \neq 0\} \]

On \( u_0 = \{u_0 \neq 0\} \), we have \( s(yu_0) = \{z = 0\} \)

\[ u_4 = \{u_4 \neq 0\} \quad \text{in} \quad s(yu_4) = \{x \neq 0\} \]

\[ x = \frac{3}{4} \quad \text{in} \quad z = \frac{u_4}{u_0} \quad \Rightarrow \quad \text{defn of } \theta(1) \]

\textbf{Rem:} The proposition seems to say that \( \exists \) bijection
\[ y \in X \text{ hypersurface } \iff \text{ line bundles } L \to X \text{ with section } \theta_0. \]

Not quite true, need to consider vanishing order of \( s \)

along \( x + \{s = 0\} \).
§ 8 Divisors on curves, Riemann-Roch.

Notation: A curve is a compact manifold of dim 1.

Defn: A divisor on a curve $X$ is a finite formal sum

$$D := \sum_{i=1}^{k} a_i P_i$$

where $P_i \in X$ is a point and $a_i \in \mathbb{Z}$.

The divisor is effective if all $a_i \geq 0$.

The degree of the divisor $D$ is $\deg D := \sum_{i=1}^{k} a_i$.

To a divisor $D$ we can associate a holomorphic line bundle.

cover $X$ with discs $D = U_\alpha$ s.t. every $U_\alpha$ contains at most one $P_i$. If $P_i \in U_\alpha$

$$V := \bigcup_{\alpha} U_\alpha$$

we consider the meromorphic function $f_\alpha: U_\alpha \to \mathbb{C}$

induced by $(z - P_i)^{a_i}$ (so $f_\alpha$ holomorphic) iff $a_i \geq 0$

since for $\alpha < \beta$ $U_\alpha \cap U_\beta$ never contains any point $P_i$

we obtain that $g_{\alpha \beta}: U_\alpha \cap U_\beta \to \mathbb{C}^*$

$$\left( \frac{f_\alpha}{f_\beta} \right) |_{U_\alpha \cap U_\beta}$$
is holomorphic.
As in §7 we see that \((\varphi_{\alpha \beta})_{\alpha < \beta}\) is a cocycle.

associate line bundle. on a compact curve \(X\)

Vice versa if \(L \rightarrow X\) holom. line bundle \(\mathcal{L}\) and \(s: X \rightarrow L\) section to we define a divisor \(D\) (associated to \(L\) and \(s\)) by \(p \in \mathcal{D}\) iff \(s(p) = 0\)

and \(\alpha_s = \text{vanishing order of } s \text{ in } p\).

NB: The divisor \(D\) depends on the choice of the section:

\[X = \mathbb{P}^1, \quad L = G^1(1)\]

sections correspond to linear forms \(ax_0 + bx_1\) vanishing in \((-b:a)\)

Prop: let \(X\) be a compact curve, \(L \rightarrow X\) line bundle

and \(s_1, s_2\) two non-zero sections of \(L\).

Denote by \(D_1, D_2\) the effective divisors associated \(s_1, s_2\)

Then we have \(\deg D_1 = \deg D_2\).

Proof: let \(U_\alpha\) be trivialising cover for \(L\) denote by \(s_i|_U: U_\alpha \rightarrow \mathbb{C}\) the holom. fcts corresponding \(s_i\).

then \(s_\alpha = \frac{s_1|_U}{s_2|_U}\) is merom. on \(U_\alpha\)

and on \(U_\alpha \cap U_\beta\) we have
\[\varphi_\lambda = \frac{\lambda_\alpha \alpha_\lambda}{\lambda_{\alpha}} = \frac{\lambda_\alpha \beta \lambda_{\beta}}{\lambda_{\beta}} = \frac{\lambda_\beta \lambda_{\alpha}}{\lambda_{\beta}} = \varphi_\beta.\]

\[\Rightarrow \text{ get morphism } \varphi: X \to \mathbb{C}\]

\[\text{ex. sheet } \Rightarrow \text{ get morphism } \varphi: X \to \mathbb{P}^1, \text{ and by construction } D_1 = \varphi^*(0) \quad \text{ex. sheet }\]

\[D_2 = \varphi^*(\infty) \quad \Rightarrow \deg D_1 = \deg \varphi = \deg D_2.\]
As in §7, we see that \((g_{\alpha\beta})_{\alpha\beta}\) is an isomorphism of the associated line bundle \(L(D)\).

**Definition:** Let \(X\) be a curve and \(D\) a divisor on \(X\). We denote by \(Q_x(D)\), the sheaf of sections of the line bundle \(L(D)\). We set

\[
H^0(X, D) = H^0(X, Q_x(D))
\]

\[
H^n(X, D) = H^n(X, Q_x(D))
\] for each cohomology sheaf \(Q_x(D)\).

Through Riemann-Roch

Let \(X\) be a compact curve and \(D\) a divisor on \(X\). Then the vector spaces \(H^i(X, Q_x)\) have finite dimension (over \(\mathbb{C}\))

\[
H^0(X, Q_x(D)) \quad H^1(X, Q_x(D))
\]

and we have

\[
h^0(X, Q_x(D)) - h^1(X, Q_x(D)) = 1 - g(x) + \deg D.
\]

\[
h^n(X, Q_x)
\]
Corollary: Let \( X \) be a compact curve s.t. \( g(X) = 0 \). Then \( X \cong \mathbb{P}^1 \).

Proof: Fix an arbitrary point \( x \in X \). Then \( D = x \) is a divisor of degree 1, so by RR

\[
h^0(X, O_D) - h^1(X, O_D) = 2. \implies h^0(X, O_D) \geq 2.
\]

\( O_D \) has a section vanishing exactly on \( x \) and " \( s_1 \) " on divisor \( D \neq x \).

Since \( \deg D_2 = \deg D = 1 \) we see that \( D_2 = x_2 \).

Since \( x \neq x_2 \) the sections \( s_1, s_2 \) define non-constant morphism \( \phi : X \rightarrow \mathbb{P}^1 \).

s.t. \( x = \phi^*(0) \), \( x_2 = \phi^*(\infty) \).

In particular \( \deg \phi = 1 \) so \( \phi \) is isomorphism. \( \square \)