The anticanonical system of Fano manifolds of index \( n - 3 \)
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Introduction

A Fano variety is an $n$-dimensional $\mathbb{Q}$-Gorenstein projective variety $X$ with ample anticanonical divisor $-K_X$. The index of a Fano variety $X$ is

$$i(X) = \sup \{ t \in \mathbb{Q} \mid -K_X \sim_{\mathbb{Q}} tH, \ H \text{ ample, Cartier} \}.$$ 

It is well known that $i(X) \leq n + 1$. If $X$ has log terminal singularities, the Picard group $\text{Pic}(X)$ is torsion free. Therefore the Cartier divisor $H$ such that $-K_X \sim_{\mathbb{Q}} i(X)H$ is determined up to isomorphism. It is called the fundamental divisor of $X$. By the Kawamata-Viehweg vanishing theorem,

$$h^i(X, H) = 0 \ \forall \ i > 0.$$ 

It is natural to investigate the existence of global sections for the fundamental divisor, that is the non vanishing of the group $H^0(X, H)$.

Once we know that $H^0(X, H) \neq 0$, the next natural question arising is what kind of singularities a general element in $|H|$ may have. It is said that a Fano variety has good divisors if the generic member of the fundamental divisor has at worst the same type of singularities as $X$.

In this work we give a detailed proof of a theorem taken from [7] that deals with the case of Fano varieties of dimension four:

**Theorem 0.0.1** (Kawamata). Let $X$ be a projective variety of dimension 4 with at most Gorenstein canonical singularities. Assume that $-K_X \sim H$ is ample. Then the following hold:

- $H^0(X, H) \neq 0$;
- let $Y \in |H|$ be a general member, then $(X, Y)$ is plt. Hence $Y$ has only canonical singularities.

The first point gives an answer to the problem of existence of global sections and the second to the problem of determining the singularities of the general member of the anticanonical system. The proof of the Theorem 0.0.1 is dealt with in the second Chapter.

In Chapter 3 we consider generalizations of Kawamata’s theorem to higher dimension, more precisely we study smooth Fano varieties of index
$n - 3$. By the Kobayashi-Ochiai criterion, if $i(X) \geq n$ then $X$ is isomorphic to a hyperquadric or to a projective space. Smooth Fano varieties of index $n - 1$ have been classified by Fujita and smooth Fano varieties of index $n - 2$ by Mukai. Then the varieties of index $n - 3$ are a class of varieties of “big” index far from being understood. The structure of a general element $Y \in |H|$ should play an important role in their classification. In this context, we propose the following theorem:

**Theorem 0.0.2.** Let $X$ be a smooth Fano variety of dimension $n \geq 4$ and index $n - 3$, with $H$ the fundamental divisor.

1. If $n \leq 5$ then $h^0(X, H) \geq n - 2$.

2. If $n = 6$ and the Picard number is one then $h^0(X, H) \geq n - 2$.

3. If $n \geq 7$ and the tangent bundle $T_X$ is $H$-semistable, then $h^0(X, H) \geq n - 2$.

4. Suppose that $h^0(X, H) \geq 1$ and let $Y \in |H|$ be a general member, then $(X, Y)$ is plt.

As in the statement of Theorem 0.0.1 the first points establish the non-vanishing of the global sections group and the last discuss the regularity of the general member.

The first three points of the theorem result from the following steps:

- first we prove the following formula

$$\chi(X, H) = \frac{H^n}{24} (-n^2 + 7n - 8) + \frac{c_2(X)H^{n-2}}{12} + n - 3.$$  

- If $n = 4, 5$ the result follows from the semipositivity of the tangent bundle $T_X$ [15] and the inequality of Miyaoka [13].

- If the tangent bundle $T_X$ is $H$-semistable, from the Bogomolov inequality we obtain $h^0(X, H) \geq n - 2$.

- finally in [5] Hwang proved that a Fano variety of dimension six and Picard number equal to one has semistable tangent bundle.

An important remark is that the semistability of the tangent bundle is conjectured for all Fano varieties of Picard number equal to one.

Finally, in the last Chapter, we discuss the following famous theorem

**Theorem 0.0.3** (Shokurov [17]). Let $X$ be a smooth Fano variety of dimension three. Then $Y \in \lvert -K_X \rvert$ general is smooth.

We give a simplified proof by using an argument taken from [4].
Chapter 1

Preliminaries

We will use the standard notation from [3], everything is defined over \( \mathbb{C} \). In the following \( \equiv, \sim \) and \( \sim_Q \) will indicate numerical, linear and \( \mathbb{Q} \)-linear equivalence of divisors.

**Definition 1.0.4.** [3, p. 241] Let \( X \) be a proper scheme of dimension \( n \). A dualizing sheaf for \( X \) is a coherent sheaf \( \omega_X^0 \) on \( X \), together with a trace morphism \( t: H^n(X, \omega_X^0) \to \mathbb{C} \), such that for all coherent sheaf \( F \) on \( X \), the natural pairing

\[
\text{Hom}(F, \omega_X^0) \times H^n(X, F) \to H^n(X, \omega_X^0)
\]

followed by \( t \) gives an isomorphism

\[
\text{Hom}(F, \omega_X^0) \to H^n(X, F).
\]

**Definition 1.0.5.** Let \( X \) be a nonsingular variety and \( \Omega_X \) its cotangent sheaf. The canonical sheaf of \( X \) is

\[
K_X^c = \det \Omega_X.
\]

**Corollary 1.0.6.** If \( X \) is a projective nonsingular variety, then the dualizing sheaf \( \omega_X^0 \) is isomorphic to the canonical sheaf \( K_X^c \).

**Theorem 1.0.7** (Serre duality). [3, p. 244] Let \( X \) be a scheme that is projective Cohen Macaulay of equidimension \( n \) over \( \mathbb{C} \). Then for any locally free sheaf \( F \) over \( X \) there are natural isomorphisms

\[
H^i(X, F) \cong H^{n-i}(X, F^\vee \otimes \omega_X^0).
\]

**Remark 1.0.8.** Let \( X \) be a normal variety, let \( K_{X_{\text{reg}}} \) be the canonical divisor of \( X_{\text{reg}} \). We can write \( K_{X_{\text{reg}}} = \sum a_i D_i \) where \( D_i \) is an irreducible subvariety of codimension one in \( X_{\text{reg}} \). In this case we define the canonical divisor of \( X \) as \( K_X = \sum a_i D_i \) where \( D_i \) is the closure in \( X \) of \( D_i \). Since \( X \) is nonsingular in codimension one, \( K_X \) is the only Weil divisor on \( X \) such that \( K_X|_{X_{\text{sing}}} = K_{X_{\text{sing}}} \).
Definition 1.0.9. A normal projective variety $X$ is said to be $\mathbb{Q}$-Gorenstein if the Weil divisor $K_X$ is $\mathbb{Q}$-Cartier.

Proposition 1.0.10 (Adjunction formula). [3, p. 182] Let $X$ be a normal projective variety and let $Y$ be an effective Cartier divisor. Let $\omega_X$ be the dualizing sheaf on $X$. Then the dualizing sheaf of $Y$ is given by the formula

$$\omega_Y \sim (\omega_X \otimes \mathcal{O}_X(Y))|_Y.$$ 

Theorem 1.0.11 (Bertini’s theorem). [3, p. 179] Let $X$ be a nonsingular closed subvariety of $\mathbb{P}_C^n$. Then there exists a hyperplane $H \subseteq \mathbb{P}_C^n$, not containing $X$ and such that the scheme $X \cap H$ is regular at every point. Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$, considered as a projective space.

Proposition 1.0.12. [2] Let $X$ and $X'$ be schemes and $f : X \to X'$ be a proper morphism. Let $L$ be a line bundle on a scheme $X$, let $\alpha$ be a k-cycle on $X'$. Then

$$f_*(c_1(f^*L) \cdot \alpha) = c_1(L) \cdot f_*(\alpha).$$

1.1 Positivity

Definition 1.1.1. [10, p. 24] Let $X$ be a projective variety, and $L$ a line bundle on $X$.

- $L$ is very ample if there exists a closed embedding of $X$ into some projective space $\mathbb{P}^N$ such that

$$L = \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X.$$ 

- $L$ is ample if $L^\otimes m$ is very ample for some $m > 0$.

A Cartier divisor $D$ is ample or very ample if the corresponding line bundle $\mathcal{O}_X(D)$ is so.

Theorem 1.1.2. [10, p. 33] Let $L$ be a line bundle on a projective scheme $X$. Then $L$ is ample if and only if

$$\int_V c_1(L)^{\text{dim}(V)} > 0$$

for every positive dimensional irreducible subvariety $V \subseteq X$ (including the irreducible components of $X$).

Definition 1.1.3. [10, p. 51] Let $X$ be a projective variety. A Cartier divisor $D$ on $X$ is nef if

$$(D \cdot C) \geq 0$$

for all irreducible curves $C \subseteq X$. 

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Definition 1.1.4. [15, p. 1] A vector bundle \(E\) on a projective manifold \(X\) of dimension \(n\) is generically nef if the following holds. Given any ample vector bundles \(H_j, 1 \leq j \leq n - 1\), let \(C\) be a curve cut out by general elements in \(|m_jH_j|\) for \(m_j \gg 0\), then the restriction \(E|_C\) is nef.

Theorem 1.1.5. [15, p. 14] Let \(X\) be a projective manifold with \(-K_X\) semiample. Then \(T_X\) is generically nef.

Theorem 1.1.6. [13, p. 468] Let \(X\) be a normal projective variety of dimension \(n\) smooth in codimension two. Let \(E\) be a torsion free sheaf on \(X\), with first Chern class being a numerically effective \(\mathbb{Q}\)-Cartier divisor. Assume that \(E\) is generically nef, then for every \(h_1, \ldots, h_{n-1}\) ample divisors on \(X\) the inequality
\[c_2(E)h_1 \cdots h_{n-1} \geq 0\]
holds.

Theorem 1.1.7. [13, p. 468] Let \(X\) be a normal projective variety of dimension \(n\) and \(\rho: X' \to X\) an (arbitrary) resolution. Assume that \(X\) is smooth in codimension two and the canonical divisor is \(\mathbb{Q}\)-Cartier and numerically effective. Then, for any numerically effective Cartier divisors \(H_1, \ldots, H_{n-2}\), the inequality
\[c_2(X') (\rho^* H_1) \cdots (\rho^* H_{n-2}) \geq 0\]
holds.

1.2 Singularities of pairs

Definition 1.2.1. [11, p. 143] Let \(D = \sum D_i\) be a divisor on a manifold \(X\). It is said to have simple normal crossings (and \(D\) is said a SNC divisor) if each \(D_i\) is smooth, and \(D\) is defined in a neighborhood of any point by an equation in local analytic coordinates of type
\[z_1 \cdots z_k = 0\]
for some \(k \leq n\). A \(\mathbb{Q}\)-divisor \(D = \sum a_iD_i\) has simple normal crossing support if \(\sum D_i\) is SNC.

Definition 1.2.2. [9, p. 5] Let \(D = \sum a_iD_i\) be a \(\mathbb{Q}\)-divisor on a variety \(X\). A log resolution of \(D\) (or of the pair \((X, D)\)) is a projective birational morphism
\[\mu: X' \to X,\]
such that \(X'\) non singular, the exceptional locus \(\text{Exc}(\mu)\) has codimension one and the divisor \(\mu^* D + \text{Exc}(\mu)\) has simple normal crossing support. Here \(\text{Exc}(\mu)\) denotes the sum of all the exceptional divisors of \(\mu\).
Definition 1.2.3. [11, p. 182] A pair \((X, \Delta)\) consists of a normal variety \(X\), together with a Weil \(\mathbb{Q}\)-divisor \(\Delta = \sum a_i \Delta_i\) on \(X\) with \(a_i \in (0, 1]\), such that the \(\mathbb{Q}\)-divisor \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier on \(X\).

Definition 1.2.4. [10, p. 183] The multiplier ideal of the divisor \(D\) for the pair \((X, \Delta)\) is the sheaf

\[
\mathcal{J}((X, \Delta), D) = \mu_* \mathcal{O}_{X'}(K_{X'} - \mu^*(K_X + \Delta + D))
\]

Theorem 1.2.5 (Nadel vanishing theorem). [11, p. 191] Let \((X, \Delta)\) be a pair and let \(D\) be a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(X\).

- If \(\mu : X' \rightarrow X\) is a log resolution of \((X, D + \Delta)\), then
  \[
  R^j \mathcal{O}_{X'}(K_{X'} - \mu^*(K_X + \Delta + D)) = 0 \quad \forall j > 0.
  \]
- Let \(N\) be any integral Cartier divisor on \(X\) such that \(N - (K_X + \Delta + D)\) is nef and big. Then
  \[
  H^j(X, \mathcal{O}_X(N) \otimes \mathcal{J}((X, \Delta), D)) = 0 \quad \forall i > 0.
  \]

Definition 1.2.6. [6, p. 16] Let \((X, \Delta)\) be a pair, \(\Delta = \sum a_i \Delta_i\) with \(a_i \in \mathbb{R}\). Suppose that \(m(K_X + \Delta)\) is Cartier for \(m > 0\). Let \(f : Y \rightarrow X\) be a birational morphism, \(Y\) normal. We can write

\[
K_Y \equiv f^*(K_X + \Delta) + \sum a(E_i, X, \Delta)E_i.
\]

where \(E_i \subseteq Y\) are distinct prime divisors and \(a(E_i, X, \Delta) \in \mathbb{R}\). Furthermore we adopt the convention that a nonexceptional divisor \(E\) appears in the sum if and only if \(E = f_{*}^{-1}D_i\) for some \(i\) and then with coefficient \(a(E, X, \Delta) = -a_i\).

The \(a(E_i, X, \Delta)\) are called discrepancies.

Definition 1.2.7. [9, p. 52] Let \((X, \Delta)\) be a pair. We set

\[
\text{discrep}(X, \Delta) = \inf \{ a(E, X, \Delta) \mid \text{E exceptional divisor over } X \}
\]

and

\[
\text{totaldiscrep}(X, \Delta) = \inf \{ a(E, X, \Delta) \mid \text{E divisor over } X \}.
\]

Lemma 1.2.8. [9, p. 53] Let \((X, \Delta)\) be a pair, then either \(\text{discrep}(X, \Delta) = -\infty\) or \(-1 \leq \text{totaldiscrep}(X, \Delta) \leq \text{discrep}(X, \Delta) \leq 1\).

Definition 1.2.9. [9, p. 56] A pair \((X, \Delta)\) is defined to be

- klt (kawamata log terminal) if \(\text{discrep}(X, \Delta) > -1\) and \(|\Delta| \leq 0\)
- plt (purely log terminal) if \(\text{discrep}(X, \Delta) > -1\)
- lc (log canonical) if \( \text{discrep}(X, \Delta) \geq -1 \).

**Remark 1.2.10.** [11, p. 165] Since \( \mathcal{J}(X, \Delta) = \mathcal{O}_X \) if and only if
\[
\text{ord}_E(K_X - \mu^*(K_X + \Delta)) > -1 \quad \forall \ E \subseteq X',
\]
the pair \( (X, \Delta) \) is klt if and only if \( \mathcal{J}(X, \Delta) = \mathcal{O}_X \) and \( (X, \Delta) \) is lc if and only if \( \mathcal{J}(X, (1 - \varepsilon)\Delta) = \mathcal{O}_X \) for all \( 0 < \varepsilon < 1 \) and \( \mathcal{J}(X, \Delta) \neq \mathcal{O}_X \).

**Lemma 1.2.11.** [9, p. 57] Let \( (X, \Delta) \) be a pair and \( \Delta' \) an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor. If \( (X, \Delta) \) is klt, then \( (X, \Delta + \varepsilon \Delta') \) is klt for \( 0 \leq \varepsilon < 1 \).

In the klt case the Nadel theorem gives the Kawamata-Viehweg vanishing theorem.

**Corollary 1.2.12.** Let \( (X, \Delta) \) be a klt pair. Let \( N \) be any integral Cartier divisor on \( X \) such that \( N - (K_X + \Delta) \) is nef and big. Then
\[
H^i(X, \mathcal{O}_X(N)) = 0 \quad \forall \ i > 0.
\]

**Remark 1.2.13.** Let \( (X, \Delta) \) be a klt pair and let \( A \) be an ample line bundle on \( X \). Suppose that \( h^0(X, A) \neq 0 \), then \( D \in |A| \) is connected. In fact, if we consider the exact sequence of cohomology groups associated to the exact sequence of sheaves
\[
0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,
\]
by the Theorem 1.2.12 he have \( H^1(X, \mathcal{O}_X(-D)) = 0 \). Thus we have an exact sequence
\[
0 \to H^0(X, \mathcal{O}_X(-D)) \to H^0(X, \mathcal{O}_X) \to H^0(D, \mathcal{O}_D) \to 0.
\]
We know that \( H^0(X, \mathcal{O}_X) \) is the set of the constant functions on \( X \), and \( H^0(X, \mathcal{O}_X(-D)) \subseteq H^0(X, \mathcal{O}_X) \) is the set of constant functions on \( X \) that are zero on \( D \) and so \( H^0(X, \mathcal{O}_X(-D)) = \{0\} \) and \( h^0(D, \mathcal{O}_D) = h^0(X, \mathcal{O}_X) = 1 \). Thus \( D \) is connected.

**Definition 1.2.14.** [6, p. 61] A variety \( X \) is said to have rational singularities if, for any \( \mu : X' \to X \) resolution of singularities, we have
\[
R^i\mu_*\mathcal{O}_{X'} = 0 \quad \forall \ i > 0.
\]

**Theorem 1.2.15** (Elkik, Flenner). [6, p. 61] Let \( X \) be a normal variety. Assume that \( (X, \Delta) \) is a klt pair. Then \( X \) has rational singularities.

**Remark 1.2.16.** As a direct consequence of Theorem 1.2.15, if \( X \) is a variety with rational singularities, we can compute the cohomology groups of a Cartier divisor on \( X \) on a smooth variety. Let \( L \) be a Cartier divisor on \( X \), \( \mu : X' \to X \) a resolution of singularities. By the Leray spectral sequence, we have
\[
H^i(X', \mu^*L) = H^i(X, L) \quad \forall \ i \in \{0, \ldots, n\}.
\]
**Definition 1.2.17.** Let \((X, \Delta)\) be a klt pair, \(D\) an effective \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor. The log canonical threshold of \(D\) for \((X, \Delta)\) is

\[ \operatorname{lct}(X, \Delta, D) = \sup \{ t \in \mathbb{R}^+ | (X, \Delta + tD) \text{ is LC} \}. \]

**Remark 1.2.18.** The threshold is also

\[ \operatorname{lct}(X, \Delta, D) = \sup \{ t | (X, \Delta + tD) \text{ is klt} \}. \]

We define a pair to be properly log canonical if it is LC and not klt. Thus if \(c = \operatorname{lct}(X, \Delta, D)\), the pair \((X, \Delta + cD)\) is properly log canonical.

**Remark 1.2.19.** [11, p. 166] The log canonical threshold is a rational number and the supremum appearing in the definition is actually a maximum. In fact let \((X, \Delta)\) be a pair and \(D \geq 0\) a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor. Let \(f : X' \to X\) be a log resolution of \(D\) and write

\[ f^*(K_X + \Delta + tD) = f^*(K_X + \Delta) + tf^*D = K_{X'} + B_{\Delta} + tB_D. \]

We write \(B_{\Delta} + tB_D = \sum_{i \in I} (b_i + ta_i)E_i + \tilde{D}\) where \(\tilde{D}\) is sum of non exceptional divisors and \(E_i\) is exceptional for all \(i \in I\), with \(I\) a finite set. The coefficient \(b_i\) refers to \(B_{\Delta}\) and \(a_i\) to \(B_D\). Since \((X, \Delta)\) is klt, we have \(b_i < 1\) for all \(i \in I\). We have

\[ b_i + ta_i \leq 1 \Leftrightarrow t \leq \frac{1 - b_i}{a_i} \]

and \(c = \min_{i \in I} \left\{ \frac{1-b_i}{a_i} \right\} \in \mathbb{Q}\).

**Definition 1.2.20.** Let \((X, \Delta)\) be a pair that is lc, \(f : X' \to X\) a log canonical resolution. Let \(E \subseteq X'\) be a divisor on \(X'\) of discrepancy \(-1\). Such a divisor is called a log canonical place. The closure of \(f(E)\) is called center of log canonicity of the pair \((X, \Delta)\) and is denoted \(\text{Center}_X(E)\). If we write

\[ K_{X'} \equiv \mu^*(K_X + \Delta) + E, \]

we equivalently can define a place as an irreducible component of \([-E]\). We denote \(\text{CLC}(X, \Delta)\) the set of all centers.

**Remark 1.2.21.** If \((X, \Delta)\) is log canonical,

\[ \{ x \mid (X, \Delta) \text{ is not klt near } x \} = \cup_{E} \text{Center}_X(E) \]

where the union runs over the places.

**Proposition 1.2.22.** Let \((X, \Delta)\) be a klt pair and \(D\) an effective \(\mathbb{Q}\)-divisor \(\mathbb{Q}\)-Cartier divisor such that \((X, \Delta + D)\) is log canonical. Then \(\text{CLC}(X, D)\) is a finite set and if \(W_1, W_2 \in \text{CLC}(X, D)\), all the irreducible components of \(W_1 \cap W_2\) are in \(\text{CLC}(X, D)\).
Lemma 1.2.23. Let $X$ be a normal variety and $\Delta$ a divisor on $X$ such that $(X, \Delta)$ is klt. Let $H$ be an ample Cartier divisor on $X$ and $Y \in |H|$ a general element. Suppose that $(X, \Delta + Y)$ is not plt and let $c$ be the log canonical threshold. Then the union of all the centers of log canonicality of $(X, \Delta + cY)$ is contained in the base locus of $|H|$. 

Proof. Let $\mu: X' \to X$ be a log resolution of the pair $(X, \Delta)$. We have

$$K_{X'} \equiv \mu^*(K_X + \Delta) + \sum_{i \in I} a_i E_i \quad \text{and} \quad a_i > -1 \ \forall \ i \in I.$$ 

If $h^0(X, H) = 1$ the statement is trivial so we can suppose $h^0(X, H) > 1$. If $h^0(X, H) > 1$, then by Remark 1.2.16, we have $h^0(X', \mu^*H) > 1$. By Bertini’s Theorem 1.0.11 there exists $\mu^*Y \in |\mu^*H|$ that meets properly and transversally all the $E_i$ that are not contained in the base locus. Then $\mu^*Y = Y' + F$, where $\mu(F) \subseteq Bs|H|$. We can write

$$K_{X'} \equiv \mu^*(K_X + \Delta + Y) - Y' + \sum a_i E_i - F.$$ 

The divisor $-Y' + \sum a_i E_i$ has simple normal crossing support on $\mu^{-1}(X\setminus Bs|H|)$ because of the choice of $Y$. If we set $X'_0 = \mu^{-1}(X\setminus Bs|H|)$, we have

$$K_{X'_0} \equiv (\mu^*(K_X + \Delta + Y) - Y' + \sum a_i E_i|_{X'_0}).$$ 

Then $\mu$ is a log resolution of the pair $(X\setminus Bs|H|, \Delta + Y\setminus Bs|H|)$, which is plt. Finally, if $(X\setminus Bs|H|, \Delta + Y\setminus Bs|H|)$ is plt, so is $(X\setminus Bs|H|, \Delta + cY\setminus Bs|H|)$.

Remark 1.2.24. Let $(X, \Delta)$ be a properly lc pair. Let $Z$ the union of all centers in $X$. Let $\mu: X' \to X$ be a birational morphism such that $X'$ is smooth,

$$K_{X'} = \mu^*(K_X + \Delta) - E - F,$$

the coefficients of $F$ are smaller than one and $\text{Supp}(E + F)$ is a simple normal crossing divisor. Then

$$\mathcal{J}(X, \Delta) = \mu_* \mathcal{O}_{X'}(K_{X'} - [\mu^*(K_X + \Delta)])$$

$$= \mu_* \mathcal{O}_{X'}(K_{X'} - [K_{X'} + E + F])$$

$$= \mu_* \mathcal{O}_{X'}(-E - |F|)$$

$$= \mu_* \mathcal{O}_{X'}(-E) = \mathcal{I}_Z.$$ 

Definition 1.2.25. Let $(X, \Delta)$ be a log canonical pair. A center $W$ is said to be exceptional if there exists a log resolution $\mu: X' \to X$ for the pair $(X, \Delta)$ such that:

- there exists only one place $E_W \subseteq X'$ whose image in $X$ is $W$;
- for all place $E' \neq E_W$, we have $\mu(E) \cap W = \emptyset$. 

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1.2.1 Tie break lemmas

**Theorem 1.2.26.** [1, p. 71] Let \((X, \Delta)\) be a klt pair and \(D\) an effective \(\mathbb{Q}\)-divisor \(\mathbb{Q}\)-Cartier such that \((X, \Delta + D)\) is properly lc. Let \(W\) be a minimal center for the pair \((X, \Delta + D)\) and \(H\) an ample Cartier divisor on \(X\). For all rational number \(0 < r \ll 1\) there exist \(c_1, c_2 \in \mathbb{Q}\), \(0 < c_1, c_2 \leq r\) and an effective \(\mathbb{Q}\)-divisor \(A \sim_{\mathbb{Q}} c_1 H\) such that the pair \((X, \Delta + (1 - c_2)D + A)\) is log canonical and \(W\) is an exceptional center of log canonicity for it.

**Theorem 1.2.27** (Kawamata, [8]). Let \((X, D)\) be a lc pair and \(W\) an exceptional center. Let \(H\) be an ample divisor and \(\varepsilon > 0\) a rational number. Then \(W\) is normal and there is an effective \(\mathbb{Q}\)-divisor \(B_W\) on \(W\) such that

- \((W, B_W)\) is a klt pair;
- \(K_W + B_W \sim_{\mathbb{Q}} (K_X + D + \varepsilon H)|_W\).

**Theorem 1.2.28.** [6, p. 263] Let \(X\) be a normal variety and \(S \subseteq X\) an irreducible Cartier divisor. Let \(B\) be an effective \(\mathbb{Q}\)-divisor and assume that \(K_X + S + B\) is \(\mathbb{Q}\)-Cartier. Then \((X, S + B)\) is plt near \(S \Leftrightarrow (S, B|_S)\) is klt.

Assume in addition that \(B\) is \(\mathbb{Q}\)-Cartier and \(S\) is klt. Then \((X, S + B)\) is LC near \(S \Leftrightarrow (S, B|_S)\) is LC.

### 1.3 Riemann Roch Theorems

**Theorem 1.3.1** (Riemann Roch). Let \(X\) be a nonsingular projective variety of dimension 3, with Chern classes \(c_1(X), c_2(X)\). Then for any Cartier divisor \(D\),

\[
\chi(X, D) = \frac{1}{12} D(D - K_X)(2D - K_X) + \frac{1}{12} Dc_2(X) + \frac{1}{24} c_1(X)c_2(X)
\]

\[= \frac{1}{6} D^3 - \frac{1}{4} D^2 K_X + \frac{1}{12} D(K_X^2 + c_2(X)) + \frac{1}{24} c_1(X)c_2(X)\]

**Remark 1.3.2.** We can define the intersection product \((D_1 \cdot \ldots \cdot D_n)\) if one of the divisors is just a Weil divisor. In fact, suppose that \(D_1\) is an irreducible hypersurface and \(D_2 \ldots D_n\) are Cartier divisors, then

\[
(D_1 \cdot \ldots \cdot D_n): = (D_2|_{D_1} \cdot \ldots \cdot D_n|_{D_1})
\]
**Lemma 1.3.3.** Let $X$ be a normal projective variety of dimension 3 with at most rational singularities.
Then for any Cartier divisor $D$,
\[ \chi(X,tD) = \frac{t^3}{6}D^3 - \frac{t^2}{4}D^2K_X + at + \chi(X,O_X) \]
for a suitable rational number $a$.

**Proof.** First we remark that the above expression makes sense by Remark 1.3.2.
Let $\mu: X' \to X$ be a resolution of singularities, and set $\bar{D} = \mu^*D$. Then
\[ \chi(X,tD) = \chi(X',t\bar{D}) \quad \forall t \in \mathbb{N} \]
by Remark 1.2.16. By Theorem 1.3.1 we have
\[ \chi(X',t\bar{D}) = \frac{t^3}{6}\bar{D}^3 - \frac{t^2}{4}\bar{D}^2K_{X'} + \frac{t}{12}\bar{D}(K_{X'}^2 + c_2(X')) + \frac{1}{24}c_1(X')c_2(X'). \]
Thus we have
\[ a = \frac{D(K_{X'}^2 + c_2(X'))}{12} \]
\[ \frac{1}{24}c_1(X')c_2(X') = \chi(X',O_{X'}) = \chi(X,O_X), \]
where the first equality is by the Riemann-Roch formula in the smooth case and the first is by the Theorem 1.2.15. Since $\mu$ is birational, $\bar{D}^3 = D^3$. So it remains to show that $\bar{D}^2K_{X'} = D^2K_X$. We have $\bar{D} = \mu^*(D)$ by definition and $\mu_*K_{X'} = K_X$, where $\mu_*K_{X'}$ is the push forward of $K_{X'}$ as a cycle. By the Projection formula 1.0.12 we have $\mu^*(D)^2K_{X'} = D^2\mu_*K_{X'}$. \qed

### 1.4 Stability

**Definition 1.4.1.** Let $X$ be a projective manifold and $\mathcal{E}$ a nonzero torsion-free coherent sheaf on $X$. Let $H$ be an ample divisor on $X$. We define the slope $\mu(\mathcal{E})$ with respect to $H$ by
\[ \mu(\mathcal{E}) = \frac{c_1(\mathcal{E})H^{n-1}}{\text{rk}(\mathcal{E})} \in \mathbb{Q}. \]

The bundle $\mathcal{E}$ is said to be $H$-semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for every nonzero subsheaf $\mathcal{F} \subseteq \mathcal{E}$.
The bundle $\mathcal{E}$ is said to be $H$-stable if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for every subsheaf $\mathcal{F} \neq 0, \mathcal{E}$.

The following theorem is a corollary of the Bogomolov inequality on smooth surfaces.
Theorem 1.4.2. [14, p. 69] Let $X$ be a projective manifold and let $H$ be
an ample divisor on $X$. Let $E$ a vector bundle of rank $r$ on $X$. Then
\[ c_2(E)H^{n-2} \geq \frac{r - 1}{2r} c_1(E)^2 H^{n-2}. \]
Chapter 2

A theorem of Kawamata

In this chapter we give a detailed proof of the following result:

**Theorem 2.0.3** (Kawamata, [7]). Let $X$ be a projective variety of dimension four with at most Gorenstein canonical singularities. Assume that $H \sim -K_X$ is ample. Then the following hold:

- $H^0(X,H) \neq 0$;
- let $Y \in |H|$ be a general member, then $(X,Y)$ is plt. Hence $Y$ has only canonical singularities.

Let us start with some preliminary results.

**Lemma 2.0.4.** Let $X$ be a normal projective Gorenstein variety of dimension $n$ et least two and let $Y$ be an effective Cartier divisor such that $(X,Y)$ is plt. Then $Y$ is irreducible.

**Proof.** We show that $Y$ is irreducible by induction on $\dim X$.

Suppose $\dim X = 2$.

We have $(X,Y)$ plt and $Y = \sum_{i=1}^{k} D_i$. Suppose $k \geq 2$. From the Remark 1.2.13 we know that $Y$ is connected. Then there are $D_1, D_2$ such that $D_1 \cap D_2 \neq \emptyset$ and let $Z = D_1 \cap D_2$, $\dim Z = 0$. There are two cases.

**Case 1** Suppose that $Z \cap X^{\text{smooth}} \neq \emptyset$.

Let $\pi: X' \to X$ be the blow up of $p \in Z \cap S^{\text{smooth}}$. Thus

$$K_{X'} = e^*(K_X + Y) + (1 - m_1 - m_2)E$$

where $E$ is the exceptional divisor, $m_i = \text{mult}_pD_i$, $m_i > 0$. So $1 - m_1 - m_2 \leq -1$ and $(X,Y)$ cannot be plt.

**Case 2** Suppose that $Z \cap X^{\text{smooth}} = \emptyset$.

Let $\pi: X' \to X$ be the minimal resolution for the surface. Then

$$K_{X'} \sim_{Q} \pi^*K_X - \sum b_iF_i$$
with $b_i \geq 0$. Moreover we have
\[ \pi^* Y = Y' + \sum c_i F_i \]
and, if $\pi(F_i) \in \mathbb{Z}, c_i > 0$. Observe that, since $Y$ is Cartier, $c_i$ is integer for all $i$. Let $\mu: X'' \to X$ be a log resolution of $(X, Y)$ obtained by composing $\pi$ with a log resolution $\lambda$ of the pull back of $Y$ in $X'$. Then we have
\[ K_{X''} \sim_{\mathbb{Q}} \mu^*(K_X + Y) - \sum D'_i - \sum (b_i + c_i) F'_i + \sum a_i E_i \]
where $E_i$ is $\lambda$-exceptional and $F'_i$ is the strict transform of $F_i$. But there is some $i$ such that $c_i \geq 1$ so that $-(b_i + c_i) \leq -1$. This is a contradiction with the fact that $(X, Y)$ is plt.

Suppose now dim $X > 2$. Let $\mu: X' \to X$ be a log resolution for the pair $(X, Y)$. We have
\[ K_{X'} = \mu^*(K_X + Y) + \sum_{i \in I} a_i E_i \text{ and } a_i \geq -1 \ \forall \ i \in I, \]
and $a_i > -1$ if $E_i$ is an exceptional divisor. Since $Y$ is integral, for the non exceptional divisors we have $a_i = -1$. We have $Y = \sum_{i=1}^k D_i$ and we want to prove that $k = 1$. Let $H$ be an hyperplane section that meets properly all the irreducible components of $\mu(\text{Exc}(\mu))$. Then $\mu^*H$ coincides with the strict transform $H'$ and
\[ K_{X'} + H' = \mu^*(K_X + Y + H) + \sum a_i E_i. \]

From the Adjunction formula 1.0.10 we get
\[ K_{H'} = \mu^*(K_H + Y|_H) + \sum a_i(E_i \cap H). \]
The divisor $Y|_H$ is ample and then connected by Remark 1.2.13. Since $H$ is ample, $H \cap D_i \neq \emptyset$ for all $i \in I$. The pair $(H, Y|_H)$ is plt and then we can apply the induction hypothesis, $Y|_H$ is irreducible and $k = 1$. \hfill $\square$

**Lemma 2.0.5.** Let $X$ be a normal projective Gorenstein variety and $Y$ an effective Cartier divisor such that $(X, Y)$ is plt. Then $Y$ is Gorenstein canonical.

**Proof.** From Lemma 2.0.4 we know that $Y$ is irreducible. Then we can apply the Proposition 1.2.28 with $S = Y$, $B = 0$. We obtain that $Y$ is klt. This means that, if $\mu: Y' \to Y$ is a resolution of singularities for $Y$, then we have
\[ K_{Y'} = \mu^*K_Y + E \]
and all the discrepancies, that appear in the decomposition of $E$ as sum of irreducible divisors, are smaller than one. But $E = K_{Y'} - \mu^*K_Y$ is sum of Cartier divisors, so it is integer and all its coefficients are at most one. Thus $Y$ is canonical. \hfill $\square$
We need the following result for the proof of Kawamata’s theorem.

**Theorem 2.0.6.** Let $X$ be a normal projective variety, $B$ an effective $\mathbb{Q}$-divisor on $X$ such that the pair $(X, B)$ is klt, and $D$ a Cartier divisor on $X$. Assume that $D$ is nef, $D - (K_X + B)$ is nef and big and $D^3 = 0$.

Then $H^0(X, D) \neq 0$.

For a proof of the Theorem 2.0.6 we refer to [7].

**Lemma 2.0.7.** Let $X$ be a normal Gorenstein projective variety and $Y$ an effective Cartier divisor. Suppose that $(X, Y)$ is a non plt pair with and let $c$ be its log canonical threshold. If all the minimal centers of $(X, cY)$ have codimension one, then $c \leq 1/2$.

**Proof.** A pair $(X, Y)$ is not plt if there exists a log canonical resolution $\mu: X' \to X$ such that $\mu^*(K_X + Y) = K_{X'} + B_Y$ and one of these two cases occurs

- there exists an exceptional divisor on $X'$ whose discrepancy is at least one;
- there exists a non exceptional divisor on $X'$ whose discrepancy is strictly bigger than one.

A non exceptional divisor is a divisor which appears in the expression of $Y$ as sum of irreducible divisors. Moreover, since $Y$ is integer, the second case can be rephrased writing that there exists a non exceptional divisor on $X'$ whose discrepancy is at least two. If all centers have codimension one these centers are not the image of any exceptional divisor. Then we are always in the second case, so there exists a divisor over $X$ whose discrepancy is at least two and $c \leq 1/2$. \[\Box\]

**Proof of the Theorem 2.0.3.**

Let $m$ be the smallest positive integer such that $H^0(X, mH) \neq 0$. We take a general member $Y \in |mH|$. The proof consists in showing that the two possibilities

1. $(X, Y)$ plt and $m > 1$,
2. $(X, Y)$ not plt

are not possible. Hence the only possibility is that $(X, Y)$ is plt and $m = 1$. In this case, Lemma 2.0.5 implies that $Y$ is Gorenstein canonical and the Adjunction formula 1.0.10 says that

$$K_Y \sim (K_X + Y)|_Y \sim 0.$$
**First case:** assume that \((X, Y)\) is plt and \(m > 1\). Then by Lemma 2.0.5, \(Y\) is Gorenstein canonical. We have

\[
K_Y \sim (K_X + Y)|_Y \sim (m - 1)H|_Y,
\]

so from the standard exact sequence

\[
0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0
\]

we obtain

\[
0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X((m - 1)H) \to \mathcal{O}_Y(K_Y) \to 0.
\]

We compute now the Euler characteristics of the three sheaves. From the Theorem 1.2.5, since \(H\) is ample, we get

\[
h^i(X, -H) = 0 \quad \forall i < 4.
\]

Serre duality implies

\[
h^i(X, -H) = h^i(X, K_X) = h^0(X, \mathcal{O}_X) = 1
\]

and \(\chi(X, -H) = 1\). Again by Theorem 1.2.12,

\[
h^i(X, (m - 1)H) = h^i(X, mH + K_X) = 0 \quad \forall i > 0
\]

because \(mH\) is ample, moreover \(h^0(X, (m - 1)H) = 0\) by minimality of \(m\). So \(\chi(X, (m - 1)H) = 0\).

Finally we show that \(\chi(Y, K_Y) = 0\). Let \(\rho: \tilde{Y} \to Y\) be a resolution of singularities of \(Y\). Let \(E\) be a divisor on \(\tilde{Y}\) such that

\[
K_{\tilde{Y}} = \rho^*K_Y + E.
\]

By [13, p. 473] there exists a resolution such that \(\rho(E)\) is a finite set, so that

\[
c_2(\tilde{Y})E = 0 \quad \text{and} \quad \rho^*K_{\tilde{Y}} \cdot K_{\tilde{Y}}^2 = \rho^*K_{\tilde{Y}}^3 = K_{\tilde{Y}}^3.
\]

If we apply the Formula 1.3.3, we obtain

\[
\chi(Y, K_Y) = \frac{K_Y^3}{6} - \frac{K_Y^3}{4} + \frac{\rho^*K_Y^2}{12} (K_{\tilde{Y}}^2 + c_2(\tilde{Y})) + \chi(Y, \mathcal{O}_Y)
\]

\[
= \frac{1}{12} (\rho^*K_Y c_2(\tilde{Y})) + \chi(Y, \mathcal{O}_Y).
\]

The equality \(\chi(Y, K_Y) = -\chi(Y, \mathcal{O}_Y)\) gives the formula

\[
\chi(Y, K_Y) = \frac{1}{24} (\rho^*K_Y c_2(\tilde{Y})).
\]
Since $K_Y$ is ample on $Y$, $\rho^*K_Y$ is nef on $\tilde{Y}$. Then we can apply the Theorem 1.1.7 that says that $\rho^*K_Y|_{\tilde{Y}} \geq 0$. Thus $\chi(Y, K_Y) \geq 0$. So we get a contradiction because the exactness of the sequence implies

$$\chi(X, -H) + \chi(Y, K_Y) = \chi(X, (m - 1)H).$$

**Second case:** assume that $(X, Y)$ is not plt and $m \geq 1$.

Let $c$ be the log canonical threshold of the pair $(X, Y)$, $c = \text{lct}(X, Y)$. We have $c \leq 1$ because $(X, Y)$ is not plt and so not klt. The pair $(X, cY)$ is properly LC, so $\text{CLC}(X, cY) \neq \emptyset$.

Let $W$ be a minimal center. We apply Theorem 1.2.26 with $\Delta = 0$, $H$, $D = cY$ and $r \ll 1$. We obtain a log canonical pair $(X, (1 - c_2)cY + A)$ where $A \sim_{\mathbb{Q}} c_1H$ and $W$ is an exceptional center for this pair. Then we apply Theorem 1.2.27 to $(1 - c_2)cY + A$, with $H$ as the ample divisor and $\varepsilon \ll 1$. Let $c' = (1 - c_2)c$, $\eta = \varepsilon + c_1$. There exists $B_W$ on $W$ such that $(W, B_W)$ is a klt pair and

$$K_W + B_W \sim_{\mathbb{Q}} (K_X + c'Y + A + \varepsilon H)|_W \sim_{\mathbb{Q}} (-1 + mc' + \eta)H|_W.$$

So, given a log resolution $\mu: X' \to X$, we have

$$\mu^*(K_X + c'Y + A) = K_{X'} + E + E' + F$$

where $E$ is the only place over $W$, $E'$ is the sum of all the others places, $|F| = 0$, $\mu(E) \cap \mu(E') = \emptyset$ and $E + E' + F$ has SNC support. Let $Z$ be the union of all centers in $X$. We consider the exact sequence

$$0 \to \mathcal{I}_Z(mH) \to \mathcal{O}_X(mH) \to \mathcal{O}_Z(mH) \to 0.$$ 

The divisor $mH - (K_X + c'Y + A)$ is ample because

$$mH - (K_X + c'Y + A) \sim_{\mathbb{Q}} (m + 1 - c'm - c_1)H$$

and $(1 - c')m + 1 - c_1 > 0$ because $c_1 \ll 1$ and $1 - c' \geq 0$. By the Nadel vanishing theorem 1.2.5, since $mH - K_X$ is integer and such that $mH - K_X - (c'Y + A)$ is ample, we have

$$H^i(X, \mathcal{O}_X(mH) \otimes \mathcal{J}(X, c'Y + A)) = 0 \quad \forall i > 0.$$ 

In particular $H^1(X, \mathcal{O}_X(mH) \otimes \mathcal{J}(X, c'Y + A)) = 0$.

By Remark 1.2.24, applied with $\Delta = c'Y + A$, we have

$$\mathcal{J}(X, c'Y + A) = \mathcal{I}_Z.$$ 

Therefore

$$H^1(X, \mathcal{I}_Z(mH)) = H^1(X, \mathcal{O}_X(mH) \otimes \mathcal{J}(X, c'Y + A)) = 0.$$ 

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and the homomorphism $H^0(X, \mathcal{O}_X(mH)) \to H^0(Z, \mathcal{O}_Z(mH))$ is surjective. Since $W$ is exceptional,

$$\mathcal{O}_Z = \mathcal{O}_W \oplus \mathcal{O}_Z \setminus W$$

and

$$H^0(Z, \mathcal{O}_Z(mH)) = H^0(W, \mathcal{O}_W(mH)) \oplus H^0(Z, \mathcal{O}_Z \setminus W(mH)).$$

By Lemma 1.2.23 the union of all the centers of log canonicity of $(X, cY)$ is contained in the base locus of $|Y|$. Since $Z$ is in the base locus of $|Y|$, the morphism

$$H^0(X, \mathcal{I}_Z(mH)) \to H^0(X, \mathcal{O}_X(mH))$$

is an isomorphism. So, if we prove that

$$H^0(W, \mathcal{O}_W(mH)) \neq 0,$$

we get a contradiction. There are now two possibilities.

**Case a**: there exists a minimal center of codimension at least 2;

**Case b**: all the minimal centers are of codimension 1.

**Case a**: assume that there exists a minimal center $W$ of codimension at least 2. Let $B_W$ be as before, such that $B_W$ is effective, $(W, B_W)$ is klt and

$$K_W + B_W \sim \mathbb{Q} r'H|_W$$

where $r' = c'm - 1 + \eta$.

Then $mH|_W$ is such that

$$mH|_W - (K_W + B_W) \sim \mathbb{Q} (m - r')H|_W$$

is ample because $(1 - c')m + 1 - \eta > 0$ (since $c' \leq 1$ and $\eta \ll 1$). Moreover, since $W$ is of dimension at most 2, it follows from Theorem 2.0.6 that $H^0(W, mH|_W) \neq 0$.

**Case b**: assume that all the minimal centers are of codimension 1. Then from the Lemma 2.0.7 we have $c \leq 1/2$. We set $r' = c'm - 1 + \eta$, so that $K_W + B_W \sim \mathbb{Q} r'H|_W$. Set

$$p(t) = \chi(W, tH|_W), \quad d = (H|_W)^3 > 0 \quad \text{and} \quad \delta = B_W(H|_W)^2.$$

Since $H|_W$ is ample and $B_W$ is effective we have $\delta \geq 0$. Then by Lemma 1.3.3 we have

$$p(t) = \frac{d}{3}t^3 + \frac{-r'd + \delta}{4}t^2 + bt + c.$$

We compute $p(-1) = \chi(W, -H|_W)$. The Nadel vanishing 1.2.5 implies that

$$h^i(W, -H|_W) = 0 \quad \forall \ i < 3,$
then $p(-1) = -h^3(W, -H_W) \leq 0$.

For the computation of $p(m - 1)$ we use the Nadel vanishing 1.2.5. We can write $(m - 1)H_W \sim Q (m - 1 - r')H_W + K_W + B_W$, where $(m - 1 - r')H_W$ is ample. Then if we set $L = (m - 1)H_W - K_W$, by Nadel vanishing 1.2.5

$$h^i(W, O_W(L + K_W) \otimes J(W, B_W)) = 0 \quad \forall \ i > 0.$$ 

Finally $J(W, B_W) = O_W$ because $(W, B_W)$ is klt and $L + K_W = (m-1)H_W$. So we obtain $p(m - 1) = h^0(W, (m - 1)H_W) \geq 0$. The same proof shows that $p(m) = h^0(W, mH_W)$.

A tedious but straightforward computation shows that

$$p(m) = \frac{m + 1}{12}[d(4m - 4 - 3r') + 3\delta] + \frac{-p(-1) + (m+1)p(m-1)}{m}.$$ 

By the discussion above we have $-p(-1) + (m+1)p(m-1) \geq 0$, and, since $\delta \geq 0$,

$$d(4m - 4 - 3r') + 3\delta \geq d(4m - 4 - 3r') = d(4m - 4 - 3(c'm - 1 + \eta)) = d(4 - 3c)m - 1 - 3\eta \geq d(4 - 3/2 - 1 - 3\eta) > 0.$$ 

Thus $p(m) = h^0(W, mH_W) > 0$ and the theorem is proved.
Chapter 3

Generalizations

3.1 Fano varieties of index $n - 3$

Let $X$ be a smooth Fano variety of dimension $n \geq 4$ and index $n - 3$, with $H$ the fundamental divisor. Then we have $-K_X \equiv (n - 3)H$. In this section we will give an expression of $\chi(X, H)$ in terms of $n$, $c_2(X)$ and $H^n$.

We recall that we have the general Riemann-Roch formula:

$$\chi(X, tH) = \frac{H^n}{n!} t^n + \frac{-K_X H^{n-1}}{2(n-1)!} t^{n-1} + \frac{(K_X^2 + c_2(X)) H^{n-2}}{12(n-2)!} t^{n-2}$$

$$+ \frac{-K_X c_2(X) H^{n-3}}{24(n-3)!} t^{n-3} + \cdots + \chi(X, \mathcal{O}_X).$$

For $j \in \{-1 \ldots -(n-4)\}$ we have, by Kodaira vanishing and Serre duality 1.0.7,

$$\chi(X, jH) = (-1)^n h^n(X, jH) = (-1)^n h^0(X, K_X - jH)$$

$$= (-1)^n h^0(X, -(n-3+j)H) = 0.$$

Hence we can write

$$\chi(X, tH) = \frac{H^n}{n!} \prod_{j=1}^{n-4} (t + j)(t^4 + at^3 + bt^2 + ct + d)$$

and to achieve our purpose we need to calculate the coefficients $a, b, c, d$. By Serre duality 1.0.7, the polynomial $\chi(X, tH)$ has the following symmetry property:

$$\chi(X, tH) = (-1)^n \chi(X, K_X - tH) = (-1)^n \chi(X, -(n-3+t)H) \ \forall \ t \in \mathbb{Z}.$$

If we set

$$Q(t) = \frac{H^n}{n!} \prod_{j=1}^{n-4} (t + j)$$

$$P(t) = t^4 + at^3 + bt^2 + ct + d,$$
we have \( Q(t) = (-1)^nQ(-(n - 3 + t)) \), so \( P(t) = P(-(n - 3 + t)) \) for all \( t \in \mathbb{Z} \).

If we compare \( P(t) \) and \( P(-(n - 3 + t)) \) we obtain

\[
P(-(n - 3 + t)) = (n - 3 + t)^4 - a(n - 3 + t)^3 + b(n - 3 + t)^2 - c(n - 3 + t) + d
\]

\[
= t^4 + 4t^3(n - 3) + 6t^2(n - 3)^2 + 4t(n - 3)^3 + (n - 3)^4
\]

\[
- at^3 - 3at^2(n - 3) - 3at(n - 3)^2 - (n - 3)^3
\]

\[
+ bt^2 + 2bt(n - 3) + b(n - 3)^2
\]

\[
- ct - c(n - 3)
\]

\[
= t^4 + at^3 + bt^2 + ct + d
\]

That gives the equalities

\[
a = 2(n - 3) \quad c = (n - 3)(b - (n - 3)^2).
\]

We can write

\[
Q(t) = t^{n-4} + \sum_{i=1}^{n-3} a_i t^{n-4-i}
\]

where

\[
a_{n-4} = (n - 4)!
\]

\[
a_1 = \sum_{j=1}^{n-4} j = \frac{(n-4)(n-3)}{2};
\]

\[
a_2 = \sum_{1 \leq i < j \leq n-4} ij = \frac{(n-3)(n-4)(3n-10)(n-5)}{24}
\]

By comparing the coefficients of the degree zero monoms in the two expressions of \( \chi(X, tH) \), we obtain

\[
d = \frac{n(n-1)(n-2)(n-3)\chi(X, \mathcal{O}_X)}{H^n} = \frac{n(n-1)(n-2)(n-3)}{H^n}
\]

because for a smooth Fano variety \( \chi(X, \mathcal{O}_X) = 1 \).

It remains the computation of \( b \). By comparing the coefficients of the degree \( n - 2 \) monoms in the two expressions of \( \chi(X, tH) \), we obtain

\[
\frac{H^n}{n!} (b + aa_1 + a_2) = \frac{(K_X^2 + c_2(X))H^{n-2}}{12(n-2)!} = \frac{(n - 3)^2H^n + c_2(X)H^{n-2}}{12(n-2)!}
\]

so

\[
b = -aa_1 - a_2 + \frac{n(n-1)(n-3)^2}{12} + \frac{n(n-1)c_2(X)H^{n-2}}{H^n}
\]

\[
= (n - 3) \left[ -(n - 3)(n - 4) - \frac{(n-4)(3n-10)(n-5)}{24} + \frac{n(n-1)(n-3)}{12} \right]
\]

\[
+ \frac{n(n-1)c_2(X)H^{n-2}}{H^n}
\]

\[
= -n^4 + 8n^3 + 9n^2 - 160n + 264 + \frac{n(n-1)c_2(X)H^{n-2}}{H^n}.
\]
We set \( p(n) = -n^4 + 8n^3 + 9n^2 - 160n + 264 \). By replacing \( a, b, c, d \) with the expressions that we obtained, we have

\[
\chi(X, H) = \frac{H^n}{n!} (n - 3)! \left[ 1 + 2(n - 3) + p(n) + \frac{n(n-1)}{12} c_2(X) H^{n-2} \right] \\
+ (n - 3) \left( p(n) + \frac{n(n-1)}{12} c_2(X) H^{n-2} - (n - 3)^2 \right)
\]

\[
= \frac{H^n}{n!} \frac{(n-3)!}{24} \left[ 24 + 48(n - 3) + (n - 2)p(n) - 24(n - 3)^3 \right] \\
+ \frac{c_2(X) H^{n-2}}{12} + n - 3 \\
= \frac{H^n}{n!} \frac{(n-3)!}{24} (-n^4 + 10n^3 - 31n^2 + 38n - 16) + \frac{c_2(X) H^{n-2}}{12} + n - 3.
\]

Moreover we have

\[-n^4 + 10n^3 - 31n^2 + 38n - 16 = (n-1)(n-2)(-n^2 + 7n - 8)\]

so we obtain

**Proposition 3.1.1.** Let \( X \) be a smooth Fano variety of dimension \( n \geq 4 \) and index \( n - 3 \), with \( H \) fundamental divisor. Then

\[
\chi(X, H) = \frac{H^n}{24} (-n^2 + 7n - 8) + \frac{c_2(X) H^{n-2}}{12} + n - 3.
\]

**Proposition 3.1.2.** Let \( X \) be a smooth Fano variety of dimension \( n \), with dimension \( n = 4, 5 \), and index \( n - 3 \). Then \( h^0(X, H) \geq n - 2 \).

**Proof.** We remark that if \( n = 4, 5 \) then \(-n^2 + 7n - 8 > 0\). By Theorem 1.1.5 the tangent bundle \( T_X \) is generically nef, hence we can apply Theorem 1.1.6 and we have that \( c_2(X) H^{n-2} \geq 0 \). Thus

\[
h^0(X, H) = \chi(X, H) = \frac{H^n}{24} (-n^2 + 7n - 8) + \frac{c_2(X) H^{n-2}}{12} + n - 3 \\
\geq \frac{H^n}{24} (-n^2 + 7n - 8) + n - 3 \geq n - 2.
\]

\( \square \)

**Proposition 3.1.3.** Let \( X \) be a smooth Fano variety of dimension \( n \) and index \( n - 3 \), with \( H \) fundamental divisor. Suppose that the tangent bundle \( T_X \) is \( H \)-semistable. Then \( h^0(X, H) \geq n - 2 \).

**Proof.** By the Kodaira vanishing theorem, we have \( \chi(X, H) = h^0(X, H) \). Since \( X \) is smooth, the tangent bundle is locally free and satisfies the conditions of Theorem 1.4.2. In our situation the inequality becomes

\[
c_2(X) H^{n-2} \geq \frac{n-1}{2n} c_1(X)^2 H^{n-2} = \frac{(n-1)(n-3)^2}{2n} H^n.
\]
Thus
\[ h^0(X, H) = \chi(X, H) = \frac{H^n}{24}(-n^2 + 7n - 8) + \frac{c_2(X)H^{n-2}}{12} + n - 3 \]
\[ \geq \frac{H^n}{24} \left(-n^2 + 7n - 8 + \frac{(n-1)(n-3)^2}{n}\right) + n - 3 \]
\[ = H^n \left(\frac{7n-9}{24}\right) + n - 3 \geq n - 2. \]

\[ \Box \]

The condition that the tangent bundle is semistable has been conjectured for all Fano manifolds with Picard number equal to one. Moreover it is verified if dim \( X \) is equal to six.

**Theorem 3.1.4** (Hwang, [5]). *Fano 6-folds with Picard number 1 have semistable tangent bundles.*

Now we look for some result about the regularity of a general section \( Y \in |H| \). We first recall the following well known result that we quote from [16, p. 32].

**Theorem 3.1.5.** *Let \( X \) be an \( n \)-dimensional log Fano variety of index \( i(X) \). Let \( H \) be a fundamental divisor on \( X \).*

1. *If \( i(X) > n - 2 \), then*

\[ h^0(X, O_X(H)) = \frac{1}{2} H^n(i(X) - n + 3) + n - 1. \]

2. *If \( i(X) = n - 2 \), and \( X \) has at most canonical Gorenstein singularities, then*

\[ h^0(X, O_X(H)) = \frac{H^n}{2} + n. \]

**Lemma 3.1.6.** *\([4]\) Let \((X, \Delta)\) be a log Fano variety with index \( i(X) \geq 1 \) and \( H \) a fundamental divisor, *

\[ -(K_X + \Delta) \sim_{\mathbb{Q}} i(X)H. \]

*Suppose that \( h^0(X, H) \geq 1 \), and let \( Y \in |H| \) be a general member. Let \( c \) be the log canonical threshold of the pair \((X, \Delta + Y)\). Suppose \((X, \Delta + Y)\) non plt and let \( W \) be a minimal center of \((X, \Delta + cY)\). Then*

\[ \dim W \geq i(X) - c + 3. \]

*Proof.* Let \( W \) be a minimal center. By Theorem 1.2.26 we can find some rational numbers \( c_1, c_2 \ll 1 \) and an effective divisor \( A \sim_{\mathbb{Q}} c_1H \) such that \( W \) is exceptional for the pair \((X, \Delta + (1 - c_2)cY + A)\). Moreover we can find
$c_1, c_2$ such that $(X, \Delta + (1 - c_2)cY + A)$ is not plt.
By Theorem 1.2.27 for every $\varepsilon > 0$ there exists $B_W$ divisor on $W$ such that $(W, B_W)$ is a klt pair and
\[
K_W + B_W \sim_Q (K_X + \Delta + (1 - c_2)cY + A + \varepsilon H)|_W
\]
\[
\sim_Q (-i(X) + (1 - c_2)c + c_1 + \varepsilon)H|_W.
\]
Set $\eta = -c_2c + c_1 + \varepsilon$, and $\eta$ is arbitrary small and
\[
K_W + B_W \sim_Q -i(X) - c - \eta)H|_W. \tag{3.1.1}
\]
Let $Z$ be the union of all centers of log canonicity of the pair $(X, \Delta + (1 - c_2)cY + A)$. The pair is not plt then we can apply the Lemma 1.2.23 and $Z$ is contained in the base locus of $|H|$. Let $\mathcal{I}_Z$ be the ideal sheaf of $Z$. We consider the exact sequence
\[
0 \to \mathcal{I}_Z(H) \to \mathcal{O}_X(H) \to \mathcal{O}_Z(H) \to 0.
\]
By Remark 1.2.24,
\[
H^1(X, \mathcal{I}_Z(H)) = H^1(X, \mathcal{O}_X(H) \otimes \mathcal{J}(X, \Delta + (1 - c_2)cY + A)) = 0
\]
where the vanishing of the first cohomology group follows from the Nadel Theorem 1.2.5. Thus we obtain the short exact sequence
\[
0 \to H^0(X, \mathcal{I}_Z(H)) \to H^0(X, \mathcal{O}_X(H)) \to H^0(Z, \mathcal{O}_Z(H)) \to 0.
\]
Since $Z$ is contained in the base locus of $|H|$, 
\[
H^0(X, \mathcal{I}_Z(H)) \cong H^0(X, \mathcal{O}_X(H))
\]
and $h^0(Z, \mathcal{O}_Z(H)) = 0$.
Since $W$ is exceptional, we have
\[
h^0(Z, \mathcal{O}_Z(H)) = h^0(W, \mathcal{O}_W(H)) + h^0(Z|_W, \mathcal{O}_{Z|W}(H))
\]
If $W$ has dimension one or two, by Theorem 2.0.6 applied to $D = H|_W$ we obtain $h^0(W, \mathcal{O}_W(H)) \neq 0$.
Suppose now that $\dim W$ is at least three. From the Theorem 3.1.5 it follows that if $i(W) > \dim W - 3$ then $h^0(W, \mathcal{O}_W(H)) \neq 0$.
Thus by Formula 3.1.1 we obtain
\[
\dim W - 3 \geq i(W) \geq i(X) - c - \eta.
\]
Since the inequality holds for $\eta$ arbitrary small, we obtain the statement. \qed

**Corollary 3.1.7.** Let $X$ be a smooth Fano variety of index $n - 3$ with $H$ fundamental divisor such that $h^0(X, H) \geq 1$. Let $Y \in |H|$ be a general element. Then $(X, Y)$ is plt.
Proof. Suppose that \((X, Y)\) is not plt. Let \(c\) be the log canonical threshold of \((X, Y)\). By Lemma 1.2.23 the pair \((X, cY)\) is plt out of the base locus of \(|H|\). Since the discrepancy of the strict transform of \(Y\) is \(-1\), we have \(c \leq 1\). Since \((X, cY)\) is properly lc there exist a minimal center \(W\). By Lemma 3.1.6 we have
\[
\dim W \geq n - c,
\]
hence \(\dim W \geq n - 1\). If there is a center of dimension \(n - 1\) such that along it \((X, cY)\) is not plt, then \(c < 1/2\) by Lemma 2.0.7, thus we have \(\dim W \geq n - 1/2\).

\[\Box\]

3.2 An inductive approach

The result that we prove in this section is a natural extension of Theorem 2.0.3.

Conjecture 3.2.1. Let \((X, \Delta)\) be a klt pair that is log Fano, i.e. there exists an ample Cartier divisor \(H\) on \(X\) such that
\[
-(K_X + \Delta) \sim_{Q} \alpha H
\]
with \(\alpha > 0\). Then we have \(H^0(X, \mathcal{O}_X(H)) \neq 0\).

Conjecture 3.2.2. Let \(X\) be a normal projective variety. Suppose that there exists an ample Cartier divisor \(H\) such that for every \(\alpha > 0\) there exists an effective \(Q\)-divisor \(\Delta\) such that the pair \((X, \Delta)\) is klt and
\[
K_X + \Delta \sim_{Q} \alpha H.
\]
Then we have \(H^0(X, \mathcal{O}_X(H)) \neq 0\).

Theorem 3.2.3. Let \((X, \Delta)\) be a klt pair that is log Fano, i.e. there exists an ample Cartier divisor \(H\) on \(X\) such that
\[
-(K_X + \Delta) \sim_{Q} H.
\]
Set \(n = \dim X\). Suppose that
\begin{itemize}
  \item \(h^0(X, H) \geq 1\);
  \item the Conjecture 3.2.1 holds for varieties of dimension at most \(n - 1\);
  \item the Conjecture 3.2.2 holds for varieties of dimension at most \(n - 2\).
\end{itemize}
If \(Y \in |H|\) is a general element, the pair \((X, \Delta + Y)\) is plt.
Proof. We argue by contradiction and we assume that \((X, \Delta + Y)\) is not plt. Let \(c\) be the log canonical threshold of the pair \((X, \Delta + Y)\), \(c = \text{lct}(X, \Delta + Y)\). We have \(c \leq 1\) because \((X, \Delta + Y)\) is not plt and so not klt. The pair \((X, \Delta + cY)\) is properly LC, so \(\text{CLC}(X, \Delta + cY) \neq \emptyset\).

Let \(W\) be a minimal center. We apply Theorem 1.2.26 with \(H = -(K_X + \Delta)\), \(D = cY\) and \(r \ll 1\). We obtain a log canonical pair \((X, \Delta + (1-c_2)cY + A)\) where \(A \sim Q c_1H\) and \(W\) is an exceptional center for this pair. Then we apply Theorem 1.2.27 to \(\Delta + (1-c_2)cY + A\), with \(H\) as the ample divisor, \(\varepsilon \ll 1\). Set \(c' = (1-c_2)c\), \(\eta = \varepsilon + c_1\).

The Theorem 1.2.27 gives the existence of \(B_W\) divisor on \(W\) such that \((W, B_W)\) is a klt pair and

\[
K_W + B_W \sim_Q (K_X + \Delta + c'Y + A + \varepsilon H)|_W \sim_Q (K_X + \Delta + c'Y + \eta H)|_W.
\]

We consider the exact sequence

\[
0 \to \mathcal{I}_Z(H) \to \mathcal{O}_X(H) \to \mathcal{O}_Z(H) \to 0.
\]

The divisor \(-(K_X + \Delta) - (K_X + \Delta + c'Y + A)\) is ample because

\[-(K_X + \Delta) - (K_X + \Delta + c'Y + A) \sim_Q (2 - c' - c_1)H\]

and \(2 - c' - c_1 > 0\) because \(c_1 \ll 1\) and \(1 - c' > 0\).

By the Nadel vanishing theorem 1.2.5, since \(H\) is integer and such that \(H - (K_X + \Delta + c'Y + A)\) is ample, we have

\[
H^1(X, \mathcal{O}_X(H) \otimes \mathcal{J}(X, \Delta + c'Y + A)) = 0.
\]

From the Remark 1.2.24, applied with \(\Delta = \Delta + c'Y + A\), we have

\[
\mathcal{J}(X, \Delta + c'Y + A) = \mathcal{I}_Z.
\]

Then

\[
H^1(X, \mathcal{I}_Z(H)) = H^1(X, \mathcal{O}_X(H) \otimes \mathcal{J}(X, \Delta + c'Y + A)) = 0
\]

and the homomorphism \(H^0(X, \mathcal{O}_X(H)) \to H^0(Z, \mathcal{O}_Z(H))\) is surjective.

Since \(W\) is exceptional,

\[
\mathcal{O}_Z = \mathcal{O}_W \oplus \mathcal{O}_{Z \setminus W}
\]

and

\[
H^0(Z, \mathcal{O}_Z(H)) = H^0(W, \mathcal{O}_W(H)) \oplus H^0(Z, \mathcal{O}_{Z \setminus W}(H)).
\]

By Lemma 1.2.23 the union of all the centers of log canonicity of \((X, \Delta + cY)\), is contained in the base locus of \(|Y|\).

Since \(Z\) is in the base locus of \(|Y|\), the morphism

\[
H^0(X, \mathcal{I}_Z(H)) \to H^0(X, \mathcal{O}_X(H))
\]

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is an isomorphism. So, if we prove that
\[ H^0(W, \mathcal{O}_W(H)) \neq 0, \]
we get a contradiction.

We will prove that \( W \) satisfies the hypothesis of 3.2.2 or of 3.2.1 with \( H|_W \) the ample divisor that appears in the conjectures. Recall that we have
\[ K_W + B_W \equiv ((1 - c_2)c - 1 + c_1 + \varepsilon)H|_W. \]

There are two cases:

1. \( c < 1 \);
2. \( c = 1 \).

In the first case, we have \( (1 - c_2)c - 1 + c_1 + \varepsilon = c - 1 + c_1 - c_2c + \varepsilon. \) Since \( c - 1 < 0 \) and \( |c_1 - c_2c| < r \) we can apply the Theorems 1.2.26 and 1.2.27 with \( r \) and \( \varepsilon \) such that \( r + \varepsilon < 1 - c. \) The pair \( (W, B_W) \) obtained in this way is log Fano.

In the second case \( (1 - c_2)c - 1 + c_1 + \varepsilon = c_1 - c_2 + \varepsilon. \) If for some \( r \) the Theorem 1.2.26 gives \( c_1, c_2 \) such that \( c_1 - c_2 < 0 \), we can find \( \varepsilon \) such that \( c_1 - c_2 + \varepsilon < 0 \) and \( (W, B_W) \) is log Fano. Suppose now that for all \( r > 0 \) we find \( c_1, c_2 \) such that \( c_1 - c_2 \geq 0 \). We apply Theorem 1.2.26 with \( r \leq \alpha \) and Theorem 1.2.27 with \( \varepsilon = \alpha - c_1 + c_2 \). Then \( K_W + B_W \sim_{\mathbb{Q}} \alpha H. \)

Finally, we need to show that if all the minimal centers are of dimension \( n - 1 \) then we are in the hypothesis of Conjecture 3.2.1. In this case in fact, by Lemma 2.0.7, we have \( c \leq 1/2 \) and then it follows from the above discussion. 

\( \square \)
Chapter 4

A theorem of Shokurov

Let $X$ be a Fano manifold of dimension three. A theorem of Shokurov states that a general element of $|-K_X|$ is smooth. This chapter is entirely dedicated to the proof of this fact.

**Remark 4.0.4.** Let $X$ be a smooth Fano variety of dimension three. Then by Theorem 3.1.5 we have $h^0(X, -K_X) \geq 4$.

The proof of the following proposition follows the same lines as that in [7, p. 178].

**Proposition 4.0.5.** Let $X$ be a normal projective variety of dimension two with at most Gorenstein canonical singularities. Assume that $K_X \sim 0$ and let $H$ be an ample divisor. Let $Y \in |H|$ be a general member. Then $(X, Y)$ is lc.

**Proof.** Assume that $(X, Y)$ is not lc. Let $c$ be the threshold for $(X, Y)$, so that $c < 1$ and $(X, cY)$ is properly lc. Let $W$ a minimal center for the pair $(X, cY)$. By Theorem 1.2.26 there exists $c_1, c_2$ arbitrary small such that the pair $(X, (1 - c_2)cY + A)$ is lc and $W$ is exceptional for it, where $A$ is an effective $\mathbb{Q}$-divisor such that $A \sim_Q c_1H$. By Theorem 1.2.27 for all $\varepsilon > 0$ there exists $B_W$ divisor on $W$ with $(W, B_W) \text{ klt}$ and

$$K_W + B_W \sim_Q (K_X + (1 - c_2)cY + A + \varepsilon H)|_W.$$

Let $Z$ be the union of all centers of log canonicity. We consider the exact sequence

$$0 \to \mathcal{I}_Z(H) \to \mathcal{O}_X(H) \to \mathcal{O}_Z(H) \to 0$$

By Lemma 1.2.24 we have $\mathcal{I}_Z = \mathcal{J}(X, (1 - c_2)cY + A)$. Thus

$$\mathcal{I}_Z(H) = \mathcal{J}(X, (1 - c_2)cY + A) \otimes \mathcal{O}_X(H)$$

and

$$H - (K_X + (1 - c_2)cY + A) \sim_Q (1 - (1 - c_2)c - c_1)H.$$
Since $c < 1$ and $c_1, c_2 \ll 1$, the divisor $H = (K_X + (1 - c_2)cY + A)$ is ample. Hence we can apply the Nadel vanishing 1.2.5 and obtain $H^1(X, \mathcal{I}_Z(H)) = 0$. Hence there is a short exact sequence

$$0 \to H^0(X, \mathcal{I}_Z(H)) \to H^0(X, \mathcal{O}_X(H)) \to H^0(Z, \mathcal{O}_Z(H)) \to 0.$$ 

Since $Z \subseteq \text{Bs}[H]$ by Lemma 1.2.23 we have $H^0(X, \mathcal{I}_Z(H)) \cong H^0(X, \mathcal{O}_X(H))$ and $H^0(Z, \mathcal{O}_Z(H)) = 0$. Since

$$H|_W - (K_W + B_W) \sim_{\mathbb{Q}} [1 - (1 - c_2)c - c_1 - \varepsilon]H|_W$$

and $1 - (1 - c_2)c - c_1 - \varepsilon > 0$ for $\varepsilon \ll 1$, the divisor $H|_W - (K_W + B_W)$ is ample on $W$. Hence we can apply Theorem 2.0.6 and have $H^0(W, \mathcal{O}_W(H)) \neq 0$, that is not possible. Thus $(X,Y)$ is lc. 

**Theorem 4.0.6 (Shokurov [17]).** Let $X$ be a smooth Fano variety of dimension three. Then $Y \in | - K_X |$ general is smooth.

**Proof.** This proof has been taken from a work of Andreas Höring, [4]. Set $H = -K_X$. By the Bertini Theorem 1.0.11, since $X$ is smooth, the singular locus of a general section of $|H|$ is contained in the base locus, so we will study $\text{Bs}[H]$.

**Step 1** Let $Y_1 \in |H|$ a general section, then the pair $(X, Y_1)$ is plt by the analogous of Theorem 2.0.3 in dimension three and $Y_1$ is Gorenstein canonical by Lemma 2.0.5. We take $\mu : X' \to X$ a log resolution for this pair,

$$K_{X'} \sim \mu^*(K_X + Y_1) - Y_1' + \sum a_j E_j.$$ 

Let $Z \in |H|_{Y_1}|$ be a general element, such that the pair $(Y_1, Z)$ is log canonical, as in the Proposition 4.0.5, and $Y_2$ such that $Z = Y_1 \cap Y_2$. We write

$$\mu^*Y_2 = Y_2' + \sum b_j E_j.$$ 

If $Y_2$ is singular along $\mu(E_i)$, then $b_i \geq 2$. We have the following equality

$$K_{X'} \sim \mu^*(K_X + Y_1 + Y_2) - Y_1' - Y_2' + \sum (a_j - b_j) E_j,$$

by taking restrictions to $Y_1'$ and by Adjunction formula we obtain

$$K_{Y_1'} \sim \mu^*(K_{Y_1} + Z) - Z' + \sum (a_j - b_j) E_j|_{Y_1'},$$

where $Z'$ is the strict transform of $Z$. Since $(Y_1, Z)$ is lc we have $a_i - b_i \geq -1$. Thus $a_i > 0$ and $Y_1$ is terminal, hence smooth along $\mu(E_i)$.

Let $\mathcal{H} \subseteq |H|$ be the open set of the sections $Y$ such that $(X, Y)$ is plt and $Y_{\text{sing}} \subseteq \text{Bs}[H]$. The preceding argument shows that, given an irreducible subvariety of $X$, if there exists a section of $\mathcal{H}$ that is singular along it, then this section is the only one with the property.
**Step II** We know that $h^i(X, H) = 0$ for all $i > 0$ by Kawamata-Viehweg vanishing applied to $L = 2H$. We know from the Lemma 4.0.4 that $h^0(X, H) \geq 4$ and $h^1(X, \mathcal{O}_X) = 0$ because $X$ is Fano. Then we have an exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, H) \to H^0(Y, H|_Y) \to 0.$$ 

Thus $h^0(Y, H|_Y) \geq 3$. Moreover, if $Z \in |H|_Y$, then $Z = Y \cap Y_1$ with $Y_1 \in |H|$. Since a general section of $|H|$ is plt and then irreducible by Lemma 2.0.4, the linear system $|H|$ has no fixed divisorial component. In particular the subvariety $Z$ is a curve, and the base locus of $|H|$ has dimension at most one.

**Step III** Set

$$\mathcal{U} = \{(Y, p) \in \mathcal{H} \times X \mid p \in Y\}.$$ 

If we call $p_1$, $p_2$ the restrictions of the projections, we have $p_2(\mathcal{U}^{sing}) \subseteq \text{Bs}|H|$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{U}^{sing} & \to & \text{Bs}|H| \\
\downarrow & & \downarrow \\
\mathcal{U} & \to & X \\
\downarrow & & \downarrow \\
p_1 & \to & \mathcal{H}.
\end{array}$$

The fiber of $p_1$ over a section $Y$ is isomorphic to $Y$ viewed as a subscheme of $X$. We prove now that $p_1|_{\mathcal{U}^{sing}} : \mathcal{U}^{sing} \to \mathcal{H}$ is not surjective. We argue by contradiction. If the application is surjective, we have $\dim \mathcal{U}^{sing} \geq \dim \mathcal{H}$ and $\dim \mathcal{H} \geq 3$ by the Remark 4.0.4. Hence the map

$$\begin{array}{ccc}
\mathcal{U}^{sing} & \to & \text{Bs}|H| \\
(Y, q) & \mapsto & q
\end{array}$$

has fibers of positive dimension because $\dim \text{Bs}|H| \leq 1$. This is not possible because we have proven that given any point there is at most one section that is singular in it. Hence $\mathcal{U}^{sing} \to \mathcal{H}$ is not surjective. Finally, the restriction of $p_1$

$$p_1 : \mathcal{U}\backslash p_1^{-1}p_1(\mathcal{U}^{sing}) \to \mathcal{H}\backslash p_1(\mathcal{U}^{sing})$$

is a morphism of nonsingular varieties, hence it is generically smooth. Since it is generically smooth, a generic fiber is smooth.
Bibliography


