

Introduction to the Rapid Decay property

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ABSTRACT. This is an introduction to the Rapid Decay property, with a survey of known results and equivalent definitions of this property. We also discuss in details the easy case when $G = \mathbf{Z}$. Everything in this paper is well-known by different sets of people.

In this paper, G will be a countable group, that we assume finitely generated by a finite generating set S . The *word length* of an element $\gamma \in G$ is the minimal number of elements of S needed to express γ . The group G acts on $\ell^2(G)$ via the (left) regular representation, and one extends this action to $\mathbf{C}G$ and the bounded operators on $\ell^2(G)$ (see Section 2 for the definitions).

DEFINITION 0.1. A finitely generated group G has *the Rapid Decay property* if there are constants C and D such that for any $R \in \mathbf{N}$ and any $f \in \mathbf{C}G$ such that f is supported on elements shorter than R , the following inequality holds:

$$\|f\|_* \leq CR^D \|f\|_2, \quad RD(1)$$

where $\|\cdot\|_*$ is the operator norm of elements in $\mathbf{C}G$ acting on $\ell^2(G)$ via the linearization of the left regular representation and $\|\cdot\|_2$ is the ℓ^2 -norm on $\mathbf{C}G$ (see Section 2).

Because of the constant C , the above inequality is interesting only for large R 's. One can notice that whereas the left hand side of the above inequality directly depends on the group law (via the regular representation) the right hand side only depends on the geometry of the group in a ball of radius R . Depending on how the group operation allows to combine γ 's and μ 's in G to obtain elements $\gamma\mu^{-1}$ of length shorter than R , one may or may not be able to control the left hand side. Groups with the Rapid Decay property are exactly the ones for which we can control the left hand side. More equivalent definitions of the Rapid Decay property will be discussed in Section 4.

The terminology *Rapid Decay* comes from Connes' original definition, equivalent to RD(1) above, which is as follows (see Definitions 2.5 and 2.9 for all the definitions).

DEFINITION 0.2. A group has the *Rapid Decay property* if

$$H_\ell^\infty(G) \subseteq C_r^*(G) \quad RD(2)$$

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that is, the set of rapidly decaying functions on G is contained in the reduced C^* -algebra of G .

Indeed, it is a classical result that any abelian and unital C^* -algebra is the algebra of continuous functions on a compact space (its dual), so philosophically, a non-abelian unital C^* -algebra (such as, for instance, the reduced C^* -algebra of a finitely generated group), should correspond to the continuous functions on a non-commutative space. So, for a finitely generated group G , admitting that the reduced C^* -algebra of G corresponds to continuous functions, it is natural to look for the smooth functions. In Section 3 we shall thoroughly see the case of the group \mathbf{Z} . Even though this case is very easy, it illustrates nicely Connes' definition.

Amenable groups is an important class of groups that give us the simplest examples of groups without the Rapid Decay property, that is amenable groups with super-polynomial growth. An example is the following semi-direct product:

$$\mathbf{Z}^2 \times \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{Z} \simeq \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n \\ b & \end{pmatrix} \mid (a, b) \in \mathbf{Z}^2, n \in \mathbf{Z} \right\} < SL_3(\mathbf{Z}).$$

From the definition of Rapid Decay property it is clear that the property is inherited by subgroups with the induced length, and hence having an amenable subgroup of super-polynomial growth is an obstruction to the Rapid Decay property, showing that for instance $SL_3(\mathbf{Z})$ does not have that property. The above example also shows that in general the Rapid Decay property is not stable under taking extensions, however, in a short exact sequence of finitely generated groups

$$\{e\} \rightarrow Z \rightarrow G \rightarrow Q \rightarrow \{e\}$$

then G has the Rapid Decay property if and only if Q has the Rapid Decay property and Z has the Rapid Decay property for the induced length from G , see [24] for the general statement, and Remark 4.3 for a discussion on the Rapid Decay property and length functions.

Overview of the situation

Here is a non-exhaustive and redundant list of what is known regarding the property of Rapid Decay:

Examples of discrete groups with the Rapid Decay property.

- Groups of polynomial growth [34]
- Free groups [29]
- Hyperbolic groups [34], [30]
- Coxeter groups [15]
- Discrete cocompact subgroups of
 - $SL_3(\mathbf{Q}_p)$ [50]
 - $SL_3(\mathbf{R})$ and $SL_3(\mathbf{C})$ [36]
 - $SL_3(\mathbb{H})$ and $E_{6(-26)}$ [14]
 - products of the above [14]
- All rank one lattices [15] and more generally hyperbolic group relatively to subgroups with the Rapid Decay property [21]
- Cocompact cubical CAT(0) groups [15]
- Mapping class groups [5]

- Braid groups [3] and [5]
- Large type Artin groups [20]
- 3-manifold groups not containing *Sol* [26]
- Wise non-Hopfian group [4]
- Small cancellation groups [2] and [46].

Examples of discrete groups without the Rapid Decay property.

- Amenable groups with superpolynomial growth [34] (for instance Intermediate growth groups)
- Groups containing amenable subgroups with superpolynomial growth, so for instance to name a few:
 - the Baumslag-Solitar groups $BS(n, m)$
 - \mathbf{Z}^∞ , Thompson’s group
 - $SL_n(\mathbf{Z})$, $n \geq 3$ and more generally non-uniform lattices in higher rank
 - $GL_2(\mathbb{F}_p[t, t^{-1}])$ (see [15])
- $\mathbf{Z} * \mathbf{Z}^2 * \mathbf{Z}^3 * \dots$ (see [51] for a length on that group)

Examples of groups for which the Rapid Decay property is open.

- $\text{Out}(\mathbf{F}_n)$, $n \geq 3$
- Cocompact lattices in $SL_n(\mathbf{Q}_p)$ or $SL_n(\mathbf{R})$, $n \geq 4$ and more generally semisimple Lie groups, see Conjecture 1.1 below
- Artin groups
- Cocompact CAT(0) groups

According to Jolissaint in [34], the Rapid Decay property is preserved by free products and some amalgamated products, as well as some central extensions and semi-direct products. Moreover according to Ciobanu, Holt and Rees in [19], the Rapid Decay property is preserved by graph products, hence one gets more examples combining the above. However it is unknown if the Rapid Decay property is preserved under quasi-isometries, although the methods used so far to establish the Rapid Decay property are.

Actually the situation is much simpler than what the above list expresses: among finitely presented groups, the only known obstruction to the Rapid Decay property is to contain an amenable subgroup of super-polynomial growth for the induced length, phenomena easily observed in the presence for instance of an exponentially distorted copy of \mathbf{Z} . In the case of finitely generated groups however, Sapir in [51] constructs a 2-generator groups without the Rapid Decay property and without super-polynomial growth amenable subgroups. We refer to Sapir in [51] for another survey of the Rapid Decay property, and to Garnzarek’s minicourse notes [25] for another introduction to the Rapid Decay property.

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1. Short historical survey and applications

First established for free groups by Haagerup in [29], the Rapid Decay property has been introduced and studied as such by Jolissaint in [34], who notably established it for groups of polynomial growth, and for classical hyperbolic groups. The extension to Gromov hyperbolic groups is due to de la Harpe in [30]. One of the earliest definitions was given by Connes in [22] in a non-commutative geometry setting, and the first important application of the Rapid Decay property is in Connes and Moscovici's work [23], proving the Novikov conjecture for Gromov hyperbolic groups. Providing the first examples of higher rank groups, Ramagge, Robertson and Steger in [50] proved that the Rapid Decay property holds for \tilde{A}_2 and $\tilde{A}_1 \times \tilde{A}_1$ groups, and Lafforgue, using a very nice quasification of their proof, established it for cocompact lattices in $SL_3(\mathbf{R})$ and $SL_3(\mathbf{C})$ in [36]. Lafforgue proved this property as part of his big results on the Baum-Connes conjecture, that we discuss a bit more below. His result on the Rapid Decay property was generalized in [14] to cocompact lattices in $SL_3(\mathbf{H})$ and $E_{6(-26)}$ as well as in a finite product of rank one Lie groups. We saw that $SL_n(\mathbf{Z})$ for $n \geq 3$ (and more generally any non-cocompact lattice in higher rank simple Lie groups as they contain exponentially distorted copies of \mathbf{Z} , see [40]) does not have the Rapid Decay property, and Valette conjectured the following.

CONJECTURE 1.1 (Valette [53]). Cocompact lattices in a semisimple Lie group (real or p -adic) have the Rapid Decay property.

In rank one the situation is different as we show with Ruane in [15]; there all lattices have the Rapid Decay property. This uses that those lattices are hyperbolic relative to polynomial growth groups. More generally, Druţu and Sapir [21] show that any hyperbolic group relative to subgroups with the Rapid Decay property, has the Rapid Decay property as well. More groups have recently been added to the list of groups with the Rapid Decay property, including mapping class groups by Behrstock and Minsky in [5] (those include braid groups [3] and [5]), large type Artin groups by Ciobanu, Holt and Rees in [20], as well as Wise non-Hopfian group by Barré and Pichot in [4] or coarse median groups, see Bowditch [9]. Some small cancellation groups satisfy the Rapid Decay property, according to Arzhantseva and Druţu [2] or Osin [46].

Methods for proving the Rapid Decay property. Unfortunately, all the methods used so far to establish the Rapid Decay property are very similar and give a bad exponent. Those methods rely on noticing that the left hand side of RD(1) is in fact a computation on triples of points (this is easier to see by looking at RD(5) in Proposition 4.1), and then reducing the computation to triples of points that are manageable. This is the reason why the Rapid Decay property is still open for lattices in $Sp_4(\mathbb{Q}_p)$ for instance (or more generally groups of Conjecture 1.1): the reduction from [50] or [36] gives triples on which the computation cannot be carried on, see Talbi's work [52] for more on the situation on p -adic lattices. Sapir's recent survey [51] gives a nice account on the similarities and differences of those methods.

Locally compact compactly generated groups. An approach to prove Conjecture [53] that so far gave only false hopes is to compare the lattice and the ambient group. Indeed, all definitions of the Rapid Decay property extend quite

easily to locally compact groups (see [34]), but in this settings the Rapid Decay property is inherited by *open* subgroups only. However, the following result is fairly elementary.

THEOREM 1.2 (Jolissaint [34]). *Let G be a locally compact group and let $\Gamma < G$ be a discrete cocompact subgroup in G , and let ℓ be a continuous length function on G . If Γ has the Rapid Decay property with respect to ℓ (restricted to Γ), then so does G (with respect to ℓ).*

As an approach to Conjecture 1.1 one could hope that the converse of the above theorem holds true, since all semisimple Lie groups have the Rapid Decay property (according to [16], it is mainly a reformulation of Kunze-Stein phenomena [32], see also [11] for a simpler proof). However such a guess seems mainly supported by a lack of counter-examples.

Radial Rapid Decay. Considering *radial* functions only has been done by Valette in [54] to establish the Rapid Decay property for radial functions for groups acting on buildings. Here we recall that given a length function on a group G , the set of radial functions (in \mathbf{CG} or even $\ell^2(G)$) is the set of functions whose values are constant on spheres, that is whose values only depend on the length of an element. This result of Valette has been extended to cocompact lattices in semisimple Lie groups for the length induced from the Riemannian metric by Perrone in [48]. However, the set of radial functions seems too small to get the full Rapid Decay property for G . For instance, in the case of locally compact non-unimodular groups, those cannot have the Rapid Decay property according to [33], but those do have a version of Rapid Decay property on radial functions, see [16]. However, in the case of discrete groups, it would be interesting to have examples of discrete groups that do satisfy the Rapid Decay inequality on radial functions but that do not have the Rapid Decay property.

The Baum-Connes conjecture. The Rapid Decay property gained attention in 2001 after Lafforgue's work on the Baum-Connes conjecture showed its importance in this context and indeed, for a large class of groups like cocompact lattices in semisimple Lie groups, the Rapid Decay property would be the last step in establishing the Baum-Connes conjecture. Lafforgue's work was the first instance where it was shown that some groups with property (T) satisfied the conjecture. We will not attempt to explain the Baum-Connes conjecture here and refer the interested reader to Valette's lecture notes [53]. We will just mention what it says, namely that, for a discrete group G , the assembly map

$$\mu_i : K_i^G(\underline{EG}) \rightarrow K_i(C_r^*(G)) \quad (BC)$$

should be an isomorphism for $i = 0, 1$, where the left hand side is the equivariant K -homology of \underline{EG} , the classifying space for proper actions for G , and is a group that algebraic topologists understand and can compute in many cases. The right hand side is the topological K -theory of C_r^*G , the reduced C^* -algebra of G , and this is a group that is interesting for analysts but that remains quite mysterious. Lafforgue in [37] showed that, for a very large class of groups (containing for instance all lattices in Lie groups, amenable groups, a-T-menable groups, hyperbolic groups, CAT(0) groups) a modified version of the Baum-Connes conjecture holds true, where the reduced C^* -algebra is replaced by any *unconditional completion*, that is a Banach algebra completion of the group ring that only depends on the absolute

values of elements. The Rapid Decay property is providing such a completion that in addition is known to have the same K-theory as the one of the reduced C*-algebra (see Proposition 2.12). Notice that despite having the Rapid Decay property, the Baum-Connes conjecture is still open for mapping class groups because nobody has yet proved that they belong to Lafforgue's class (or that they are *strongly bolic*).

Random walks. A symmetric random walk on G is given by a finitely supported symmetric function f with support generating G and such that $\sum_{\gamma \in G} f(\gamma) = 1$. For such a random walk, the return probability of the random walk after $2n$ steps is given by $f^{(2n)}(e)$, the $2n$ convolution of f with itself. The asymptotics of the return probability are very interesting, however as soon as the group is non-amenable (which according to Kesten amounts to the spectral radius of f being strictly smaller than 1), the decay is exponential. A finer information is hence given by

$$\varphi_f : \mathbf{N} \rightarrow \mathbf{R}, n \mapsto \rho_f^{-2n} f^{(2n)}(e) \quad (RW)$$

where ρ_f denotes the spectral radius of f , that is the operator norm of f acting on $\ell^2(G)$. This asymptotics in (RW) can depend on the function and not only of the group, as shown by Cartwright's example [13], see also [55]. This asymptotics has been computed by [38] in the case of free groups, by [12] in the case of free products of \mathbf{Z}^n 's and recently by Gouëzel in the case of hyperbolic groups [27]. In the cases where G has the Rapid Decay property, we can get bounds for this asymptotics from a very simple remark from [16], Section 7.

THEOREM 1.3 (Chatterji, Pittet, Saloff-Coste). *Let G be a group with the Rapid Decay property. Then, for every f a finitely supported symmetric probability measure on G , there exists $c_f > 0$ such that for any $n \in \mathbf{N}$ we have*

$$\varphi_f(n) \geq c_f n^{-2D}.$$

for any D satisfying the inequality $RD(1)$.

PROOF. Since f is symmetric, $f^{(2n)}(e) = \|f^{(n)}\|_2^2$ and $\rho_f^{2n} = \|f^{(n)}\|_*^2$. Moreover, $f^{(n)}$ is supported on a ball of radius dn where d is the diameter of the support of f . Then the inequality from $RD(1)$ gives us that

$$\rho_f^{2n} \leq C d^{2D} n^{2D} f^{(2n)}(e)$$

and we conclude setting $c_f = C^{-1} d^{-2D}$. □

In case where $\rho_f < 1$ (which is exactly when G is non-amenable), then there is a constant $c_f > 0$ so that $\varphi_f(n) \leq c_f n^{-1}$ (see [18]). Hence it is very interesting to determine the best possible Rapid Decay degree of a group G , as it will give a bound on the possible asymptotic behaviors of random walks on G . However, in the case of hyperbolic groups, Gouëzel in [27] shows that the exact asymptotic is $3/2$, whereas the Theorem 1.3 above will only give a bound between 1 and 3. Notice that $3/2$ is the Rapid Decay degree that can be deduced from most proofs of the Rapid Decay property for hyperbolic groups (for instance [36] or [30]) but Nica shows in [45] that indeed this is the smallest Rapid Decay degree for hyperbolic groups.

Other applications. Other applications of the Rapid Decay property include

- Connes and Moscovici’s work proving the Novikov conjecture for Gromov hyperbolic groups use the Rapid Decay property for hyperbolic groups [23].
- Brodzki and Niblo’s result that groups that satisfy the property of Rapid Decay with respect to a conditionally negative length function have the metric approximation property [8].
- De la Harpe, Robertson and Valette in [31] established that in a group G with the Rapid Decay property, then $\rho(\chi_S) = |S|^{1/2}$ if and only if S generates a free semi-group, where $\rho(\chi_S)$ is the spectral radius of the characteristic function of a finite subset S of G .
- The Rapid Decay property for hyperbolic groups has been used by Grigorchuk and Nagnibeda in [28] to compute the convergence radius of the complete growth series of a hyperbolic group.
- Nagnibeda and Pak in [41] show that a finitely generated non-amenable group with the Rapid Decay property has infinitesimally small spectral radius.
- Further use of the Rapid Decay property can be found in the work of Nevo [42] and [43], or Antonescu and Christensen in [1].

2. Basic definitions

DEFINITION 2.1. A *length function* on a discrete group G is a function $\ell : G \rightarrow \mathbf{R}_+$ satisfying:

- (1) $\ell(\gamma) = \ell(\gamma^{-1})$ for any $\gamma \in G$.
- (2) $\ell(\gamma\mu) \leq \ell(\gamma) + \ell(\mu)$ for any $\gamma, \mu \in G$.
- (3) $\ell(e) = 0$, where $e = 1$ denotes the identity element in G .

The constant map $G \rightarrow \mathbf{R}_+, \gamma \mapsto 0$ is the *trivial length*. If G is generated by some finite subset S , then the *algebraic word length* $\ell_S : G \rightarrow \mathbf{N}$ is a length function on G , where, for $\gamma \in G$, $\ell_S(\gamma)$ is the minimal length of γ as a word on the alphabet $S \cup S^{-1}$, that is,

$$\ell_S(\gamma) = \min\{n \in \mathbf{N} \mid \gamma = s_1 \dots s_n, s_i \in S \cup S^{-1}\}.$$

Length functions and displacement maps for isometric actions on metric spaces are basically the same thing and given a metric space (X, d) on which a group G acts by isometries (ie $d(\gamma \cdot x, \gamma \cdot y) = d(x, y)$ for any $x, y \in X$), then

$$\ell(\gamma) := d(x_0, \gamma \cdot x_0)$$

is a length function, for any fixed $x_0 \in X$. Conversely, given a length function ℓ on G , if $N = \{\gamma \in G \mid \ell(\gamma) = 0\}$, then note that N is a subgroup of G , and we set $X = G/N$ and $d(\gamma N, \mu N) = \ell(\mu^{-1}\gamma)$. One can check that (X, d) is a well-defined metric space on which G acts by isometries, and from the definition ℓ is the displacement map based at N .

A length function ℓ' *dominates* another length function ℓ if there are two constants $A \geq 1$ and $B \geq 0$ such that

$$\ell(\gamma) \leq A\ell'(\gamma) + B,$$

for any $\gamma \in G$. We say that ℓ and ℓ' are *equivalent* if they dominate each other.

REMARK 2.2. *It is straightforward to check that if G is finitely generated, any word length dominates any other length and hence two word lengths are always equivalent (as long as the generating sets are finite).*

DEFINITION 2.3. We denote by \mathbf{CG} the *complex group ring* (or algebra) of the group G . It is a complex vector space with basis indexed by the elements of G , and the multiplicative structure is given by the group law. Analysts view \mathbf{CG} as the set of functions $f : G \rightarrow \mathbf{C}$ with finite support, it is a ring for pointwise addition and convolution:

$$f * g(\gamma) = \sum_{\mu \in G} f(\mu)g(\mu^{-1}\gamma). \quad (f, g \in \mathbf{CG}, \gamma \in G).$$

Algebraists tend to think of \mathbf{CG} as formal linear complex combinations of elements in G , that is formal finite sums $\sum_{\gamma \in G} f_{\gamma}\gamma$ with the f_{γ} 's in \mathbf{C} , and the multiplicative structure is given by:

$$\left(\sum_{\gamma \in G} f_{\gamma}\gamma\right)\left(\sum_{\mu \in G} g_{\mu}\mu\right) = \sum_{\gamma, \mu \in G} f_{\gamma}g_{\mu}\gamma\mu.$$

Those two models are easily seen to be equal identifying the algebraist's element $\gamma \in \mathbf{CG}$ with the function $\delta_{\gamma} : G \rightarrow \mathbf{C}$ taking value one in γ and zero elsewhere and checking that $\delta_{\gamma} * \delta_{\mu} = \delta_{\gamma\mu}$, so that a finitely supported function f is just the finite combination $\sum_{\gamma \in G} f(\gamma)\delta_{\gamma}$. We shall without further warnings use both models, with a preference for the analyst's one.

It is a standard fact that any Hilbert space admits an orthonormal basis, so that any infinite dimensional Hilbert space with countable basis is isomorphic to $\ell^2(\mathbf{N})$. We shall only consider separable Hilbert spaces, so in fact we shall only be dealing with \mathbf{C}^n 's or $\ell^2(\mathbf{N})$. However, in the case of a discrete group G , writing $\ell^2(G)$ has the advantage of recording the unitary representation of G on $\ell^2(G)$.

DEFINITION 2.4. The *(left) regular representation* of G is the map

$$\begin{aligned} G &\rightarrow \mathcal{U}(\ell^2(G)) \\ \gamma &\mapsto \{\xi \mapsto \gamma(\xi) := \delta_{\gamma} * \xi\} \end{aligned}$$

This means that an element $\gamma \in G$ shifts $\xi \in \ell^2(G)$ by pre-composition, since $\gamma(\xi)(g) = \delta_{\gamma} * \xi(g) = \xi(\gamma^{-1}g)$. Here $\mathcal{U}(\ell^2(G))$ denotes the unitary operators of $\ell^2(G)$, and one can check that indeed, $\delta_{\gamma}^* = \delta_{\gamma^{-1}}$, so that $\delta_{\gamma} * \delta_{\gamma}^* = \delta_e$ which is the identity operator. The *coefficients of the regular representation* are then given by

$$\{\langle \gamma(\xi), \eta \rangle\}_{\gamma \in G},$$

where $\xi, \eta \in \ell^2(G)$ are of norm one.

Extending the regular representation by linearity induces an injective map

$$\begin{aligned} \mathbf{CG} &\rightarrow \mathcal{B}(\ell^2(G)) \\ f &\mapsto \sum_{\gamma \in G} f(\gamma)\delta_{\gamma} \end{aligned}$$

which is just the left convolution by f . Here $\mathcal{B}(\ell^2(G))$ denotes the algebra of bounded operators on $\ell^2(G)$, the norm of such an operator being given by

$$\|f\|_* = \sup\{\|f * \xi\|_2 \mid \xi \in \ell^2(G), \|\xi\|_2 = 1\}.$$

For $\xi \in \ell^2(G)$ one has:

$$\|f * \xi\|_2 = \left\| \sum_{\gamma \in G} f(\gamma) \delta_\gamma * \xi \right\|_2 \leq \sum_{\gamma \in G} |f(\gamma)| \|\delta_\gamma * \xi\|_2 = \sum_{\gamma \in G} |f(\gamma)| \|\xi\|_2 = \|f\|_1 \|\xi\|_2,$$

so that the supremum over the ξ 's of norm one will never exceed the right hand side of the above inequality and taking a limit in $\ell^1(G)$ we hence deduce that

$$\|f\|_* \leq \|f\|_1,$$

for all $f \in \ell^1(G)$.

DEFINITION 2.5. The *reduced C^* -algebra of G* , denoted by C_r^*G is the closure (for the operator norm) of \mathbf{CG} acting on $\ell^2(G)$ in its left regular representation, namely

$$C_r^*G = \overline{\mathbf{CG}}^{\|\cdot\|_*} \subseteq \mathcal{B}(\ell^2(G)).$$

REMARK 2.6. The inner product on $\ell^2(G)$ is given by

$$\langle \xi, \eta \rangle = \sum_{\gamma \in G} \xi(\gamma) \overline{\eta(\gamma)},$$

hence $f * \xi(e) = \sum_{\gamma \in G} f(\gamma) \xi(\gamma^{-1}) = \langle f, \xi^* \rangle$. Taking $\xi = \delta_e$ one gets that $\|f\|_2 \leq \|f\|_* \leq \|f\|_1$ showing that $\ell^1(G) \subseteq C_r^*G \subseteq \ell^2(G)$.

These are in general very rough estimates of $\|f\|_*$ unless f is positive and G is amenable and indeed Leptin's characterization of amenability is the following.

THEOREM 2.7 (Leptin [39]). *Let G be a group, then G is amenable if and only if for any $f \in \mathbf{R}^+G$ then $\|f\|_* = \|f\|_1$.*

At this stage and using Definition 0.1 as well as Leptin's characterization of amenability (Theorem 2.7 above) we can show the following.

THEOREM 2.8 (Jolissaint [34]). *Let G be a finitely generated amenable group. Then G has the Rapid Decay property if and only if it has polynomial growth.*

PROOF. Let $r \geq 0$ and denote by χ_r the characteristic function of a ball of radius r . First assume that G has polynomial growth (and hence is amenable). For f supported on a ball of radius r we compute

$$\|f\|_* \leq \|f\|_1 = \langle \chi_r, |f| \rangle \leq \|\chi_r\|_2 \|f\|_2,$$

and conclude since $\|\chi_r\|_2 = \sqrt{\sharp B(r)}$ is a polynomial by assumption. The last inequality is just Cauchy-Schwarz inequality.

Conversely assume that G is amenable with the Rapid Decay property, hence

$$\|\chi_r\|_* = \|\chi_r\|_1 \leq P(r) \|\chi_r\|_2.$$

Since $\|\chi_r\|_1 = \sharp B(r) = \|\chi_r\|_2^2$ we conclude that $\sqrt{\sharp B(r)} \leq P(r)$ and hence G has polynomial growth. \square

DEFINITION 2.9. Consider for f in \mathbf{CG} and ℓ a length function on G a *weighted ℓ^2 norm*, depending on a parameter $s \geq 0$ and given by

$$\|f\|_{\ell, s} = \sqrt{\sum_{\gamma \in G} |f(\gamma)|^2 (1 + \ell(\gamma))^{2s}}.$$

The *s*-Sobolev completion of \mathbf{CG} is the completion of \mathbf{CG} with respect to this norm and is denoted by $H_\ell^s(G)$. The *functions of rapid decay* are given by

$$H_\ell^\infty(G) = \bigcap_{s \geq 0} H_\ell^s(G)$$

An alternate definition of the Rapid Decay property, giving an upper bound for the operator norm, is the following.

DEFINITION 2.10. Let G be a finitely generated group, the G has the *Rapid Decay property* if there are constants $c \geq 1$ and $s \geq 0$ such that, for every $f \in C_r^*G$

$$\|f\|_* \leq c\|f\|_{\ell,s} \tag{RD(3)}$$

for one (hence any) algebraic length ℓ on G . (Notice that RD(3) is equivalent to $H_\ell^s(G) \subseteq C_r^*G$.)

REMARK 2.11. *The Rapid Decay property depends on the choice of a length on G . However, because of Remark 2.2 it is straightforward to see that if G finitely generated has the Rapid Decay property with respect to any length, then it will have it for the word length as it dominates all the others. Moreover, the group G will have the Rapid Decay property for one length if and only if it has it for any equivalent length.*

If $s = 0$ or $\ell = 0$, then $H_\ell^s(G) = \ell^2(G)$. The $H_\ell^s(G)$'s are Hilbert spaces for the scalar product

$$\langle f, g \rangle_{\ell,s} = \sum_{\gamma \in G} f(\gamma) \overline{g(\gamma)} (1 + \ell(\gamma))^{2s},$$

$H_\ell^\infty(G)$ is a Fréchet space, but a priori none of them are algebras unless G has the Rapid Decay property, in which case, Lafforgue's adaptation of a result of Jolissaint [35] gives the following.

PROPOSITION 2.12 (Lafforgue [36]). Let G be a group with the Rapid Decay property for a length function ℓ . Then for s big enough (to satisfy the inequality RD(3)), the Hilbert space $H_\ell^s(G)$ is in fact a Banach sub-algebra of C_r^*G , which is dense and has the same K-theory as C_r^*G .

REMARK 2.13. *Taking the complex group algebra is a functor from the category of groups to the one of complex algebras, and it can be viewed both as a covariant or contravariant functor. In fact, algebraists tend to think of a covariant functor, meaning that given a group homomorphism $\varphi : G \rightarrow \Lambda$ one gets an algebra homomorphism $\varphi_* : \mathbf{CG} \rightarrow \mathbf{C}\Lambda$ by $\varphi_*(\sum_{\gamma \in G} f_\gamma \gamma) = \sum_{\gamma \in G} f_\gamma \varphi(\gamma)$, whereas analysts see a contravariant functor with $\varphi^* : \mathbf{C}\Lambda \rightarrow \mathbf{CG}$ by $\varphi^*(f) = f \circ \varphi$. The same holds for ℓ^1 , but not the reduced C^* -algebra, and this is an important problem in the Baum-Connes conjecture, as the rest of the objects defining it are all functorial.*

3. The case $G = \mathbf{Z}$

Let us look at the Rapid Decay property in the case where the group is \mathbf{Z} . Using inequality RD(1) it is straightforward to show that \mathbf{Z} has the Rapid Decay property, as for $f \in \mathbf{CG}$ supported on elements of length less than R , since the operator norm is always bounded above by the ℓ^1 norm, we have that

$$\|f\|_* \leq \|f\|_1 = \sum_{n=-R}^R |f(n)| \leq \sqrt{\sum_{n=-R}^R 1} \sqrt{\sum_{n=-R}^R |f(n)|^2} = \sqrt{2R+1} \|f\|_2$$

The last inequality is Cauch-Schwartz inequality, and the same computation in fact establishes the Rapid Decay property for groups with polynomial volume growth, and the degree D being half of the growth, see Theorem 2.8.

Let us now look at Connes' original definition for the case $G = \mathbf{Z}$. To do so, first let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ denote the circle and $\mathcal{L}(S^1)$ denote the Laurent polynomials over the circle, that is, finite sums of the form

$$A = \sum_{n \in \mathbf{Z}} A_n z^n.$$

We denote by $L^2(S^1)$ the square integrable functions (for the Lebesgue measure on S^1), by $\mathcal{C}(S^1)$ the continuous functions and by $\mathcal{C}^\infty(S^1)$ the smooth functions. We shall see that the Fourier transform is an algebra isomorphism $\mathbf{CZ} \simeq \mathcal{L}(S^1)$, that extends to an isometric isomorphism of Hilbert spaces $\ell^2(\mathbf{Z}) \simeq L^2(S^1)$, and an isometric isomorphism of C^* -algebras $C_r^*(\mathbf{Z}) \simeq \mathcal{C}(S^1)$ which restricted to $H^\infty(\mathbf{Z})$ has image $\mathcal{C}^\infty(S^1)$.

Let us start by recalling that $\mathcal{L}(S^1)$ is an algebra for pointwise addition and multiplication, and notice that $\mathcal{L}(S^1)$ is a prehilbert space, as it is a complex vector space and

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{L}(S^1) \times \mathcal{L}(S^1) &\rightarrow \mathbf{C} \\ (A, B) &\mapsto \int_{S^1} A(z) \overline{B(z)} dz \end{aligned}$$

defines an inner product, that by definition extends to the Hilbert space

$$L^2(S^1) = \{F : S^1 \rightarrow \mathbf{C} \mid \int_{S^1} |F(z)|^2 dz < \infty\}$$

Moreover, $\{z^n\}_{z \in \mathbf{Z}}$ is an orthonormal basis for $\mathcal{L}(S^1)$ since

$$\int_{S^1} z^n dz = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

and a Hilbert basis for $L^2(S^1)$.

DEFINITION 3.1. Let G be a finitely generated abelian group, its *Pontryagin dual* is the compact abelian group

$$\hat{G} = \text{Hom}(G, S^1).$$

The *Fourier transform* is the map

$$\begin{aligned} \mathcal{F} : \mathbf{C}G &\rightarrow \mathcal{C}(\hat{G}) \\ f &\mapsto \mathcal{F}(f) : \{x \mapsto \sum_{\gamma \in G} f(\gamma)x(\gamma)\}. \end{aligned}$$

In case where $G = \mathbf{Z}$, then $\hat{G} = S^1$ since a homomorphism is then determined by its value on 1, and the Fourier transform reads

$$\begin{aligned} \mathcal{F} : \mathbf{CZ} &\rightarrow \mathcal{C}(S^1) \\ f &\mapsto \mathcal{F}(f)(z) = \sum_{n \in \mathbf{Z}} f(n)z^n, \end{aligned}$$

which is an isomorphism from \mathbf{CZ} to $\mathcal{L}(S^1)$ since it is a bijection on the basis.

Moreover, since

$$\mathcal{F}(\delta_n * \delta_m) = \mathcal{F}(\delta_{n+m}) = z^{n+m} = z^n z^m = \mathcal{F}(\delta_n)\mathcal{F}(\delta_m)$$

we conclude that \mathcal{F} is an algebra isomorphism. We have hence proved the following.

PROPOSITION 3.2. The algebras \mathbf{CZ} and $\mathcal{L}(S^1)$ are isomorphic.

Let us now turn to $\ell^2(\mathbf{Z})$ and $L^2(S^1)$. It can seem obvious now that the Fourier transform \mathcal{F} extends to an isometric isomorphism between $\ell^2(\mathbf{Z})$ and $L^2(S^1)$, but there are some subtle points worth being mentioned. Indeed if \mathbf{CZ} is clearly dense in $\ell^2(\mathbf{Z})$, it is less clear why $\mathcal{L}(S^1)$ should be dense in $L^2(S^1)$. It is hence straightforward to check that the Fourier transform \mathcal{F} extends to an isometric injective linear map $\mathcal{F} : \ell^2(\mathbf{Z}) \rightarrow L^2(S^1)$, but surjectivity is a bit subtle. Indeed, to show that \mathcal{F} is surjective, it is tempting to pick $F \in L^2(S^1)$ and try to express it as a series $\sum_{n \in \mathbf{Z}} a_n z^n$, where a such that $a(n) = a_n$ belongs to $\ell^2(\mathbf{Z})$. We even know that the a_n 's should be the *Fourier coefficients* of F , given by

$$a_n = \int_{S^1} F(z) z^{-n} dz,$$

since the map $\mathcal{I} : \mathcal{L}(S^1) \rightarrow \mathbf{CZ}$, $A \mapsto \mathcal{I}(A)(n) = \int_{S^1} A(z) z^{-n} dz$ is an inverse for \mathcal{F} on \mathbf{CZ} . As such, this doesn't quite work (but almost) because there are continuous functions whose Fourier series diverge at some points (those examples have been provided by Du Bois-Reymond in 1876, see e.g. [6] p. 50).

However if we admit the better known fact that the space of smooth functions $C^\infty(S^1)$ is dense in $L^2(S^1)$, we can use the following particular case of Dirichlet's Theorem (see [6] for the general case) to show that $\mathcal{L}(S^1)$ is dense in $C^\infty(S^1)$ and hence in $L^2(S^1)$ as well.

THEOREM 3.3 (Dirichlet). *The Fourier series of an $F \in C^\infty(S^1)$ uniformly converges to F at every point.*

Let us now turn to the C^* -algebras and recall that $\mathcal{C}(S^1)$ is a C^* -algebra for pointwise addition and multiplication and for the supremum norm, given by

$$\|F\| = \sup\{|F(z)|, z \in S^1\} \quad (F \in \mathcal{C}(S^1)).$$

It is a standard application of the Stone-Weierstraß theorem that the space $\mathcal{L}(S^1)$ is dense in $\mathcal{C}(S^1)$ for the supremum norm (but notice that in view of Dirichlet's theorem and the examples of continuous functions having Fourier expansions divergent in some point, we know that the polynomials approaching pointwise a continuous function is not necessarily a partial sum of its Fourier expansion).

LEMMA 3.4. *Take $F \in \mathcal{C}(S^1)$ and consider the operator*

$$\begin{aligned} L_F : L^2(S^1) &\rightarrow L^2(S^1) \\ \xi &\mapsto F\xi. \end{aligned}$$

then its operator norm $\|L_F\|_$ is equal to $\|F\|$.*

PROOF. Let $z_0 \in S^1$ be such that $|F(z_0)| = \|F\|$. By definition, $\|L_F\|_* = \sup\{\|F\xi\|_2 \mid \xi \in L^2(S^1), \|\xi\|_2 = 1\}$. Moreover, for $\xi \in L^2(S^1)$ we have that

$$\|F\xi\|_2 = \sqrt{\int_{S^1} |F(z)\xi(z)|^2 dz} \leq \|F\| \sqrt{\int_{S^1} |\xi(z)|^2 dz} = \|F\| \|\xi\|_2,$$

so that $\|L_F\|_* \leq \|F\|$. To approach $\|F\|$ with elements of the form $\|F\xi\|_2$ and $\|\xi\|_2 = 1$, take $\epsilon > 0$ and let U_ϵ be a neighborhood of z_0 such that $|F(z)| > |F(z_0)| - \epsilon > 0$ for any $z \in U_\epsilon$ (if $\|F\| = 0$ it's obvious). Define

$$\xi_\epsilon(z) = \begin{cases} |U_\epsilon|^{-1/2} & \text{if } z \in U_\epsilon \\ 0 & \text{otherwise} \end{cases}$$

Then $\|\xi_\epsilon\|_2 = 1$ and $\|F\xi_\epsilon\|_2 \geq |F(z_0)| - \epsilon = \|F\| - \epsilon$. □

It is now straightforward to deduce the following.

PROPOSITION 3.5. The algebra isomorphism $\mathcal{F} : \mathbf{CZ} \rightarrow \mathcal{L}(S^1)$ extends to an isometric isomorphism $C_r^*(\mathbf{Z}) \rightarrow \mathcal{C}(S^1)$.

We now end this discussion by looking at smooth functions. Notice that, for $F \in \mathcal{C}^1(S^1)$, if $F(z) = \sum_{n \in \mathbf{Z}} A_n z^n$, its derivative is given by

$$F'(z) = \sum_{n \in \mathbf{Z}} in A_n z^n,$$

so that in case where $F \in \mathcal{C}^k(S^1)$, the k -th derivative is given by

$$F^{(k)}(z) = \sum_{n \in \mathbf{Z}} i^k n^k A_n z^n.$$

Since everything lives in $L^2(S^1)$, we have that $F \in \mathcal{C}(S^1)$ is actually in $\mathcal{C}^\infty(S^1)$ if and only if all its derivatives are in $L^2(S^1)$, which means that

$$\sum_{n \in \mathbf{Z}} |n|^{2k} |A_n|^2 < \infty$$

for any $k \in \mathbf{N}$. With a minor change of $|n|$ by $(1 + \ell_{\mathbf{Z}}(n))$ (up to a constant, depending on the chosen finite generating set for \mathbf{Z}) we see that $F \in \mathcal{C}^\infty(S^1)$ if and only if it is the Fourier transform of an element in $H^\infty(\mathbf{Z})$. To summarize, we have obtained the following identifications:

$$\begin{aligned} \mathbf{Z} &\longleftrightarrow S^1 \\ \mathbf{CZ} &\simeq \mathcal{L}(S^1) \\ H^\infty(\mathbf{Z}) &\simeq \mathcal{C}^\infty(S^1) \\ C_r^*(\mathbf{Z}) &\simeq \mathcal{C}(S^1) \\ \ell^2(\mathbf{Z}) &\simeq L^2(S^1) \end{aligned}$$

In this case, the Rapid Decay property for \mathbf{Z} in the sense of the inclusion RD(2) amounts to the well-known inclusion $\mathcal{C}^\infty(S^1) \subseteq \mathcal{C}(S^1)$ because smooth functions are in particular continuous.

REMARK 3.6. *It is less clear to understand what $\ell^1(\mathbf{Z})$ corresponds to under the Fourier transform, but we can see that the Fourier coefficients of a smooth element belongs to $\ell^1(\mathbf{Z})$: Since $F \in \mathcal{C}^\infty(S^1)$ if and only if its Fourier coefficients are in $H^\infty(\mathbf{Z})$, it is enough to see that an element $f \in H^\infty(\mathbf{Z})$ has finite ℓ^1 norm. But for $f \in H^\infty(\mathbf{Z})$, it implies in particular that*

$$\sum_{n \in \mathbf{Z}} |f(n)|^2 (1 + |n|)^4 < \infty$$

and hence there is $n_0 \in \mathbf{N}$ such that for any $n \geq n_0$ one has that

$$(|f(n)| + |f(-n)|)^2 \leq 2(|f(n)|^2 + |f(-n)|^2) < \frac{1}{(1 + |n|)^4},$$

showing that $f \in \ell^1(\mathbf{Z})$ and hence $H^\infty(\mathbf{Z}) \subseteq \ell^1(\mathbf{Z})$. This argument can be extended to polynomial growth groups, but not to all groups with the Rapid Decay property as $H_\ell^\infty(G)$ is included in $\ell^1(G)$ precisely when the group has polynomial growth.

4. Equivalent definitions of property RD

In this section we shall see several equivalent definitions of the Rapid Decay property, giving us a bigger flexibility for using that property. See also [10] for more equivalent definitions.

PROPOSITION 4.1. Let G be a discrete group, endowed with a length function ℓ . Then the following are equivalent:

- (1) There are constants C and D such that for any $R \in \mathbf{N}$ and any $f \in \mathbf{C}G$ such that f is supported on elements shorter than R , the following inequality holds:

$$\|f\|_* \leq CR^D \|f\|_2, \quad \text{RD(1)}$$

- (2) The following containment holds

$$H_\ell^\infty(G) \subseteq C_r^*(G) \quad \text{RD(2)}$$

- (3) There exists constants $c, s \geq 0$ such that, for each $f \in \mathbf{C}G$ one has

$$\|f\|_* \leq c \|f\|_{\ell, s} \quad \text{RD(3)}$$

- (4) There are constants C and D such that for any $R \in \mathbf{N}$ and any $f, g \in \mathbf{C}G$ such that f is supported on elements shorter than R , the following inequality holds:

$$\|f * g\|_* \leq CR^D \|f\|_2 \|g\|_2, \quad \text{RD(4)}$$

- (5) There are constants C and D such that for any $R \in \mathbf{N}$ and any $f, g, h \in \mathbf{C}G$ such that f is supported on elements shorter than R , the following inequality holds:

$$|f * g * h(e)| \leq CR^D \|f\|_2 \|g\|_2 \|h\|_2 \quad \text{RD(5)}$$

- (6) There are constants M and $k \geq 0$ such that for any $\xi, \eta \in \ell^2(G)$ of norm one, the coefficients of the regular representation satisfy

$$\sum_{\gamma \in G} \frac{|\langle \gamma(\xi), \eta \rangle|^2}{(1 + \ell(\gamma))^{2k}} \leq M \quad \text{RD(6)}$$

- (7) There are constants C and D such that for any $R \in \mathbf{N}$ and for any $\xi, \eta \in \ell^2(G)$ of norm one, we have that the coefficients of the regular representation satisfy

$$\sum_{\gamma \in B_R} |\langle \gamma(\xi), \eta \rangle|^2 \leq CR^{2D}. \quad \text{RD(7)}$$

Condition RD(6) was first given by Breuillard at an AIM conference on the property of Rapid Decay, and RD(7) is due to Perrone in [47].

REMARK 4.2. *There are variations on each equivalent conditions:*

- (1) *One can always choose $f \in \mathbf{R}_+G$ instead of \mathbf{CG} . Indeed, to go from \mathbf{R}_+G to \mathbf{CG} , for instance in $RD(1)$, we write $f \in \mathbf{CG}$ as $f = f_1 - f_2 + i(f_3 - f_4)$ with $f_k \in \mathbf{R}_+G$ and the supports of f_k and f_{k+1} disjoint for $k = 1, 3$, then $\|f\|_2^2 = \sum_{k=1}^4 \|f_k\|_2^2$ and thus*

$$\|f\|_* \leq \sum_{k=1}^4 \|f_k\|_* \leq CR^D \sum_{k=1}^4 \|f_k\|_2 \leq \sqrt{4}CR^D \|f\|_2.$$

- (2) *All those conditions are expressed in terms of the regular representation (see Definition 2.4) but most conditions can be reformulated in terms of any unitary representation with some obvious changes.*
- (3) *The length does not need to be a word length, a similar string of equivalences holds for other length (even in case where G is not finitely generated), and the proofs are easily adapted (the length may not have integer values only).*

The multiplicative constants are unimportant and vary from one point to another. The degrees are more interesting and do vary as well. It is the same degree D for points (1), (4), (5) and (7), and the exact relations between D , s and k are as follows. Given s , it is obvious from the proof of $RD(1)$ implies $RD(3)$ that one can take $D = s$. However given D , it is less obvious that in fact one can take $s = D + \epsilon$ for any $\epsilon > 0$. This is done by Nica in [44]. Similarly, from the proof of $RD(6)$ implies $RD(3)$, given k one can take $k = s$, and adapting Nica's argument to the implication of $RD(6)$ would allow to chose $k = s + \epsilon$ for any $\epsilon > 0$. In the same vein, Perrone in [47] shows that $RD(7)$ implies $RD(6)$ with $k = D + \epsilon$ for any $\epsilon > 0$. According to Nica in [44], the degree in the sense of $RD(1)$ has to be greater than $1/2$ and knowing the exact degree has interesting consequences in view of Theorem 1.3, but methods for establishing the Rapid Decay property do not always give the best estimate for that degree.

PROOF. We will show that (1) \iff (3), then (2) \iff (3), and (1) \implies (4) \implies (5) \implies (1), then (3) \iff (6), and finally (1) \iff (7).

(3) \iff (1): Take $f \in \mathbf{CG}$ with support contained in B_R (a ball of radius R), we have assuming (3) that:

$$\|f\|_* \leq c\|f\|_{\ell, s} = c \sqrt{\sum_{\gamma \in B_R} |f(\gamma)|^2 (R+1)^{2s}} \leq c(R+1)^s \|f\|_2 \leq CR^s \|f\|_2$$

and thus (1) is satisfied, for $D = s$ and C depending on c and s .

Conversely, we denote, for $n \in \mathbf{N}$ by $S_n = \{\gamma \in G \mid \ell(\gamma) = n\}$ the sphere of radius n and compute, for $f \in \mathbf{CG}$:

$$\|f\|_* = \left\| \sum_{n=0}^{\infty} f|_{S_n} \right\|_* \leq \sum_{n=0}^{\infty} \|f|_{S_n}\|_*.$$

So, assuming (1) we have that

$$\begin{aligned}
\|f\|_* &\leq \sqrt{|f(e)|} + \sum_{n=1}^{\infty} Cn^D \|f|_{S_n}\|_2 \leq \sum_{n=0}^{\infty} K(n+1)^D \|f|_{S_n}\|_2 \\
&\leq K \sum_{n=0}^{\infty} (n+1)^{-1} (n+1)^{D+1} \|f|_{S_n}\|_2 \\
&\leq K \underbrace{\sqrt{\sum_{n=0}^{\infty} (n+1)^{-2}}}_{\pi/\sqrt{6}} \sqrt{\sum_{n=0}^{\infty} (n+1)^{2D+2} \|f|_{S_n}\|_2^2} \\
&\leq c \sqrt{\sum_{n=0}^{\infty} \sum_{\gamma \in S_n} |f(\gamma)|^2 (\ell(\gamma) + 1)^{2D+2}} = c \|f\|_{\ell, D+1}.
\end{aligned}$$

Hence we get (3) for $s = D + 1$.

(2) \iff (3): First notice that RD(3) is equivalent to $H_\ell^s(G) \subseteq C_r^*G$, so that (3) implies (2) is obvious since $H_\ell^\infty(G) \subseteq H_\ell^s(G)$, hence we will be looking at the other implication. Let us first prove that the graph of the inclusion $H_\ell^\infty(G) \rightarrow C_r^*G$ is closed. Indeed, let $\{f_n\}_{n \in \mathbb{N}}$ in $H_\ell^\infty(G)$ tend to f in $H_\ell^\infty(G)$ and the image under the inclusion $\{f_n\}_{n \in \mathbb{N}}$ tend to g in C_r^*G , we need to prove that $f = g$. Since operator convergence implies weak $*$ -convergence, $\{\langle f_n * \xi, \eta \rangle\}_{n \in \mathbb{N}}$ converges to $\langle g * \xi, \eta \rangle$ for any $\xi, \eta \in \ell^2(G)$. Since $\{f_n\}_{n \in \mathbb{N}}$ tends to f in $\ell^2(G)$ as well, it implies that $\{f_n * \xi * \eta(e)\}_{n \in \mathbb{N}}$ converges to $f * \xi * \eta(e)$ for any $\xi, \eta \in \mathbf{CG}$. According to Remark 2.6, $f * \phi * \psi(e) = \langle f * \phi, \psi^* \rangle$, so we conclude that $f = g$. The closed graph theorem in the generality of Proposition 1, Chapter I page 20 of [7], applied to the Fréchet space $H_\ell^\infty(G)$ and to C_r^*G (viewed as a Banach space) then implies that the inclusion RD(2) is continuous. This by definition amounts to the existence of an $s > 0$ and $c > 1$ such that $\|f\|_* \leq c \|f\|_{\ell, s}$ for any $f \in H_\ell^\infty(G)$, and hence in particular for any $f \in \mathbf{CG}$.

(1) \implies (4) \implies (5) \implies (1): That (1) implies (4) is obvious since for any $g \in \mathbf{CG}$, by definition of the operator norm of f we have that

$$\frac{\|f * g\|_2}{\|g\|_2} \leq \|f\|_*.$$

That (4) implies (5) follows from Cauchy-Schwartz inequality:

$$|f * g * h(e)| \leq \sum_{\gamma \in G} |f * g(\gamma) h^*(\gamma)| \leq \|f * g\|_2 \|h\|_2$$

where $h^*(\gamma) = \overline{h(\gamma^{-1})}$. To see that (5) implies (1) it is enough to define, for $\gamma \in G$

$$h(\gamma) = \frac{f * g(\gamma^{-1})}{\|f * g\|_2},$$

and notice that in that case $f * g * h(e) = \|f * g\|_2$ and $\|h\|_2 = 1$. Then, for any $\epsilon > 0$, by the definition of the operator norm and by density of \mathbf{CG} in $\ell^2(G)$, there

is $g_\epsilon \in \mathbf{CG}$ such that:

$$\|f\|_* - \epsilon \leq \frac{\|f * g_\epsilon\|_2}{\|g_\epsilon\|_2}$$

and since the above inequality holds for any $\epsilon > 0$ we recover point (1).

(3) \iff (6): Let us first assume RD(3) and take $k = s + 1$. For any $n \in \mathbf{N}$ we denote by S_n the sphere of radius n , that is the elements in G of length exactly equal to n . We set

$$a_n(\gamma) = \chi_{S_n}(\gamma) \frac{\overline{\langle \gamma(\xi), \eta \rangle}}{(1+n)^{2k}}$$

and compute

$$\begin{aligned} \sum_{\gamma \in S_n} \frac{|\langle \gamma(\xi), \eta \rangle|^2}{(1+\ell(\gamma))^{2k}} &= \sum_{\gamma \in S_n} \langle \gamma(\xi), \eta \rangle a_n(\gamma) = \langle a_n(\xi), \eta \rangle \leq \|a_n\|_* \\ &\leq c \|a_n\|_{\ell, s} = c \sqrt{\sum_{\gamma \in S_n} \frac{|\langle \gamma(\xi), \eta \rangle|^2}{(1+\ell(\gamma))^{4k}} (1+\ell(\gamma))^{2s}} \\ &= c \frac{1}{(1+n)} \sqrt{\sum_{\gamma \in S_n} \frac{|\langle \gamma(\xi), \eta \rangle|^2}{(1+\ell(\gamma))^{2k}}}. \end{aligned}$$

Where c is a constant that does not depend on ξ and η . Hence,

$$\sum_{\gamma \in G} \frac{|\langle \gamma(\xi), \eta \rangle|^2}{(1+\ell(\gamma))^{2k}} = \sum_{n=1}^{\infty} \sum_{\gamma \in S_n} \frac{|\langle \gamma(\xi), \eta \rangle|^2}{(1+\ell(\gamma))^{2k}} \leq \sum_{n=1}^{\infty} c \frac{1}{(1+n)^2} \leq M.$$

Conversely, assuming RD(6), we take $f \in \mathbf{CG}$ supported on a ball of radius R and $\xi \in \ell^2(G)$ of norm one and so that $f(\xi) \neq 0$ (for instance any Dirac mass would do), then, for $\eta = f(\xi)/\|f(\xi)\|_2$ we have that (using Cauchy-Schwartz inequality):

$$\begin{aligned} \|f(\xi)\|_2 &= \langle f(\xi), \eta \rangle = \sum_{\gamma \in G} |f(\gamma)| \langle \gamma(\xi), \eta \rangle = \sum_{\gamma \in G} \frac{\langle \gamma(\xi), \eta \rangle}{(1+\ell(\gamma))^k} |f(\gamma)| (1+\ell(\gamma))^k \\ &\leq \sqrt{\sum_{\gamma \in G} \frac{\langle \gamma(\xi), \eta \rangle}{(1+\ell(\gamma))^k}} \|f\|_{\ell, k} \leq M \|f\|_{\ell, k}. \end{aligned}$$

Since the operator norm of f is the supremum over all $\xi \in \ell^2(G)$ of norm one of the left hand side of the above inequality and M does not depend on ξ , we deduce RD(3), with $s = k$.

(1) \iff (7): To prove the direct implication, take $\xi, \eta \in \ell^2(G)$ of norm one, $R \geq 1$ and define

$$f(\gamma) = \chi_{B_R} \overline{\langle \gamma(\xi), \eta \rangle},$$

we compute

$$\begin{aligned} \sum_{\gamma \in B_R} |\langle \gamma(\xi), \eta \rangle|^2 &= \sum_{\gamma \in B_R} \langle \gamma(\xi), \eta \rangle f(\gamma) = \langle f(\xi), \eta \rangle \leq \|f\|_* \leq CR^D \|f\|_2 \\ &= CR^D \sqrt{\sum_{\gamma \in B_R} |\langle \gamma(\xi), \eta \rangle|^2}, \end{aligned}$$

and RD(7) follows. Conversely assuming RD(7), we take $f \in \mathbf{CG}$ and $\xi \in \ell^2(G)$ of norm one and so that $f(\xi) \neq 0$ (for instance any Dirac mass would do), then, for $\eta = f(\xi)/\|f(\xi)\|_2$ we have that (using Cauchy-Schwartz inequality):

$$\begin{aligned} \|f(\xi)\|_2 &= \langle f(\xi), \eta \rangle = \sum_{\gamma \in G} f(\gamma) \langle \gamma(\xi), \eta \rangle = \sum_{\gamma \in B_R} f(\gamma) \langle \gamma(\xi), \eta \rangle \\ &\leq \|f\|_2 \sqrt{\sum_{\gamma \in B_R} |\langle \gamma(\xi), \eta \rangle|^2} \leq CR^D \|f\|_2. \end{aligned}$$

Since the operator norm of f is the supremum over all $\xi \in \ell^2(G)$ of norm one of the left hand side of the above inequality and C does not depend on ξ , we deduce RD(1), with the same degree D . \square

REMARK 4.3.

- (1) Using either RD(1) or RD(3), we immediately see that any subgroup H of G has the Rapid Decay property for the induced length. Indeed, since if H is a subgroup of G , $f \in \mathbf{CH}$ supported in a ball of radius r can be viewed in \mathbf{CG} , supported in a ball of radius r as well and the right hand side doesn't change, whereas on the left hand side, the operator norm of f acting on $\ell^2(H)$ is smaller than the operator norm of f acting on $\ell^2(G)$ since there are more elements in $\ell^2(G)$ to evaluate f on.
- (2) According to Remark 2.2, the word length dominates any other length, so balls in the word length are the smallest, hence if a group has the Rapid Decay property with respect to any length, then in particular it will have the Rapid decay property with respect to any word length. This is the reason why we often omit specifying the length, assuming that we only deal with the word length.

EXAMPLE 4.4. For a discrete group G , the map $\ell_0 : G \rightarrow \mathbf{R}_+$ defined by $\ell_0(1) = 0$ and $\ell_0(\gamma) = K \in \mathbf{R}_+$ for any $\gamma \in G$ is a length function. But G has the Rapid Decay property with respect to ℓ_0 if and only if G is finite. Indeed, if G has the Rapid Decay property with respect to ℓ_0 , then there exists a constant C such that for any $f, g \in \mathbf{CG}$ then $\|f * g\|_2 \leq C \|f\|_2 \|g\|_2$, which implies that $\ell^2(G)$ is an algebra. This can happen if and only if G is finite, see [49]. The same statement holds if we just assume ℓ_0 to be bounded.

References

- [1] Cristina Antonescu and Erik Christensen, *Metrics on group C^* -algebras and a non-commutative Arzelà-Ascoli theorem*, J. Funct. Anal. **214** (2004), no. 2, 247–259, DOI 10.1016/j.jfa.2004.04.015. MR2083302
- [2] G. Arzhantseva, C. Druţu. *Geometry of infinitely presented small cancellation groups, Rapid Decay and quasi-homomorphisms*. arXiv:1212.5280
- [3] Sylvain Barré and Mikael Pichot, *The 4-string braid group B_4 has property RD and exponential mesoscopic rank* (English, with English and French summaries), Bull. Soc. Math. France **139** (2011), no. 4, 479–502. MR2869301
- [4] Sylvain Barré and Mikael Pichot, *La propriété de décroissance rapide pour le groupe de Wise* (French, with English and French summaries), Ann. Inst. Fourier (Grenoble) **65** (2015), no. 2, 709–724. MR3449165
- [5] Jason A. Behrstock and Yair N. Minsky, *Centroids and the rapid decay property in mapping class groups*, J. Lond. Math. Soc. (2) **84** (2011), no. 3, 765–784, DOI 10.1112/jlms/jdr027. MR2855801

- [6] Rajendra Bhatia, *Fourier series*, Texts and Readings in Mathematics, Hindustan Book Agency, New Delhi, 1993. MR1657675
- [7] N. Bourbaki, *Espaces vectoriels topologiques. Chapitres 1 à 5.*, Éléments de mathématiques.
- [8] Jacek Brodzki and Graham A. Niblo, *Approximation properties for discrete groups, C^* -algebras and elliptic theory*, Trends Math., Birkhäuser, Basel, 2006, pp. 23–35, DOI 10.1007/978-3-7643-7687-1_2. MR2276913
- [9] Brian H. Bowditch, *Embedding median algebras in products of trees*, Geom. Dedicata **170** (2014), 157–176, DOI 10.1007/s10711-013-9874-x. MR3199482
- [10] Adrien Boyer, *Quasi-regular representations and rapid decay*, Potential Anal. **44** (2016), no. 2, 355–372, DOI 10.1007/s11118-015-9515-0. MR3460029
- [11] Adrien Boyer, *Semisimple Lie groups satisfy property RD, a short proof* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **351** (2013), no. 9–10, 335–338, DOI 10.1016/j.crma.2013.05.007. MR3072155
- [12] Elisabetta Candellero and Lorenz A. Gilch, *Phase transitions for random walk asymptotics on free products of groups*, Random Structures Algorithms **40** (2012), no. 2, 150–181, DOI 10.1002/rsa.20370. MR2877562
- [13] Donald I. Cartwright, *Some examples of random walks on free products of discrete groups*, Ann. Mat. Pura Appl. (4) **151** (1988), 1–15, DOI 10.1007/BF01762785. MR964500
- [14] Indira Chatterji, *Property (RD) for cocompact lattices in a finite product of rank one Lie groups with some rank two Lie groups*, Geom. Dedicata **96** (2003), 161–177, DOI 10.1023/A:1022184513930. MR1956838
- [15] I. Chatterji and K. Ruane, *Some geometric groups with rapid decay*, Geom. Funct. Anal. **15** (2005), no. 2, 311–339, DOI 10.1007/s00039-005-0508-9. MR2153902
- [16] I. Chatterji, C. Pittet, and L. Saloff-Coste, *Connected Lie groups and property RD*, Duke Math. J. **137** (2007), no. 3, 511–536, DOI 10.1215/S0012-7094-07-13733-5. MR2309152
- [17] I. Chatterji and L. Saloff-Coste. *Introduction to the property of Rapid decay*. Notes not intended for publication, available at: <http://www.aimath.org/WWN/rapiddecay/IntroRD.pdf>
- [18] I. Chatterji, Ch. Pittet and L. Saloff-Coste. *On the decay of the heat kernel*. Work in progress.
- [19] Laura Ciobanu, Derek F. Holt, and Sarah Rees, *Rapid decay is preserved by graph products*, J. Topol. Anal. **5** (2013), no. 2, 225–237, DOI 10.1142/S1793525313500052. MR3062944
- [20] Laura Ciobanu, Derek F. Holt, and Sarah Rees, *Rapid decay and Baum-Connes for large type Artin groups*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6103–6129, DOI 10.1090/tran/6532. MR3461028
- [21] Cornelia Druțu and Mark Sapir, *Relatively hyperbolic groups with rapid decay property*, Int. Math. Res. Not. **19** (2005), 1181–1194, DOI 10.1155/IMRN.2005.1181. MR2147058
- [22] Alain Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994. MR1303779
- [23] Alain Connes and Henri Moscovici, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology **29** (1990), no. 3, 345–388, DOI 10.1016/0040-9383(90)90003-3. MR1066176
- [24] Lukasz Garncarek, *Property of rapid decay for extensions of compactly generated groups*, Publ. Mat. **59** (2015), no. 2, 301–312. MR3374609
- [25] L. Garncarek. *Mini-course: property of rapid decay*. <https://arxiv.org/abs/1603.06730>
- [26] F. Gautero. *Rapid Decay and 3-manifold groups*. Preprint 2016.
- [27] Sébastien Gouëzel, *Local limit theorem for symmetric random walks in Gromov-hyperbolic groups*, J. Amer. Math. Soc. **27** (2014), no. 3, 893–928, DOI 10.1090/S0894-0347-2014-00788-8. MR3194496
- [28] Rostislav Grigorchuk and Tatiana Nagnibeda, *Complete growth functions of hyperbolic groups*, Invent. Math. **130** (1997), no. 1, 159–188, DOI 10.1007/s002220050181. MR1471889
- [29] Uffe Haagerup, *An example of a nonnuclear C^* -algebra, which has the metric approximation property*, Invent. Math. **50** (1978/79), no. 3, 279–293, DOI 10.1007/BF01410082. MR520930
- [30] Pierre de la Harpe, *Groupes hyperboliques, algèbres d’opérateurs et un théorème de Jolissaint* (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 14, 771–774. MR972078
- [31] Pierre de la Harpe, A. Guyan Robertson, and Alain Valette, *On the spectrum of the sum of generators of a finitely generated group. II*, Colloq. Math. **65** (1993), no. 1, 87–102. MR1224787
- [32] Carl Herz, *Sur le phénomène de Kunze-Stein* (French), C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A491–A493. MR0281022

- [33] Ronghui Ji and Larry B. Schweitzer, *Spectral invariance of smooth crossed products, and rapid decay locally compact groups*, *K-Theory* **10** (1996), no. 3, 283–305, DOI 10.1007/BF00538186. MR1394381
- [34] Paul Jolissaint, *Rapidly decreasing functions in reduced C^* -algebras of groups*, *Trans. Amer. Math. Soc.* **317** (1990), no. 1, 167–196, DOI 10.2307/2001458. MR943303
- [35] Paul Jolissaint, *K -theory of reduced C^* -algebras and rapidly decreasing functions on groups*, *K-Theory* **2** (1989), no. 6, 723–735, DOI 10.1007/BF00538429. MR1010979
- [36] Vincent Lafforgue, *A proof of property (RD) for cocompact lattices of $SL(3, \mathbf{R})$ and $SL(3, \mathbf{C})$* , *J. Lie Theory* **10** (2000), no. 2, 255–267. MR1774859
- [37] Vincent Lafforgue, *K -théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes* (French), *Invent. Math.* **149** (2002), no. 1, 1–95, DOI 10.1007/s002220200213. MR1914617
- [38] Steven P. Lalley, *Finite range random walk on free groups and homogeneous trees*, *Ann. Probab.* **21** (1993), no. 4, 2087–2130. MR1245302
- [39] H. Leptin, *On locally compact groups with invariant means*, *Proc. Amer. Math. Soc.* **19** (1968), 489–494. MR0239001
- [40] Alexander Lubotzky, Shahar Mozes, and M. S. Raghunathan, *Cyclic subgroups of exponential growth and metrics on discrete groups* (English, with English and French summaries), *C. R. Acad. Sci. Paris Sér. I Math.* **317** (1993), no. 8, 735–740. MR1244421
- [41] Igor Pak and Tatiana Smirnova-Nagnibeda, *On non-uniqueness of percolation on non-amenable Cayley graphs* (English, with English and French summaries), *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), no. 6, 495–500, DOI 10.1016/S0764-4442(00)00211-1. MR1756965
- [42] Amos Nevo, *Spectral transfer and pointwise ergodic theorems for semi-simple Kazhdan groups*, *Math. Res. Lett.* **5** (1998), no. 3, 305–325, DOI 10.4310/MRL.1998.v5.n3.a5. MR1637840
- [43] Amos Nevo, *On discrete groups and pointwise ergodic theory*, *Random walks and discrete potential theory (Cortona, 1997)*, *Sympos. Math.*, XXXIX, Cambridge Univ. Press, Cambridge, 1999, pp. 279–305. MR1802436
- [44] Bogdan Nica, *On the degree of rapid decay*, *Proc. Amer. Math. Soc.* **138** (2010), no. 7, 2341–2347, DOI 10.1090/S0002-9939-10-10289-5. MR2607863
- [45] B. Nica. *On operator norms for hyperbolic groups*. To appear in *Journal of Topology and Analysis*.
- [46] D. V. Osin. *Property (RD) for $C^*(1/6)$ -groups*. Preprint 2014.
- [47] M. Perrone. *Rapid Decay and weak containment of unitary representations*. Unpublished.
- [48] Mattia Perrone, *Radial rapid decay property for cocompact lattices*, *J. Funct. Anal.* **256** (2009), no. 11, 3471–3489, DOI 10.1016/j.jfa.2009.02.007. MR2514049
- [49] M. Rajagopalan, *On the L^p -space of a locally compact group*, *Colloq. Math.* **10** (1963), 49–52. MR0148792
- [50] J. Ramagge, G. Robertson, and T. Steger, *A Haagerup inequality for $\tilde{A}_1 \times \tilde{A}_1$ and \tilde{A}_2 buildings*, *Geom. Funct. Anal.* **8** (1998), no. 4, 702–731, DOI 10.1007/s000390050071. MR1633983
- [51] Mark Sapir, *The rapid decay property and centroids in groups*, *J. Topol. Anal.* **7** (2015), no. 3, 513–541, DOI 10.1142/S1793525315500247. MR3346932
- [52] Malik Talbi, *A Haagerup inequality, deformation of triangles and affine buildings*, *J. Inst. Math. Jussieu* **5** (2006), no. 2, 187–227, DOI 10.1017/S1474748005000241. MR2225041
- [53] Alain Valette, *Introduction to the Baum-Connes conjecture*, *Lectures in Mathematics ETH Zürich*, Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji; With an appendix by Guido Mislin. MR1907596
- [54] Alain Valette, *On the Haagerup inequality and groups acting on \tilde{A}_n -buildings* (English, with English and French summaries), *Ann. Inst. Fourier (Grenoble)* **47** (1997), no. 4, 1195–1208. MR1488249
- [55] Wolfgang Woess, *Random walks on infinite graphs and groups*, *Cambridge Tracts in Mathematics*, vol. 138, Cambridge University Press, Cambridge, 2000. MR1743100

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