1 Introduction

Exercise 1. Consider, for \(u \in H^2([0,1])\) and \(\lambda \in \mathbb{C}\) the equation

\[
-\frac{d^2u}{dx^2} = \lambda u. \tag{1}
\]

1. For which \(\lambda \in \mathbb{C}\) is there a nontrivial solution to (1) such that \(u(0) = u(1) = 0\)?
2. For which \(\lambda \in \mathbb{C}\) is there a nontrivial solution to (1) such that \(u(0) = u(1)\) and \(u'(0) = u'(1)\)?
3. For which \(\lambda \in \mathbb{C}\) is there a nontrivial solution to (1) such that \(u(0) = 0\)?
4. Conclusion?

Correction to Exercise 1. 1. \(\lambda = \pi^2n^2\), with \(n \in \mathbb{N}\), and the associated \(u\) are multiples of \(\sin(\pi nx)\).
2. \(\lambda = 4\pi^2n^2\), and the associated \(u\) are of the form \(\alpha \cos(2\pi nx) + \beta \sin(2\pi nx)\).
3. A general solution to (1) is of the form \(u(x) = \alpha e^{i\sqrt{\lambda}x} + \beta e^{-i\sqrt{\lambda}x}\). Therefore, there are solutions for all \(\lambda \in \mathbb{C}\), by taking \(\alpha = -\beta\).
4. Boundary conditions do matter, especially in spectral theory!

Exercise 2 (The homogeneous heat equation). Let \(g \in C_c^\infty(\mathbb{R}^n)\). Consider the equation

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\
u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{2}
\end{cases}
\]

A) Suppose that \(u \in C^0([0,\infty);S(\mathbb{R}^n)) \cap C^\infty((0,\infty);S(\mathbb{R}^n))\) satisfies (2).
1. Let \(v\) denote the Fourier transform of \(u\) in the spatial variables only, i.e. \(v(t,\xi) := \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}u(t,x)dx\). Show that we have

\[
v(t,\xi) = e^{-t|\xi|^2}(\mathcal{F}g)(\xi).\]

2. Deduce that for all \(t > 0\) and all \(x \in \mathbb{R}^n\), we have

\[
u(t,x) = \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)dy. \tag{3}
\]

B) Reciprocally, show that, if we define \(u\) by (3), then \(u \in C^0([0,\infty) \times \mathbb{R}^n) \cap C^\infty((0,\infty) \times \mathbb{R}^n)\) and \(u\) satisfies (2).

Correction to Exercise 2. A)1. We have, for every \(t > 0\),

\[
\frac{\partial v}{\partial t}(t,\xi) = \mathcal{F}\left(\frac{\partial u}{\partial t}\right)(t,\xi) = \mathcal{F}(-\Delta u)(t,\xi) = |\xi|^2v(t,\xi).
\]

The result follows by solving the differential equation.

2. We have

\[
u(t,x) = \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} v(\xi)e^{ix\cdot\xi}d\xi
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi e^{ix\cdot\xi}e^{-t|\xi|^2} \int_{\mathbb{R}^n} dy e^{-iy\cdot\xi}g(y)
= \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} dy g(y)(\mathcal{F}(e^{-t|\xi|^2}))(y - x)
= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)dy.
\]

B) Differentiating under the integral sign, we see that for any \(\eta > 0\), we have \(u \in C^\infty(\mathbb{R}^n \times [\eta,\infty))\), so that \(u \in C^\infty(\mathbb{R}^n \times (0,\infty))\). Write \(\Phi(t,x) := \frac{1}{(2\pi)^n/2} e^{-\frac{|x|^2}{4t^2}}\), so that we have

\[
\frac{\partial u}{\partial t}(t,x) - \Delta u(x,t) = \int_{\mathbb{R}^n} \left(\frac{\partial \Phi}{\partial t} - \Delta_x \Phi\right)(x-y,t)g(y)dy.
\]
Now, \( \frac{\partial \Phi}{\partial t} = \left( \frac{|x|^2}{4t^2} - \frac{n}{4t} \right) \Phi \), while \( \frac{\partial \Phi}{\partial x} = \left( \frac{x^2}{4t^2} - \frac{1}{4t} \right) \Phi \), so that \( \Delta \Phi(x, t) = \left( \frac{|x|^2}{4t^2} - \frac{n}{4t} \right) \Phi \). Therefore, we have \( \frac{\partial u}{\partial t} + \Delta u = 0 \) for all \((x, t) \in \mathbb{R}^n \times (0, \infty)\).

To conclude, let us check that \( \lim_{t \to 0} u(x, t) = g(x) \), which will give us the continuity of \( u \) in \( \mathbb{R}^n \times [0, \infty) \).

Let \( \varepsilon > 0 \). We may find \( \delta > 0 \) such that for all \( y \in B(x, \delta) \), we have \( |g(y) - g(x)| < \varepsilon \).

Note that for any \( t > 0 \), we have \( \int_{\mathbb{R}^n} \Phi(t, x) dx = 1 \). Therefore, we have

\[
g(x) = \int_{\mathbb{R}^n} \Phi(t, x - y) g(x) dy,
\]

so that

\[
|g(x) - u(t, x)| = \left| \int_{\mathbb{R}^n} \Phi(t, x - y) (g(x) - g(y)) dy \right|
\leq \int_{B(x, \delta)} \Phi(t, x - y) (g(x) - g(y)) dy + \int_{\mathbb{R}^n \setminus B(x, \delta)} \Phi(t, x - y) (g(x) - g(y)) dy
\leq \int_{B(x, \delta)} |g(x) - g(y)| dy + 2\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x, \delta)} \Phi(t, x - y) dy
\leq \varepsilon + \frac{C}{\sqrt{t}}\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\delta}^{\infty} e^{-r^2} r^{n-1} dr,
\]

which goes to zero as \( t \to 0^+ \).

2 Chapter 1

2.1 Sobolev Spaces

Exercise 3. Show that \( H^2(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d) | \Delta f \in L^2(\mathbb{R}^d) \} \), where \( \Delta f \) is understood in the sense of distributions.

Correction to Exercise 3. We only have to show that \( H^2(\mathbb{R}^d) \subset \{ f \in L^2(\mathbb{R}^d) | \Delta f \in L^2(\mathbb{R}^d) \} \), the other inclusion being trivial.

Let \( f \in S(\mathbb{R}^d) \). We have

\[
f \in H^2(\mathbb{R}^d) \iff \partial^\alpha f \in L^2(\mathbb{R}^d) \quad \forall |\alpha| \leq 2
\iff \mathcal{F}(\partial^\alpha f) \in L^2(\mathbb{R}^d) \quad \forall |\alpha| \leq 2 \text{ by the Plancherel equality}
\iff (\xi^\alpha \mathcal{F}(f)) \in L^2(\mathbb{R}^d) \quad \forall |\alpha| \leq 2
\iff (|\xi|^2 \mathcal{F}(f)) \in L^2(\mathbb{R}^d)
\iff \Delta f \in L^2(\mathbb{R}^d)
\]

Since this is true for every \( f \in S(\mathbb{R}^d) \), we deduce the result by density.

2.2 Compact operators

Exercise 4. Let \( \mathcal{H} \) be a Hilbert space, and \( (K_n) \) be a sequence of compact operators on \( \mathcal{H} \). Suppose that there exists a bounded operator \( K \in \mathcal{H} \) such that \( \|K - K_n\| \to 0 \). Show that \( K \) is compact.

Hint: Use the characterisation of pre-compact sets by covering of balls

Correction to Exercise 4. Let \( \varepsilon > 0 \). There exists \( n_0 \) such that \( \|K - K_{n_0}\| < \varepsilon/2 \). The image of the unit ball \( B(0, 1) \) by \( K_{n_0} \) is a pre-compact set, so there exists \( x_1, \ldots, x_N \in \mathcal{H} \) such that \( K_{n_0}(B(0, 1)) \subset \bigcup_{k=0}^N B(x_k, \varepsilon/2) \).

Let \( x \in B(0, 1) \). There exists \( 1 \leq k \leq N \) such that \( \|K_{n_0}(x) - x_i\| \leq \varepsilon/2 \). We have

\[
\|K(x) - x_i\| \leq \|K(x) - K_{n_0}(x)\| + \|K_{n_0}(x) - x_i\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore, \( K(B(0, 1)) \subset \bigcup_{k=0}^N B(x_k, \varepsilon) \), so that \( K \) is compact.
Exercise 5. Let $H$ be a Hilbert space, and let $K$ be a compact operator on $H$. Show that there exists a sequence $K_n$ of finite rank operators on $H$ such that $\|K - K_n\| \to 0$.

Hint: Cover $K(B(0, 1))$ by finitely many small balls, and project on the center of each ball.

Correction to Exercise 5. Let $n \geq 1$. The set $K(B(0, 1))$ is pre-compact, so we may cover it by finitely many balls of radius $\frac{1}{n}$:

$$K(B(0, 1)) \subset \bigcup_{k=0}^{N(n)} B(x_k, \frac{1}{n}).$$

We denote by $P_n$ the orthogonal projection on $\text{Span}\{(x_k, 1 \leq k \leq N(n))\}$. We define $K_n := P_n K$. This is a linear operator, and it has finite rank, so it is compact. Let $x \in B(0, 1)$. There exists $k$ such that $Kx \in B(x_k, n^{-1})$. We have

$$\| (K - K_n)(x) \| \leq \| Kx - x_k \| + \| P_n Kx - x_k \|$$

$$\leq \frac{1}{n} + \| P_n Kx - P_n x_k \| \quad \text{since} \quad P_n x_k = x_k$$

$$\leq \frac{1}{n} + \| Kx - x_k \| \quad \text{since} \quad P_n \text{ is a contraction}$$

$$\leq \frac{2}{n}.$$

This is true for every $x \in B(0, 1)$, so that $\| K - K_n \| \leq \frac{2}{n}$. The result follows.

Exercise 6 (Hilbert-Schmidt operators). Let $(X, A, \mu)$ be a measured space. Let $N = N(x, y) \in L^2((X, A, \mu)^2)$. Consider the operator $K$ on $L^2(X, A, \mu)$ given by

$$(Kf)(x) = \int N(x, y) f(y) d\mu(y).$$

Such an operator is called a Hilbert-Schmidt operator.

1. Let $N_x : y \mapsto N(x, y)$. Why does $N_x$ belong to $L^2(X, A, \mu)$ for almost every $x \in X$?
2. Check that $K : L^2 \to L^2$ is well-defined, and that $\|Kf\|_{L^2} \leq \|N\|_{L^2} \|f\|_{L^2}$.
3. Show that the operator $K$ is compact. Hint: use the fact that $Kf(x) = \langle f, N_x \rangle$.

Correction to Exercise 6. 1. This is just a consequence of Fubini’s theorem.

2. Let $f \in L^2(X, A, \mu)$. For almost every $x \in X$, $N_x$ belongs to $L^2(X, A, \mu)$, and by Cauchy-Schwartz, we have for those $x$ that $\|Kf(x)\| \leq \|N_x\|_{L^2} \|f\|_{L^2}$. Integrating in $x$, we obtain, using Fubini’s theorem, that $\|Kf\|_{L^2} \leq \|N\|_{L^2} \|f\|_{L^2}$.

3. To show that $K$ is compact, we take $f_n$ a sequence in $L^2$ converging weakly to $f \in L^2$, and show that $Kf_n$ converges strongly. The sequence $f_n$ being convergent, it is bounded by some $C > 0$.

Let $x \in X$ be such that $N_x \in L^2$. We have $Kf_n(x) = \langle f_n, N_x \rangle$, which converges to $\langle f_n, N_x \rangle = Kf(x)$. Furthermore, we have for those $x \in X$ that $\|Kf(x)\|^2 \leq C \|N_x\|^2_{L^2}$, which is an integrable function by Fubini’s theorem.

Therefore, we may apply the Lebesgue dominated convergence theorem to conclude that $Kf_n$ converges strongly to $Kf$ in $L^2$, thus concluding the proof.

2.3 Elliptic problems

Exercise 7. Let $\Omega \subset \mathbb{R}^3$ be a bounded regular open set. We want to show that the equation

$$\begin{align*}
-\Delta u &= u^3, & \text{in} \quad \Omega \\
u &= 0, & \text{on} \quad \partial \Omega
\end{align*}$$

admits a non-trivial weak solution.

1. Show that there exists a minimizer to the problem

$$\inf \left\{ \int_{\Omega} |\nabla u|^2 \, |u \in H_0^1(\Omega), \int_{\Omega} u^4 = 1 \right\}$$

$$\tag{4}$$

2. Show that if $u$ is a minimizer of (4), then there exists $\lambda \in \mathbb{R}$ such that $-\Delta u = \lambda u^3$.
3. Conclude.
Correction to Exercise 7. 1. Let \( u_n \in H^1_0(\Omega) \) be such that \( \int_\Omega u_n^4 = 1 \) for all \( n \), and

\[
\int_\Omega |\nabla u_n|^2 \to \inf \left\{ \int_\Omega |\nabla u|^2 \bigg| u \in H^1_0(\Omega), \int_\Omega u^4 = 1 \right\} =: \alpha.
\]

\((u_n)\) is bounded in \( H^1_0(\Omega) \). Recall that \( H^1_0(\Omega) \) can be compactly embedded in \( L^4(\Omega) \), so we may extract a subsequence, which we still denote by \( u_n \), which converges strongly in \( L^4(\Omega) \) towards \( v \in H^1_0(\Omega) \). In particular, we have \( \int_\Omega v^4 = 1 \), so that

\[
\int_\Omega |\nabla v|^2 \geq \inf \left\{ \int_\Omega |\nabla u|^2 \bigg| u \in H^1_0(\Omega), \int_\Omega u^4 = 1 \right\}.
\]

For any \( w \in H^1_0(\Omega) \), we have \( \langle u_n, w \rangle_{H^1_0} \to \langle v, w \rangle_{H^1_0} \). In particular, for \( w = v \), we have

\[
\|v\|_{H^1_0}^2 = \lim_{n \to \infty} \langle u_n, v \rangle_{H^1_0} \leq \lim \sup \|u_n\|_{H^1_0} \|v\|_{H^1_0} = \alpha \|v\|_{H^1_0}^2.
\]

Therefore, we have \( \|v\|_{H^1_0} \leq \alpha \), so that \( \|v\|_{H^1_0} = \alpha \). We have shown that \( v \) is a minimizer of the problem.

2. Let \( \varphi \in C^\infty(\Omega) \). \( v \) being a minimizer of the previous problem, we have for any \( \varepsilon \in \mathbb{R} \)

\[
\int_\Omega |\nabla v|^2 \leq \frac{\int_\Omega |\nabla v + \varepsilon \nabla \varphi|^2}{\left(\int_\Omega (v + \varepsilon \varphi)^4\right)^{1/2}}.
\]

Let us carry out a Taylor expansion as \( \varepsilon \to 0 \). We have

\[
\int_\Omega |\nabla v + \varepsilon \nabla \varphi|^2 = \int_\Omega |\nabla v|^2 + 2\varepsilon \int_\Omega \nabla v \cdot \nabla \varphi + O(\varepsilon^2),
\]

while

\[
\left(\int_\Omega (v + \varepsilon \varphi)^4\right)^{-1/2} = \left(1 + 4\varepsilon \int_\Omega v^3 \varphi + O(\varepsilon^2)\right)^{-1/2} \quad \text{since} \quad \int_\Omega v^4 = 1
\]

\[
= 1 - 2\varepsilon \int_\Omega v^3 \varphi + O(\varepsilon^2).
\]

Therefore,

\[
\frac{\int_\Omega |\nabla v + \varepsilon \nabla \varphi|^2}{\left(\int_\Omega (v + \varepsilon \varphi)^4\right)^{1/2}} - \int_\Omega |\nabla v|^2 = 2\varepsilon \int_\Omega \nabla v \cdot \nabla \varphi - 2\alpha \varepsilon \int_\Omega v^3 \varphi + O(\varepsilon^2).
\]

Now, this quantity must be positive for any \( \varepsilon \in \mathbb{R} \). Therefore, the terms proportional to \( \varepsilon \) must vanish, so that

\[
\int_\Omega \nabla v \cdot \nabla \varphi = \alpha \int_\Omega v^3 \varphi.
\]

In other words, \( v \) is a weak solution to \(-\Delta v = \alpha v^3\).

3. We set \( u := \sqrt{\alpha}v \). We have \(-\Delta u = -\Delta \alpha v = \alpha^{3/2}v^3\), while \( u^3 = \alpha^{3/2}v^3 \). Therefore, \(-\Delta u = u^3\). \( u \) is a nontrivial solution of our equation, since \( v \) cannot be identically zero (since \( \int_\Omega v^4 = 1 \)).

**Exercise 8** (Robin boundary conditions). Let \( \Omega \subset \mathbb{R}^d \) be an open set with smooth boundary, and let \( \alpha \in \mathbb{R} \). Consider the elliptic problem

\[-\Delta u = f u,
\]

with Robin boundary conditions

\[\partial_n u = \alpha u\]

on \( \partial \Omega \).

1. Check that a weak solution of this equation is a function \( u \in H^1(\Omega) \) which satisfies

\[
\int_\Omega f v = \int_\Omega \nabla u \cdot \nabla v - \alpha \int_{\partial \Omega} u v \quad \forall v \in H^1(\Omega).
\]
2. Suppose that $\alpha < 0$. Show that, for any $f \in L^2(\Omega)$, there exists a unique $u \in H^1(\Omega)$ which satisfies (5).

3. Show that, for any $\alpha \in \mathbb{R}$, there exists a sequence $\lambda_1^2 \leq \lambda_2^2 \leq \ldots \leq \lambda_k^2 \leq \ldots$, and an orthonormal basis $(\varphi_k^2)_{k \geq 1}$ of $L^2(\Omega)$, with $\varphi_k^2 \in C^\infty(\Omega)$ satisfying

$$
\begin{cases}
-\Delta \varphi_k^2 = \lambda_k^2 \varphi_k^2 & \text{in } \Omega \\
\partial_n \varphi_k^2 = \alpha \varphi_k^2 & \text{on } \partial \Omega
\end{cases}
$$

in the weak sense.

**Correction to Exercise 8.** 1. We need to check two things: that every strong solution is a weak solution, and that a weak solution which is regular enough is also a strong solution.

Let $u$ be a strong solution of the problem, that is, a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $-\Delta u = f$, in $\Omega$ and $\partial_n u = \alpha u$ on $\partial \Omega$. Let $v \in H^1(\Omega)$. By Green's formula, we have

$$
\int_{\Omega} \nabla u \cdot \nabla v - \alpha \int_{\partial \Omega} u v = \int_{\Omega} (-\Delta u) v + \int_{\partial \Omega} (\partial_n u) v - \alpha \int_{\partial \Omega} u v
$$

$$
= \int_{\Omega} f v,
$$

so $u$ is indeed a weak solution.

Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a weak solution of the problem. Then, using Green's formula again, we have

$$
\int_{\Omega} (-\Delta u - f) v + \int_{\partial \Omega} (\partial_n u - \alpha u) v = 0.
$$

We deduce that, for any $v \in C^\infty(\Omega)$, we have $\int_{\Omega} (-\Delta u - f) v = 0$, so that $-\Delta u = f$ in $\Omega$. We then have

$$
\int_{\partial \Omega} \alpha (\partial_n u - \alpha u) v = 0,
$$

from which we deduce that $\partial_n u = \alpha u$ on $\partial \Omega$.

2. The map $H^1(\Omega) \times H^1(\Omega) \ni (u, v) \mapsto F(u, v) := \int_{\Omega} \nabla u \cdot \nabla v - \alpha \int_{\Omega} uv$ is bilinear, symmetric, and positive definite, since $\alpha < 0$. We have $\min(1, \alpha) \|u\|_{H^1}^2 \leq F(u, u) \leq (1 + \alpha) \|u\|_{H^1}^2$, for any $u \in H^1(\Omega)$, so $F(u, v)$ defines a scalar product which is equivalent to the usual scalar product on $H^1(\Omega)$. In particular, $v \mapsto \int_{\Omega} f v$ is continuous for the norm induced by this scalar product. We may therefore apply the Riesz’ theorem to conclude that there exists a unique $u \in H^1(\Omega)$ such that for any $v \in H^1(\Omega)$, we have $F(u, v) = \int_{\Omega} f v$. The result follows.

3. In this question, we do not suppose that $\alpha < 0$, so we may not apply the previous question directly. However, we may consider the bilinear symmetric map $F_\lambda(u, v) = \lambda \int_{\Omega} \nabla u \cdot \nabla v - \alpha \int_{\Omega} uv + \int_{\Omega} w$. We know from the trace theorem that there exists $C > 0$ such that for any $v \in H^1(\Omega)$, we have $\int_{\partial \Omega} v^2 \leq C\|v\|_{H^1(\Omega)}$. Therefore, when $\lambda > C\alpha$, we have $\min((\lambda - C\alpha), 1) \|u\|_{H^1} \leq F(u, u) \leq (\lambda + C\alpha + 1) \|u\|_{H^1}$. We may therefore apply Riesz’ theorem to conclude that, for any $f \in L^2(\Omega)$, there exists a unique $T_\lambda f \in H^1(\Omega)$, such that for all $v \in H^1(\Omega)$, we have $F(\lambda(u, v) = \langle f, v \rangle_{L^2(\Omega)}$. $T_\lambda$ is then a linear continuous operator, so that, if we write $T_\lambda := \iota_{H^1(\Omega) \rightarrow L^2(\Omega)} T_\lambda$, $T_\lambda$ is a compact operator. Let us check that it is self-adjoint.

Let $f, g \in L^2(\Omega)$. We have

$$
\int_{\Omega} (\nabla T_\lambda f) \cdot \nabla T_\lambda g \quad \text{since } T_\lambda g \text{ is a weak solution}
$$

$$
= \int_{\Omega} (\nabla T_\lambda g) \cdot f \quad \text{since } T_\lambda f \text{ is a weak solution}.
$$

The operator $T_\lambda$ is thus self-adjoint. Therefore, there exists a non-increasing sequence of numbers $(\mu_k)_{k \geq 1}$ going to zero, and an orthonormal basis $(\varphi_k)_{k \geq 1}$ of $L^2(\Omega)$, such that $T_\lambda \varphi_k^2 = \mu_k \varphi_k^2$.

From the equation $T_\lambda \varphi_k^2 = \mu_k \varphi_k^2$, we deduce that

$$
-\lambda \Delta \varphi_k^2 + 1 = \frac{1}{\mu_k} \varphi_k^2,
$$

so that, setting $\lambda_k^2 := \frac{1}{\mu_k}$, we obtain the result.

### 2.4 Spectral geometry

**Exercise 9.** *(Courant’s nodal theorem)*

Let $\Omega \subset \mathbb{R}^d$ is an open set with a (regular enough) boundary. We shall denote by $\lambda_k(\Omega)$ the $k$-th
eigenvalue of the positive Laplacian $-\Delta$ with Dirichlet boundary conditions, and by $(\phi_k)_{k\in \mathbb{N}}$ an associated orthonormal basis of $L^2(\Omega)$. Recall that we have

$$\lambda_k(\Omega) = \min_{E_k \subset H^1_0(\Omega)} \max_{v \neq 0} \frac{\int_{\Omega} |\nabla v(x)|^2 \, dx}{\int_{\Omega} v(x)^2 \, dx}.$$  \hfill (6)

For each $k$, let us write $D_k := \{ x \in \Omega; \phi_k(x) \neq 0 \}$. We denote by $N_k$ the number of connected components of $D_k$.

1. Suppose that $\lambda_k(\Omega) < \lambda_{k+1}(\Omega)$. Show that $N_k \leq k$.
   
   Hint: For each connected component $D_{i,k}$, of $D_k$, $1 \leq i \leq N_k$, consider the function
   
   $$\psi_{i,k}(x) = \begin{cases} \phi_k(x) & \text{if } x \in D_{i,k} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $N_k \geq k + 1$. Using (6), show that $\lambda_{k+1} \leq \lambda_k$.

2. Deduce that

   $$N_k \leq \max\{ k' \in \mathbb{N}; \lambda_{k'} = \lambda_k \}.$$  

3. Show that

   $$N_k = O(\lambda_k^{d/2}).$$  \hfill (7)

4. Find a domain $\Omega$ and a family of eigenfunctions and eigenvalues $\phi_k, \lambda_k$ with $-\Delta \phi_k = \lambda_k \phi_k$ with $\lambda_k \rightarrow \infty$ such that

   $$N_k = o(\lambda_k^{d/2}).$$  \hfill (8)

**Correction to Exercise 9.** 1. Note that the functions $\psi_{i,k}$ do all belong to $H^1_0(\Omega)$. Let $c = (c_1, \ldots, c_{N_k}) \in \mathbb{R}^{N_k}$, and set

   $$\psi_c := \sum_{i=1}^{N_k} c_i \psi_{i,k}.$$ 

We have

$$\int_{\Omega} |\nabla \psi_c(x)|^2 \, dx = \int_{\Omega} \psi_c(x) \Delta \psi_c(x) \, dx$$

$$= \sum_{i=1}^{N_k} c_i^2 \int_{\Omega} \psi_{i,k}(x) \Delta \psi_{i,k}(x) \, dx \quad \text{since the $\psi_{i,k}$ have disjoint support}$$

$$= \sum_{i=1}^{N_k} c_i^2 \int_{\Omega} \lambda_k |\psi_{i,k}(x)|^2 \, dx$$

$$= \lambda_k \int_{\Omega} |\psi_c|^2 \, dx.$$ 

Hence, the $\psi_c$ form a vector space of dimension $N_k$, on which the Rayleigh quotient is constant equal to $\lambda_k$. If we had $N_k \geq k + 1$, we would therefore have a vector space of dimension $k + 1$ on which the Rayleigh quotient takes value $\lambda_k$, hence we would have $\lambda_{k+1} \leq \lambda_k$, contradicting the assumption that $\lambda_k < \lambda_{k+1}$.

2. Let $k \in \mathbb{N}$, and let $k'$ be the largest integer such that $\lambda_k = \lambda_{k'}$. It must be finite, thanks to Weyl’s law. $k'$ satisfies the assumptions of the previous question, so we deduce that $N_k = N_{k'} \leq k'$.

3. Let us write $N(\lambda) := \text{Card}\{ k \in \mathbb{N}; \lambda_k \leq \lambda \}$. We have $N(\lambda_k) = k' \geq N_k$ by the previous question. On the other hand, by Weyl’s law, we have $N(\lambda_k) = O(\lambda_k^{d/2})$. The result follows.

4. On the square $[0, \pi] \times [0, \pi]$, one can take $\phi_k(x, y) = \sin(x) \sin(k y)$. We have $-\Delta \phi_k = (1 + k^2) \phi_k$, but $N_k = k$.

### Chapter 2

**Exercise 10.** Let $\mathcal{H}$ be a separable Hilbert space, and let $T$ be an injective self-adjoint operator on $\mathcal{H}$. Show that $T^{-1}$ is also self-adjoint.
Correction to Exercise 10. Recall that, for \((x,y) \in \mathcal{H}^2\), we write \(J(x,y) = (y,-x)\). We then have \(\text{Gr}(T^*) = (\text{Gr}(T))^\perp\). Now, writing \(K(x,y) = (y,x)\), we clearly have \(\text{Gr}(T^{-1}) = K(\text{Gr}(T))\). Therefore, we have
\[
\text{Gr}(T^{-1})^* = (\text{Gr}(T^{-1}))^\perp = (JK(\text{Gr}(T)))^\perp = (-KJ(\text{Gr}(T)))^\perp \quad \text{since} \quad JK = -KJ
\]
\[
= K(\text{Gr}(T))^\perp \quad \text{since, for any subspace } F, \text{ we have } (K(F))^\perp = K(F^\perp)
\]
\[
= K(\text{Gr}(T)) \quad \text{since } T \text{ is self-adjoint}
\]
\[
= \text{Gr}(T^{-1}).
\]
Therefore, \(T^{-1}\) is self-adjoint.

Exercise 11. Let \(\mathcal{H}\) be a separable Hilbert space, and let \(A, B\) be self-adjoint operators on \(\mathcal{H}\). Suppose that \(\mathcal{D}(B) \subseteq \mathcal{D}(A)\) and \(Av = Bv\) for all \(v \in \mathcal{D}(B)\). Show that \(\mathcal{D}(A) = \mathcal{D}(B)\).

Correction to Exercise 11. Let \(v \in \mathcal{D}(A)\). We want to show that \(v \in \mathcal{D}(B)\). For this, we will show that \(v \in \mathcal{D}(B^*)\). The map \(\mathcal{D}(A) \ni w \mapsto \langle v, Aw \rangle\) is continuous, since \(v \in \mathcal{D}(A) = \mathcal{D}(A^*)\). Restricting its domain to \(\mathcal{D}(B)\), it will still be continuous, so that \(\mathcal{D}(B) \ni w \mapsto \langle v, Aw \rangle = \langle v, Bw \rangle\) is continuous. Therefore, \(v \in \mathcal{D}(B^*) = \mathcal{D}(B)\), so that \(\mathcal{D}(A) = \mathcal{D}(B)\).

Exercise 12. Let \(\mathcal{H}\) be a separable Hilbert space, and \(A\) be a continuous operator on \(\mathcal{H}\). If \(B\) is an operator on \(\mathcal{H}\), we define \(A + B\) to be the operator with \(\mathcal{D}(A + B) = \mathcal{D}(B)\), and \((A + B)v = Av + Bv\) if \(v \in \mathcal{D}(B)\).

1. Show that, if \(B\) is closed, then \(A + B\) is closed.
2. Show that, if \(B\) is densely defined (i.e., if \(\mathcal{D}(B)\) is dense in \(\mathcal{H}\)), then \((A + B)^* = A^* + B^*\).

Correction to Exercise 12. 1. Let \((v_n) \subseteq \mathcal{D}(A + B)\) be a sequence converging to \(v \in \mathcal{H}\), and such that \((A + B)v_n\) converges to \(w \in \mathcal{H}\). We want to show that \(v \in \mathcal{D}(A + B)\) and \(w = (A + B)v\).

A being continuous, \(Av_n\) converges to \(Av\). Therefore, \(Bv_n\) is also a convergent sequence. The operator \(B\) being closed, we deduce that \(v \in \mathcal{D}(B)\), and \(Bv_n\) converges to \(Bv\). Therefore, \(v \in \mathcal{D}(A + B)\) and \((A + B)v_n\) converges to \((A + B)v\).

2. Let \(v \in \mathcal{H}\). We have
\[
v \in \mathcal{D}((A + B)^*) \iff \mathcal{D}(A + B) \ni w \mapsto \langle v, (A + B)w \rangle\text{ is continuous}
\]
\[
\iff \mathcal{D}(B) \ni w \mapsto \langle v, Aw \rangle + \langle v, Bw \rangle\text{ is continuous since } \mathcal{D}(A + B) = \mathcal{D}(A)
\]
\[
\iff \mathcal{D}(B) \ni w \mapsto \langle v, Bw \rangle\text{ is continuous}
\]
\[
\iff v \in \mathcal{D}(B^*).
\]
Now, if \(v\) belongs to this space and \(w \in \mathcal{D}(A + B) = \mathcal{D}(B)\), we have
\[
\langle (A + B)^*v, w \rangle = \langle v, (A + B)w \rangle = \langle v, Aw \rangle + \langle v, Bw \rangle = \langle A^*v, w \rangle + \langle B^*v, w \rangle = \langle (A^* + B^*)v, w \rangle,
\]
so that \((A + B)^* = A^* + B^*\).

Exercise 13. Let \(\mathcal{H}_1, \mathcal{H}_2\) be Hilbert spaces, and let \(T_1\) be an operator on \(\mathcal{H}_1\), and \(T_2\) be an operator on \(\mathcal{H}_2\). \(T_1\) and \(T_2\) are said to be unitarily equivalent if there exists \(U : \mathcal{H}_1 \to \mathcal{H}_2\) a unitary isomorphism such that \(\mathcal{D}(T_2) = U\mathcal{D}(T_1)\), and \(UT_1U^*v = T_2v\) for all \(v \in \mathcal{D}(T_2)\).

1. Show that \(T_1\) is closed/symmetric/self-adjoint if and only if \(T_2\) is.
2. Show that \(\sigma(T_1) = \sigma(T_2)\), and \(\sigma_p(T_1) = \sigma_p(T_2)\).
3. Let \(\mathcal{H} = L^2(\mathbb{R}^d)\), and \(T\) be given by \(\mathcal{D}(T) = H^2(\mathbb{R}^d), T = -\Delta\). Show that \(\sigma(T) = [0, \infty)\).

Correction to Exercise 13. 1. Suppose that \(T_1\) is closed. Let us show that \(T_2\) is closed. Let \(x_n \in \mathcal{D}(T_2)\) converge to some \(x \in \mathcal{H}_2\), and such that \(T_2x_n\) converges to some \(y \in \mathcal{H}_2\). Then, for each \(n, z_n := U^*x_n \in \mathcal{D}(T_1)\), and \(z_n\) converges to \(z := U^*x\), since \(U^*\) is continuous. We have \(T_1z_n = T_1U^*x_n = U^*T_2x_n\), so that \(T_1z_n\) converges to \(U^*y\). \(T_1\) being closed, \(z_n\) must converge to some \(z \in \mathcal{D}(T_1)\), and \(T_1z = U^*y\).

Since \(x_n = Uz_n\), we have \(x_n \to Uz\), and \(y = UT_1z = T_2Uz = T_2x\). Therefore, \(T_2\) is closed.
Suppose that $T_1$ is symmetric. Let $v, w \in \mathcal{D}(T_2)$. We have
\[
\langle T_2v, w \rangle_{\mathcal{H}_2} = \langle T_2UU^*v, UU^*w \rangle_{\mathcal{H}_2} = \langle UT_1U^*v, UU^*w \rangle_{\mathcal{H}_2} = \langle T_1U^*v, U^*w \rangle_{\mathcal{H}_1}
\]
since $U$ is an isometry
\[
= \langle U^*v, T_1U^*w \rangle_{\mathcal{H}_1}
\]
since $T_1$ is symmetric
\[
= \langle U^*v, U^*T_2w \rangle_{\mathcal{H}_1} = \langle v, T_2w \rangle_{\mathcal{H}_2}
\]
since $U^*$ is an isometry.

Therefore, $T_2$ is symmetric.

Suppose that $T_1$ is self-adjoint. Then $T_1$ is symmetric, and so $T_2$ is also symmetric. The only thing we have to show is that $\mathcal{D}(T^*_2) \subset \mathcal{D}(T_2)$. Let $v \in \mathcal{D}(T_2)$. We have
\[
\mathcal{D}(T_2) \ni w \mapsto \langle v, T_2w \rangle_{\mathcal{H}_2}
\]
is continuous \iff $\mathcal{D}(T_2) \ni w \mapsto \langle U^*v, U^*T_2w \rangle_{\mathcal{H}_1}$ is continuous
\[
\mathcal{D}(T_2) \ni w \mapsto \langle U^*v, U^*T_2w \rangle_{\mathcal{H}_1}
\]
is continuous,
since $U^*$ is continuous. Therefore, $v \in \mathcal{D}(T_1)$ if and only if $U^*v \in \mathcal{D}(T^*_1) = \mathcal{D}(T_1)$. But $U^*v \in \mathcal{D}(T_1)$ is equivalent to $v \in \mathcal{D}(T_2)$. Therefore, $T_2$ is self-adjoint.

We have shown that the properties on $T_1$ imply the same properties on $T_2$. Of course, we deduce that properties on $T_2$ imply the same properties on $T_1$, by exchanging the role of $U$ and $U^*$.

2. Let $z \in \mathbb{C}$. We have
\[
(T_2 - z) = U(T_1 - z)U^*,
\]
so that $T_2 - z$ is invertible if and only if $T_1$ is invertible. This shows that $\rho(T_1) = \rho(T_2)$, and hence $\sigma(T_1) = \sigma(T_2)$.

Equation (9) also shows that $\ker(T_2 - z) \neq \{0\}$ if and only if $\ker(T_1 - z) \neq \{0\}$, so that $\sigma_p(T_1) = \sigma_p(T_2)$.

3. By the Plancherel equality, the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is an isometry. Furthermore, if $M_{|\xi|^2}$ denotes the operator of multiplication by $|\xi|^2$ (with the domain defined in the lecture notes), $T$ and $M_{|\xi|^2}$ are unitarily equivalent. We know that $\sigma(M_{|\xi|^2}) = [0, \infty)$, so that $\sigma(T) = [0, \infty)$.

Exercise 14. Let $\mathcal{H} = \ell^2(\mathbb{Z})$, and $(Tv)(n) := v(n + 1) + v(n - 1)$. Consider the map
\[
U : \ell^2(\mathbb{Z}) \rightarrow L^2(0, 2\pi) \quad (v(n))_n \mapsto \sum_{n \in \mathbb{Z}} v_n e^{inz}.
\]

1. Compute $UTU^*$.
2. What is $\sigma(T) \triangleq \sigma_p(T)$ ?

Correction to Exercise 14. 1. Let $f \in L^2(0, 2\pi)$. We denote by $c_n(f)$ its complex Fourier coefficients. We have $(U^*f)(n) = c_n(f)$, so that, writing $e_n(x) := e^{inx}$, we have
\[
UTU^*f = \sum_{n \in \mathbb{Z}} (c_{n+1}(f) + c_{n-1}(f)) e_n
\]
\[
= e^{-1} \sum_{n \in \mathbb{Z}} c_{n+1}(f) e_{n+1} + e_1 \sum_{n \in \mathbb{Z}} c_{n-1}(f) e_{n-1}
\]
\[
= (e^{-1} + e_1)f = M_{2\cos}f,
\]
where $M_{2\cos}$ denotes the operator of multiplication by $2\cos x$, which is continuous on $L^2(0, 2\pi)$.

2. By the previous exercise, $\sigma(T) = \sigma(M_{2\cos}) = [-2, 2]$, and $\sigma_p(T) = \sigma_p(M_{2\cos}) = \emptyset$.

4 Chapter 3

Exercise 15 (Unitary semigroup). Let $\mathcal{H}$ be a separable Hilbert space, and $T$ be a self-adjoint operator on $\mathcal{H}$. For each $t \in \mathbb{R}$, we write $U(t) = e^{-itT}$.

1. Show that $U(t) \in \mathcal{L}(\mathcal{H})$, and that $U(t)$ is unitary.
2. Show that $U(t + s) = U(t)U(s)$. 

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3. Show that \( \lim_{s \to t} U(s)v = U(t)v \) for any \( v \in \mathcal{H} \) and any \( t \in \mathbb{R} \).

4. Show that \( U(t)D(T) \subseteq D(T) \).

5. Let \( f \in D(T) \). Show that \( F : \mathbb{R} \ni t \mapsto U(t)f \in \mathcal{H} \) is in \( C^{1}(\mathbb{R};\mathcal{H}) \), and that it solves

\[
\frac{d}{dt} F(t) = TF(t).
\]

**Correction to Exercise 15.** 1. and 2. The map \( f_{1}(\lambda) := e^{-it\lambda} \) is bounded on \( \mathbb{R} \). Therefore, we may define \( U(t) := f_{1}(t) \) by the \( L^{2} \) functional calculus\(^1\), and its domain is then \( \mathcal{H} \).

By the properties of the functional calculus, we have \( f_{1}(T)f_{1}(T) = (f_{1}f_{1})(T) = f_{3}(T) \), so that \( U(t+s) = U(t)U(s) \).

In particular, \( U(t)^{-1} = U(-t) \), and, by another property of the continuous functional calculus, \( U(t) = \bar{U}(t) = f_{1}(T) = f_{3}(T) \). Therefore, \( U(t) \) is unitary.

3. The result follows from Lemma 13 in section 3.2.

4. We use again the \( L^{2} \) functional calculus. We know that there exists a countable set \( N \), a finite measure \( \mu \) on \( N \times \mathbb{R} \), and a unitary isomorphism \( U : \mathcal{H} \to L^{2}(N \times \mathbb{R}, \mu) \) such that, writing \( h(n,s) = s \), we have \( v \in D(T) \iff Uv \in D(M_{h}) \). But, by definition, we have \( (UU(t)v)(s) = e^{its}(Uv)(s) \), which is in \( D(M_{h}) \) if and only if \( Uv \) is. Therefore, \( U(t)D(T) = D(T) \).

5. To show that \( U(t)f \) is continuously differentiable for every \( t \in \mathbb{R} \), it suffices to show it for \( t = 0 \), to the previous question. By the functional calculus, we have

\[
\mathcal{U}(\frac{U(t) - \text{Id}}{t}) \mathcal{f}(s) = \frac{e^{its} - 1}{t} (\mathcal{U} \mathcal{f})(s).
\]

Now, we have \( \left| \frac{e^{its} - 1}{t} \right| \leq |s| \), and \( |s| (\mathcal{U} \mathcal{f})(s) \) is in \( L^{2}(N \times \mathbb{R}) \), so we may apply the dominated convergence theorem to conclude that \( \mathcal{U}(\frac{U(t) - \text{Id}}{t}) \mathcal{f}(s) \) converges to \(-i\mathcal{H} \mathcal{f} = \mathcal{U} \mathcal{f} \) as \( t \to 0 \). The result follows.

**Exercise 16.** Let \( \mathcal{H} = L^{2}(\mathbb{R}) \), \( T := -\frac{d^{2}}{dx^{2}} \) with \( D(T) = H^{2}(\mathbb{R}) \), and \( S := i \frac{d}{dx} \), with \( D(S) = H^{1}(\mathbb{R}) \).

1. Show that \( T = S^2 \). Do we have \( S = \sqrt{T} \)?

2. Compute explicitly \( e^{-itS} \) for \( t \in \mathbb{R} \).

3. Show that \( \cos(t \sqrt{T}) = \cos(ts) \) for all \( t \in \mathbb{R} \), and compute these operators explicitly.

4. Compute explicitly the operators \( \frac{\sin(t \sqrt{T})}{\sqrt{T}} \).

**Correction to Exercise 16.** 1. Let \( v \in D(T) \). Then \( v \in D(S), Sv \in D(T) \) and \( S^{2}v = Tv \). Therefore, \( T \subseteq S^{2} \). On the other hand, \( D(S^{2}) = \{ v \in D(S)| Sv \in D(S) \} = D(T) \), so that \( T = S^{2} \).

\( T \) is a self-adjoint operator, so we may define \( \sqrt{T} \) by the \( L^{2} \) functional calculus. We then have \( D(\sqrt{T}) = H^{1}(\mathbb{R}) \). However, \( \sqrt{T} \) will be a positive operator: for any \( v \in D(\sqrt{T}) \), we have \( \langle v, \sqrt{T}v \rangle \geq 0 \).

\( S \) is not a positive operator, since \( \langle v, Sv \rangle \) is negative when \( v(t) = e^{-it}t \). Therefore, \( S \neq \sqrt{T} \).

2. Let \( f \in H^{1}(\mathcal{H}) = D(S) \). We know thanks to the previous exercise that \( \frac{d}{dt} e^{-itS}f = (e^{-itS}f)' \). Now, the function

\[
f_{1}(s) := f(t + s)
\]

also satisfies \( \frac{d}{dt} f_{1} = f_{1}' \), and we have \( f_{0} = e^{-itS}f = f \). Therefore, we have \( e^{-itS}f = f_{1} \) for every \( f \in H^{1}(\mathbb{R}) \). Since \( H^{1}(\mathbb{R}) \) is dense in \( L^{2}(\mathbb{R}) \), the result follows for every \( f \in L^{2}(\mathbb{R}) \).

3. We have \( \cos(ts) f = \frac{e^{itS}f + e^{-itS}f}{2} = \frac{f_{t} + f_{-t}}{2} \).

Now, \( \cos(t \sqrt{T}) = \cos(ts) \sqrt{S^{2}} \). Using the \( L^{2} \) functional calculus, \( \cos(t \sqrt{S^{2}}) \) is conjugated in some \( L^{2} \) space to the multiplication by \( \cos(ts) = \cos(ts) \), since \( \cos(s) \) is an even function. Therefore, \( \cos(t \sqrt{T}) = \cos(S) \).

4. By the same argument as above, since \( s \mapsto \sin(s) \) is an even function, we have \( \frac{\sin(t \sqrt{T})}{\sqrt{T}} = \frac{\sin(ts)}{s} = \frac{1}{2} \sin(ts) \).

We have \( \sin(ts)f(x) = \frac{f(x + t) - f(x - t)}{2} \).

Now \( \frac{1}{s} \) is the inverse of \( S \): it is the operator which takes a function \( f \) to a function \( F \in L^{2}(\mathbb{R}) \) such that \( F' = if \). Let us check that \( (\sin(ts)f)(x) \) always belongs to the domain of \( S^{-1} \).

For any \( y \in \mathbb{R} \), we have

\[
\int_{-\infty}^{y} (f(x + t) - f(x - t)) dx = \int_{y-t}^{y+t} f(x) dx
\]

\(^{1}\)When \( T \) is a bounded operator, we may just use the continuous functional calculus.
Therefore,

\[ \int_{-\infty}^{y} \left| \int_{y-t}^{y} \sin(tS)f(x)dx \right|^2 dy = \frac{1}{4} \int_{-\infty}^{y} \left| \int_{y-t}^{y+t} f(x)dx \right|^2 dy \]

\leq \int_{y-t}^{y+t} ds \int_{\mathbb{R}} |f(x)|^2 dx

\leq 2||f||_{L^2}^2.

Therefore, \( i \int_{-\infty}^{y} \sin(tS)f(x)dx \) is in \( L^2 \), and its derivative is equal to \( i\sin(tS)f \). We hence have \( i \int_{-\infty}^{y} \sin(tS)f(x)dx = (S^{-1}\sin(tS)f)(y) \). Finally, we obtain that

\[ \left( \frac{\sin(t\sqrt{T})}{\sqrt{T}} \right)(y) = \frac{1}{2} \int_{y-t}^{y+t} f(x)dx. \]

**Exercise 17.** Let \( \mathcal{H} \) be a separable Hilbert space, let \( f \in \mathcal{H} \) and let \( \omega \in \mathbb{R} \). Let \( H \) be a self-adjoint operator on \( \mathcal{H} \). We shall denote by \( \sigma(H) \) its spectrum.

Consider a solution \( u \in C^1(\mathbb{R}; \mathcal{H}) \) of the equation

\[ \begin{cases} i \frac{\partial u}{\partial t} + Hu(t) = -f e^{i\omega t} \\ u(0) = 0 \end{cases} \tag{10} \]

\[ E(t) := ||u(t)||_{\mathcal{H}}^2. \]

1. Show that the solutions of (10) can be written as

\[ u(t) = \left( \frac{e^{it\omega} - e^{itH}}{\omega - H} \right)f. \]

2. Suppose that \( \omega \notin \sigma(H) \). Show that \( t \mapsto E(t) \) is bounded.

3. Suppose that \( Hf = \omega f \), \( ||f|| = 1 \). Show that \( E(t) = t^2 \).

4. Suppose that \( \omega \in \sigma_{ac}(H) \) and that the density of the spectral measure of \( f \) is continuous at \( \omega \). Show that \( E(t) \sim t^2 \) for some \( c > 0 \).

**Correction to Exercise 17.** 1. Note that the function \( \lambda \mapsto \frac{e^{it\omega} - e^{it\lambda}}{\omega - \lambda} \) is continuous and goes to zero at infinity, so we may apply the spectral theorem to it. Write \( v(t) := \left( \frac{e^{it\omega} - e^{itH}}{\omega - H} \right)f \). We have

\[ i \frac{\partial v}{\partial t} = \left( \frac{He^{itH} - \omega e^{it\omega}}{\omega - H} \right)f. \]

Therefore,

\[ i \frac{\partial v}{\partial t} + e^{i\omega t}f = \left( \frac{He^{itH} - \omega e^{it\omega} + (\omega - H)e^{it\omega}}{\omega - H} \right)f = Hu(t), \]

and the result follows, since \( v(0) = 0 \).

2. Recall that the spectral measure \( \mu_f \) is defined by

\[ \int \varphi d\mu_f = \langle \varphi(H)f, f \rangle \]

when \( \varphi \in C_0(\mathbb{R}; \mathbb{C}) \).

Hence,

\[ E(t) = \left\langle \left( \frac{e^{it\omega} - e^{it\lambda}}{\omega - \lambda} \right)f, \left( \frac{e^{it\omega} - e^{itH}}{\omega - H} \right)f \right\rangle \]

\[ = \left\langle \left( \frac{e^{it\omega} - e^{it\lambda}}{\omega - \lambda} \right)^2 f, f \right\rangle \]

\[ = \int_{\mathbb{R}} \left| \frac{e^{it\omega} - e^{it\lambda}}{\omega - \lambda} \right|^2 d\mu_f(\lambda) \]

\[ = \int_{\mathbb{R}} \frac{1}{(\omega - \lambda)^2} \left| e^{it(\omega + \lambda)/2} (e^{it(\omega - \lambda)/2} - e^{it(\lambda - \omega)/2}) \right|^2 d\mu_f(\lambda) \]

\[ = 4 \int_{\mathbb{R}} \frac{\sin^2 \left( \frac{t}{2}(\omega - \lambda) \right)}{(\omega - \lambda)^2} d\mu_f(\lambda) \]
If $\omega \notin \sigma(H)$, then the spectral measure vanishes in a neighbourhood of $\omega$. Therefore, since the denominator and the sinus are bounded, $E(t)$ is bounded.

3. If $Hf = \omega f$, then $\mu_f(\lambda)$ is a Dirac at $\omega$. We have $\frac{\sin^2 \left( \frac{1}{2}(\omega - \lambda) \right)}{(\omega - \lambda)^2} = \frac{t^2}{4} + g(\omega - \lambda)$, where $g$ is continuous and vanishes at zero. Therefore, $E(t) = t^2$.

4. By assumption, we may write $d\mu_f(\lambda) = \rho(\lambda) d\lambda + d\nu(\lambda)$, where $\rho$ is continuous, compactly supported, $\rho(\omega) \neq 0$ and $\nu$ is supported away from $\omega$. Therefore,

$$E(t) = 4 \int_\mathbb{R} \frac{\sin^2 \left( \frac{1}{2}(\omega - \lambda) \right)}{(\omega - \lambda)^2} \rho(\lambda) d\lambda + \text{ bounded term}.$$ 

By making the change of variables $x = t(\omega - \lambda)$, $d\lambda = \frac{dx}{t}$ we get

$$\int_\mathbb{R} \frac{\sin^2 \left( \frac{1}{2}(\omega - \lambda) \right)}{(\omega - \lambda)^2} \rho(\lambda) d\lambda = \int_\mathbb{R} \frac{t \sin^2 (x/2)}{x^2} \rho(\omega - x/t) dx \sim_{t \to +\infty} t \rho(\omega)/4, \text{ by the dominated convergence theorem.}$$

Therefore, $E(t) \sim \rho(\omega)t$.

5 Chapter 4

**Exercise 18.** Let $f \in C^\infty_c(\mathbb{R}^3)$. Show that, for any $\lambda \in \mathbb{R}$ and $\omega \in S^2$, we have

$$((-\Delta - \lambda^2)^{-1} f)(|x|\omega) = \frac{e^{i\lambda|x|}}{4\pi|x|} \hat{f}(-\omega) + O\left(\frac{1}{|x|^2}\right).$$

**Correction to Exercise 18.** We have

$$((-\Delta - \lambda^2)^{-1} f)(x) = \int_{\mathbb{R}^3} f(y) e^{i\lambda|x-y|} 4\pi|x-y| dy$$

Now, we have $|x - y| = |x| - \frac{y \cdot y}{|x|} + O\left(\frac{1}{|x|}\right)$, so that

$$\frac{e^{i\lambda|x-y|}}{4\pi|x-y|} = \frac{e^{i\lambda|x|} e^{i\lambda \frac{y \cdot y}{|x|}}}{4\pi|x|} \left(1 + O\left(\frac{1}{|x|}\right)\right).$$

The result follows.

**Exercise 19.** In $\mathbb{R}^3$, consider cylindrical coordinates $(r, \theta, z) \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}$. Recall that in such coordinates, the Laplacian takes the form

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. $$

Consider the potential $V \in L^\infty_{\text{comp}}(\mathbb{R}^3; \mathbb{C})$ given by

$$V(r, \theta, z) := e^{i\theta} 1_{r \leq 1} 1_{|z| \leq 1}.$$

The aim of this exercise is to show that $-\Delta + V$ has no resonances.

If $\ell \in \mathbb{Z}$, we denote by $\Pi_\ell$ the projection of the $\ell$-th Fourier mode:

$$\Pi_\ell u(r, \theta, z) := \frac{e^{i\ell \theta}}{2\pi} \int_0^{2\pi} u(r, \phi, z) e^{-i\ell \phi} d\phi.$$ 

1. Let $R > 0$. Show that there exists $C(R) > 0$ such that for all $\ell \in \mathbb{Z}$ and all $u \in L^2(\mathbb{R}^3)$ supported in $B(0, R)$ and which satisfies $\Pi_\ell u = u$, we have

$$\langle -\Delta u, u \rangle \geq C^2 \|u\|_{L^2}.$$

2. Let $\rho \in C^\infty_c(\mathbb{R}^3)$ which does not depend on $\theta$. Show that for all $\lambda \in \mathbb{C}$, there exists $C > 0$ depending on $\rho$ and $\lambda$ such that for all $\ell \in \mathbb{Z},$

$$\|\Pi_\ell (-\Delta - \lambda^2)^{-1} \rho \Pi_\ell \|_{L^2 \to L^2} \leq \frac{C(\lambda)}{1 + |\ell|}.$$
3. Show that, if \( u \) is a resonant state, there exists \( C > 0 \) such that we have for all \( \ell \in \mathbb{Z} \)

\[
\|\Pi_{j+1}u\|_{L^2} \leq \frac{C}{1 + |j|} \|\Pi_ju\|_{L^2}.
\]

4. Conclude that \(-\Delta + V\) has no resonances in \( \mathbb{C} \).

**Correction to Exercise 19.**

1. We have \( \Delta u = \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \ell^2 u + \frac{\partial^2 u}{\partial \theta^2} \), so that

\[
\langle -\Delta u, u \rangle = \int_{(0,\infty)} \int_{(0,2\pi)} \int \left( -\partial_r^2 u - r^{-1} \partial_r u + r^{-2} \ell^2 u - \partial_\theta^2 u \right) \pi r dr d\theta dz
\]

\[
= \int_{(0,\infty)} \int_{(0,2\pi)} \int (r|\partial_z u|^2 + \partial_r u \partial_r (r \pi) - \pi \partial_r u + r^{-1} \ell^2 |u|^2) dr d\theta dz
\]

\[
= \int_{(0,\infty)} \int_{(0,2\pi)} \int (r|\partial_z u|^2 + r|\partial_r u|^2 + r^{-1} \ell^2 |u|^2) dr d\theta dz
\]

\[
\geq \int_{(0,\infty)} \int_{(0,2\pi)} \int r^{-2} \ell^2 |u|^2 r dr d\theta dz
\]

\[
\geq C \ell^2 \|u\|_{L^2},
\]

since \( r \) is bounded in the support of \( u \).

2. Suppose that \( u, f \in L^2(\mathbb{R}^3) \) satisfy

\[
u = \Pi_\ell \rho (-\Delta - \lambda^2)^{-1} \rho \Pi_\ell f.
\]

Since \( \rho \) does not depend on \( \theta \), \( \Pi_\ell \) commutes with \( \rho \). \( \Pi_\ell \) does also commute with \( (-\Delta - \lambda^2)^{-1} \), as can be seen using the integral kernel of \( (-\Delta - \lambda^2)^{-1} \). Therefore, we have

\[
u = \rho (-\Delta - \lambda^2)^{-1} \rho \Pi_\ell f.
\]

By question 2, we deduce that there exists \( C(\lambda) \) such that \( \|u\|_{H^2} \leq C(\lambda) \|f\|_{L^2} \).

But \( \|u\|_{H^2} \geq \langle -\Delta u, u \rangle \geq C \ell^2 \|u\|_{L^2} \) by question 1. The result follows.

3. We have

\[
\Pi_{j+1} u = -\Pi_{j+1} e^{i\theta} \mathbf{1}_{r \leq 1} \mathbf{1}_{|z| \leq 1} (-\Delta - \lambda^2)^{-1} \rho u
\]

\[
= -\Pi_{j} \mathbf{1}_{r \leq 1} \mathbf{1}_{|z| \leq 1} (-\Delta - \lambda^2)^{-1} \rho u
\]

\[
= -\Pi_{j} \mathbf{1}_{r \leq 1} \mathbf{1}_{|z| \leq 1} (-\Delta - \lambda^2)^{-1} \rho \Pi_{j} u
\]

Therefore, by question 3, we have

\[
\|\Pi_{j+1} u\|_{L^2} \leq \|\Pi_{r \leq 1} \mathbf{1}_{|z| \leq 1} \Pi_{j} (-\Delta - \lambda^2)^{-1} \rho \Pi_{j} \|_{L^2 \rightarrow L^2} \|\Pi_{j} u\|_{L^2}
\]

\[
\leq \|\Pi_{j} \rho (-\Delta - \lambda^2)^{-1} \rho \Pi_{j} \|_{L^2 \rightarrow L^2} \|\Pi_{j} u\|_{L^2}
\]

\[
\leq C(\lambda) \frac{1}{1 + |j|} \|\Pi_{j} u\|_{L^2}.
\]

4. If \(-\Delta + V\) had a resonance, there would exist a function \( u \in L^2(\mathbb{R}^3) \) which would be non-zero and which would satisfy

\[
\|\Pi_{j+1} u\|_{L^2} \leq \frac{C}{1 + |j|} \|\Pi_{j} u\|_{L^2}.
\]

Therefore, if \( \Pi_{j} u = 0 \) for all \( j' \geq j \). Therefore, \( \Pi_{j} u \neq 0 \) for all \( j \) such that \(-j\) is large enough. On the other hand, for \(|j|\) large enough, we would have \( \|\Pi_{j+1} u\| \leq \|\Pi_{j} u\|_{L^2} \), so that \( \|\Pi_{j} u\| \) is strictly decreasing on an interval of the form \((-\infty, -j_0)\). This is not possible, since \( \|\Pi_{j} u\|_{|j| \rightarrow \infty} \rightarrow 0 \).