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Water waves for small surface tension: an approach via normal form

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Synopsis

In this paper we determine the possible crest-forms of permanent waves of small amplitude which exist on the free surface of a two-dimensional fluid layer under the influence of gravity and surface tension when the Froude number $\tilde{\lambda}$ is close to 1. The Bond number b, measuring surface tension, is assumed to satisfy $b < \frac{1}{3}$. We find one-parameter families of periodic waves of two different types, quasiperiodic waves and solitary waves with oscillations at infinity. The existence of true solitary waves is established for a sequence of systems approximating the full Euler equations in every algebraic order of $\tilde{\lambda} - 1$.

1. Introduction

Steady waves of permanent form are well known to exist on the free surface of an inviscid fluid under the influence of gravity and surface tension. In spite of the unbroken interest in this phenomenon for about a century, some of the mathematical questions are not yet answered, even for small amplitude waves, e.g. whether solitary waves exist for small values of the surface tension, i.e. if the Bond number is less than a third. Amick and the second author discovered in **[2]** that this region of the parameter space should be particularly rich in solutions. They found that the "critical phase space" is then four-dimensional. Solitary waves appear as homoclinic orbits connecting the rest-state to itself.

In this paper, we determine the possible crest-forms of small-amplitude plane steady waves which bifurcate from the quiescent state when the Froude number $c/\sqrt{\hat{g}h} =: \lambda^{-2}$ is close to one, and when the Bond number $b = T/\rho hc^2$ is a fixed positive parameter. Here $\hat{g}$ denotes the acceleration of gravity, h the depth of the quiescent layer, ρ the fluid's density, c the wave speed, and T the coefficient of surface tension. Our analysis yields a "complete" picture of solutions in the following sense: we shall show that the problem can be fully reduced to a two-(respectively four-) dimensional "reversible" dynamical system $\mathscr{S}$ for $b > \frac{1}{3}$ (respectively $b < \frac{1}{3}$). This system will be approximated by a sequence of integrable systems $\mathscr{S}_N$ which differs from the original one by terms of the order ε^N, where ε measures the amplitude of the solutions, and N tends to infinity. Every wave w which has an approximant w_N solving $\mathscr{S}_N$ will be detected, provided we can prove that w exists as a continuation of w_N when $N \to \infty$. Therefore, we shall find all waves which have nonzero "algebraic" approximations w_N and which are limits of these w_N.

In the parameter region $b > \frac{1}{3}$ we already know the answer to this classification: every solution is either a periodic wave ($\lambda < 1$) or a solitary wave of depression ($\lambda > 1$). This was first proved in **[1]** and **[2]**, see also **[19]** for a modified version which is closer to the method used here. Existence of periodic waves has been shown by Zeidler **[23]**. The classification mentioned above is implicitly contained in **[19]** and will be given again here. The situation is more complicated for $b < \frac{1}{3}$. Again, we know the existence of periodic waves due to **[23]**. What Beale calls solitary waves with capillary ripples (RSW) and which are homoclinic connections of periodic wave trains, were found in **[5]** and **[21]**. This picture will be completed by quasiperiodic waves and a wave-form which is periodic but different from the one hitherto determined (we call it of type II). It lives on a 2-torus and owes its existence to a resonance between two independent frequencies. The existence of a true solitary wave, i.e. one which decays to zero at infinity, is still an open problem if we restrict λ to a neighbourhood of 1. In this case, the approximate systems $\mathscr{S}_N$ all have solitary wave solutions, but we cannot prove that these solutions persist as $N \to \infty$. However, we have recently found true solitary waves for $b < \frac{1}{3}$, in **[14]**, when (b, λ) lies on the upper bank of the curve III in Figure 2.1 of the next section. There are waves both of elevation and of depression as numerical studies have shown in **[7]**.

The question as to whether solitary waves exist for $b < \frac{1}{3}$ and $\lambda \approx 1$ has received much attention. The only definite results we know of are for model equations. In **[12]** RSW were found for a model system, see also **[4]**. The model equation $\varepsilon^2 u^{(4)} + u'' - u + u^2 = 0$, proposed in **[2]**, has been found to have no nontrivial solution vanishing at infinity in **[3]**. Extensions were shown in **[11]** and **[8]**.

Of course all these results cannot contradict the statement given previously, that in every algebraic order of approximation of the full system solitary waves do exist. After all, the route of approximation is nonunique. For the history and an overview of currently available related results we refer to **[2]** and **[5]**. We shall show how Beale's result can be reproduced from ours. In comparision, we can only remark in our favour that the existence of RSW is a consequence of the classical implicit function theorem and needs no hard modification.

The method which we are going to apply consists of three main steps. The first is the reduction to a system of ODE's of minimal order; the next step is the construction of normal forms for the reduced vector field in the spirit of the work by Elphick *et al.* **[9, 13]**. These normal-form systems are the approximate systems in $\mathscr{S}_N$ previously mentioned. They turn out to be integrable and, thus, the class of bounded solutions is easily obtained. Finally we have to show persistence of solutions against reversible perturbations of high algebraic order in the amplitude. In this order, the three steps form the content of Sections 2, 3 and 4. Section 5 contains the proof of persistence for quasiperiodic waves.

The reduction for quasilinear elliptic systems in unbounded cylindrical domains near a solution being independent of the axial variable has been justified by Mielke in **[20]** via the centre manifold concept and optimal regularity. It is based on the idea of treating the unbounded space variable x as a timelike variable and applying dynamical system methods. This was first realised for a much simpler problem in **[18]**. It implies that all sufficiently small bounded solutions of the original Euler system (2.2) lie on a manifold which is modelled over the central part of the linearisation A in (2.5). The spectrum of A is given by the roots of the equation (2.6): $\sigma \cos \sigma - (\lambda - b\sigma^2) \sin \sigma = 0$, $\sigma \in \mathbb{C}$, whence we may obtain the dispersion relation by setting $\sigma = ik$, $k \in \mathbb{R}$. For $\lambda = 1$, $b < \frac{1}{3}$ for example, the central part Σ_0 of ΣA is given by an eigenvalue 0 of multiplicity 2 and two simple eigenvalues $\pm iq$, $q > 0$. The solution $\underline{w}$ of (2.2) can be decomposed as $\underline{w} = \underline{w}_0 + \underline{w}_1$, where $\underline{w}_0$ is the projection of $\underline{w}$, commuting with A, which is spanned by the generalised eigenfunctions to Σ_0. Moreover, $\underline{w}_1$ is a pointwise function of $\underline{w}_0$, smooth in the parameters and of lower order than linear in $\underline{w}_0$, cf. Theorem 2.2. Therefore, (2.2) can be reduced to (2.14), and normal form theory can be applied. In fact (2.14) inherits the symmetries of (2.2), in particular it is reversible again. For the case $b < \frac{1}{3}$, the explicit details can be found in (3.11) to (3.15), where (3.14) yields the normal form.

As an example, let us treat the case of solitary waves with ripples at infinity. In the lowest order the form of the free surface modulation is given by (2.3) as $S = -[W_1]$ where $[W_1]$ denotes the mean of W_1 over the layer, and $\underline{w} = (W_2(1), W_1, W_2)$. The central eigenspace is four-dimensional and spanned by φ_0, φ_1, $\varphi_\pm$ in (3.11). Set $\underline{w} = \alpha_0\underline{\varphi}_0 + \alpha_1\underline{\varphi}_1 + z\underline{\varphi}_+ + \bar{z}\underline{\varphi}_-$, $z = r \exp(i\Theta)$, then we obtain in lowest order

$$S(x) = -\alpha_0(x) + \frac{2}{q} \sinh q \cdot r \cos \Theta(x).$$

This is the normal form approximation of the RSW. The functions α_0, Θ are determined by (3.17), $r = K^{\frac{1}{2}} = |\mu|\, k^{\frac{1}{2}}$ is constant. We obtain

$$S(x) = \mu(\operatorname{sgn} \mu)\Big\{\rho^{\frac{1}{2}} \cosh^{-2}(\rho^{\frac{1}{4}} vx/2) - \tfrac{1}{3}(1 + \rho^{\frac{1}{2}}) + \frac{2}{\sqrt{6}} \frac{\sinh q}{q} (1 - \rho)^{\frac{1}{2}} \cos \Theta(x)\Big\}, \tag{1.1}$$

where $\rho = 1 - 6\,dk$ is a free parameter, $v = |\mu|\ (\frac{1}{3} - b)^{-1}$, and $\Theta(x) = (q + \mathcal{O}(\mu))x + \Theta_0 + \mathcal{O}(\mu)$, Θ_0 is a free phase. For the definition of d, see (3.15). Therefore, we have, for given $\mu = \lambda - 1$, a one-parameter family of RSW, parametrised by ρ. The homoclinic part is slowly varying in comparison to the periodic part. Its period is $2\pi/q + \mathcal{O}(\mu)$ in the physical variable. Obviously, the oscillatory part is nonzero.

The reversible solutions of the normal form system (NF-system) persist under reversible perturbations. Therefore (1.1) is the lowest order form of a RSW solution of the full Euler equations. The bounds estimating the difference between the exact – and the NF – solution are given in (4.14). The first part estimates the error of the oscillatory term and is small in any fixed power of (μ) uniformly in $x \in \mathbb{R}$. The second part bounds the difference of the homoclinic connections by powers of $|\mu|$ uniformly in x. However, the bound is not uniform with respect to the approximate solution, since ω (cf. (4.14)) is strictly less than ω_0. All these results are in agreement with Beale's analysis [**5**].

2. Basic equations and reduction

Free surface travelling waves of an inviscid fluid layer are studied under the influence of gravity and surface tension. Due to the Galilean invariance of the underlying Euler equations, the motion is steady in a moving coordinate system. Irrotational flow is considered. The equations are well known (cf. [**22, 19**]) and read

$$\begin{aligned} \operatorname{div} \underline{v} = \operatorname{curl} \underline{v} = 0, &\qquad 0 < \eta < s(\xi), \\ \tfrac{1}{2}|\underline{v}|^2 - b\kappa(s) + \lambda s = \text{constant}, & \\ v_1\, \partial_\xi s - v_2 = 0, &\qquad \eta = s(\xi), \\ v_2 = 0, &\qquad \eta = 0, \end{aligned} \tag{2.1}$$

where $\kappa(s) := \partial_\xi((1 + (\partial_\xi s)^2)^{-\frac{1}{2}}\, \partial_\xi s)$ is the curvature of the free surface $\eta = s(\xi)$ and $\tilde{D} = \{(\xi, \eta)/\xi \in \mathbb{R},\ 0 < \eta < s(\xi)\}$ is the flow domain, $\underline{v} = (v_1, v_2)$ denotes the velocity vector. All quantities are in nondimensional form and

$$\lambda = \frac{\hat{g}h}{c^2}, \qquad b = \frac{T}{\rho h c^2}$$

denote the inverse square of the Froude (respectively the Bond) number. Here, $\hat{g}$ is the gravity, h a mean depth of the layer, c the wave speed, ρ the density and T the coefficient of surface tension. Note that $T/\rho \to T$, $b/\lambda = \beta$, $\lambda^{-1} = \gamma$ yields the corresponding parameters in [**5**].

We seek bounded solutions of (2.1) in $\tilde{D}$. In fact, we restrict ourselves to small bounded solutions. Each of these solutions corresponds to a travelling wave and *vice versa*. Due to the moving coordinate system, the quiescent state of the layer is given by $\underline{v} = (1, 0)$, $s(\xi) = 1$. Let $\Psi(\xi, \eta)$ denote the stream function, which is defined by

$$\partial_\xi \Psi = -v_2, \qquad \partial_\eta \Psi = v_1, \qquad \Psi|_{\eta=0} = 0.$$

The transformation

$$x = \xi, \qquad y = \Psi(\xi, \eta) = \eta + \psi(\xi, \eta)$$

is globally invertible from $\tilde{D}$ to $D := \mathbb{R} \times (0, 1)$ as long as $|v_1 - 1|$, $|v_2|$ are small compared to 1. Introduce in addition

$$W_1 = \tfrac{1}{2}(v_1^2 + v_2^2 - 1), \qquad W_2 = v_2 v_1^{-1};$$

then (2.1) can be written as follows (cf. [**19**, p. 145])

$$\begin{aligned} \partial_x \beta &= \frac{1}{b}(1 + \beta^2)^{\frac{3}{2}}\Big(W_1(\cdot, 1) + \lambda\Big(\Big[\frac{1}{g}\Big] - 1\Big)\Big), \qquad y = 1, \\ \partial_x \underline{W} &= K(\underline{W})\, \partial_y \underline{W} \qquad \text{in } D. \end{aligned} \tag{2.2}$$

The boundary condition $W_2 = 0$ at $y = 0$ will be incorporated into the functional-analytic setting. The scalar function $\beta = \beta(x)$ is the trace of W_2 on $y = 1$. The other terms are given as follows:

$$\beta = W_2(\cdot, 1), \qquad g = \left(\frac{1 + 2W_1}{1 + W_2^2}\right)^{\frac{1}{2}},$$

$$K(\underline{W}) = \begin{pmatrix} W_2 g & -g^3 \\ g^{-1} & W_2 g \end{pmatrix}, \qquad [W_1] = \int_0^1 W_1\, dy.$$

In order to establish the relation between these new and the physical variables, observe that $g = v_1$. For $s = 1 + S$, we have, using the conservation of flux,

$$1 = \int_0^{1+S} v_1 \, d\eta = 1 + S + \int_0^1 \frac{g-1}{g} dy$$

and thus for the modulation S of the free surface

$$S = -1 + \left[\frac{1}{g}\right] = -[W_1] + \quad \text{higher order terms.} \tag{2.3}$$

Observe that (2.2) is reversible, i.e. the right-hand side of (2.2) anticommutes with the reflexion

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.4}$$

The system (2.2) is treated as a quasilinear evolution equation in

$$X = \mathbb{R} \times L_2(0, 1) \times L_2(0, 1) \quad \text{with norm } \|\cdot\|,$$

where

$$\underline{w} = \begin{pmatrix} \beta \\ \underline{W} \end{pmatrix} \in D(A) = \mathbb{R} \times H^1(0, 1) \times H^1(0, 1)$$

$$\cap \{W_2(0) = 0, \ W_2(1) = \beta\} \quad \text{with norm } \|\cdot\|_A,$$

where A denotes the linearisation of the right-hand side of (2.2) in $\underline{w} = \underline{0}$, $\lambda = 1$ for fixed $b > 0$. It is given by

$$A\underline{w} = \begin{pmatrix} \frac{1}{b}(W_1(1) - \lambda[W_1]) \\ -\partial_y W_2 \\ \partial_y W_1 \end{pmatrix}. \tag{2.5}$$

We seek solutions $\underline{w} \in C^k_{bd}(\mathbb{R}, D(A))$ of (2.2), the latter being abbreviated by

$$\partial_x \underline{w} = A(\mu, b)\underline{w} + F(\mu, b; \underline{w}), \qquad \mu = \lambda - 1. \tag{2.2'}$$

Here k is some fixed integer, $k \geqq 1$. A was defined in (2.5). A and F map $D(A)$ smoothly into X since the traces of W_1 and W_2 at $y = 0$ and 1 are well defined; $\underline{W}$ is a continuous function of y, $\partial_y \underline{W} \in L_2(0, 1)^2$.

A systematic study of possible wave forms will be given in the following sections under the assumption that the parameter λ varies in the neighbourhood of 1. Although almost all previous mathematical investigations have been undertaken under the same premise, it is by no means the only relevant choice (cf. **[14]**). To obtain more insight into this phenomenon, consider the operator A in (2.5). It acts on functions living on the cross-section $(0, 1)$ of the domain D. Its spectrum ΣA consists of eigenvalues of finite multiplicities whose locations are determined by the equation (cf. **[2]**)

$$\sigma \cos \sigma = (\lambda - b\sigma^2) \sin \sigma, \qquad \sigma \in \mathbb{C}. \tag{2.6}$$

Reversibility, i.e. $AR + RA = 0$, R from (2.4), implies the reflectional symmetry of ΣA with respect to the real axis and the imaginary axis. If $\Sigma A \cap i\mathbb{R}$ is empty, then the quiescent state $\underline{w} = 0$ is isolated in $C^0_{bd}(\mathbb{R}, X)$, i.e. no small-amplitude waves of interest exist. If one accepts that $(\partial_x - A)^{-1}$ is a bounded linear operator from $C^0_{bd}(\mathbb{R}, X)$ into $C^0_{bd}(\mathbb{R}, D(A))$, this fact is easy to establish, since the nonlinearity F in (2.2′) satisfies, if $r \in (0, 1)$ and $c(r) > 0$ is suitably chosen,

$$\|F(\underline{w})\| \leqq c(r)\, \|\underline{w}\|_A^2 \quad \text{for all } \|\underline{w}\|_A \leqq r,$$

for every μ and b. However, the above-mentioned property of $(\partial_x - A)^{-1}$ is relatively hard to obtain. Even with the resolvent estimate of the following lemma, the operators $\exp(A_+ x)$ for $x < 0$ and $\exp(A_- x)$ for $x > 0$, which are the corner-stones for the construction of that inverse, will have an x^{-1}-singularity as $x \to \pm 0$, in general, due to the fact that ΣA is infinite on both sides of the imaginary axis. The difficulties can be overcome through optimal regularity results for $\partial_x - A$, a fact which was discovered and analysed by Mielke in **[20]**.

In the (b, λ)-parameter space, the set of points where waves can bifurcate from the trivial state can therefore be determined by the requirement that the central part of ΣA, $\Sigma_0 := \Sigma A \cap i\mathbb{R}$ contains at least one eigenvalue of multiplicity two. This set includes all cases where the dimension of Σ_0 changes, in view of the reversibility of A. They are shown in Figure 2.1, where we have also indicated the location of the "critical" eigenvalues – those with minimal $|\mathrm{Re}\, \sigma|$ – thus characterising the type of bifurcations when passing through the curves I, II, and III. There are no other bifurcations below the curve III–IV. But there may exist other phenomena beyond that bound.

We finally give a parameter representation of III. Set $\sigma = iq$, $q \in \mathbb{R}$, and use

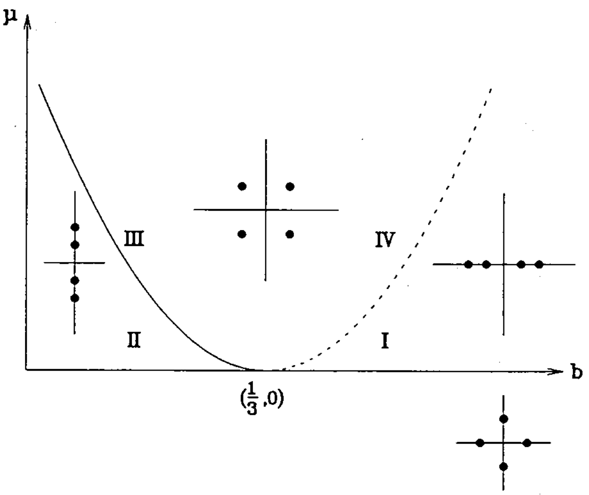

Figure 2.1. Location of bifurcation curves in parameter space.

(2.6), together with its differentiated form, to obtain

$$b=\frac{1}{2\sinh^2 q}+\frac{1}{2q\tanh q}=\frac{1}{3}-\frac{2q^2}{45}+\mathcal{O}(q^4),$$

$$\mu=\frac{q^2}{2\sinh^2 q}+\frac{q}{2q\tanh q}-1=\frac{q^4}{45}+\mathcal{O}(q^6),$$

where the latter expansion holds near $b=\frac{1}{3}$, $\lambda=1$. In this paper we treat the bifurcation along I and II.

Now we shall show how to reduce (2.2) to a system of ordinary differential equations via a centre manifold argument. An abstract version for the quasilinear case, as needed here, was given by Mielke **[20]**. The main step is a resolvent estimate on $i\mathbb{R}$.

LEMMA 2.1. *Given A as in* (2.5) *for $\lambda=1$, consider the natural complexification $\hat{A}$ in $\hat{X}=X+iX$. For every $b>0$ there exists a constant $q_0>0$, such that all $z=iq$, $q\in\mathbb{R}$, $|q|\geqq q_0$, belong to the resolvent set of $\hat{A}$. Moreover, the following inequality holds for some positive γ, which is independent of q:*

$$\|(\hat{A}-iq)^{-1}\|_{\hat{X}\to\hat{X}}\leqq\frac{\gamma}{|q|},\qquad |q|\geqq q_0.$$

Proof. Observe that $(\hat{A}-z)\underline{w}=(\beta,\underline{V})$ leads to

$$\left.\begin{array}{lr}\text{(a)} & W_1(1)-[W_1]-zbW_2(1)=b\beta,\\ \text{(b)} & -W_2'-zW_1=V_1,\\ \text{(c)} & W_1'-zW_2=V_2\end{array}\right\}\tag{2.7}$$

and $W_2(0)=0$. Squaring (b) and (c) yields

$$|\underline{W}'|_0^2+q^2|\underline{W}|_0^2-2q\,\mathrm{Im}\,W_1(1)\bar{W}_2(1)=|\underline{V}|_0^2.\tag{2.8}$$

Here $|\underline{W}|_0$ denotes the L_2-norm of $\underline{W}=(W_1,W_2)$. Multiply (2.7a) by $\bar{W}_1(1)$ and obtain

$$\left(bz-\frac{1}{z}\right)W_2(1)\bar{W}_1(1)=|W_1(1)|^2+\frac{1}{z}\bar{W}_1(1)[V_1]-b\beta\bar{W}_1(1)$$

Set $z=iq$ and use the geometric inequality

$$|\mathrm{Im}\,W_1(1)\bar{W}_2(1)|\leqq\frac{1}{bq+\dfrac{1}{q}}\left\{(1+2\varepsilon^2)\,|W_1(1)|^2+\frac{1}{4\varepsilon^2}\left(\frac{|V_1|_0^2}{q^2}+b^2\,|\beta|^2\right)\right\}$$

for any $\varepsilon>0$. To estimate $|W_1(1)|$, we multiply (2.7(c)) by y^k, for any $k\in\mathbb{N}$, and integrate

$$|W_1(1)|^2\leqq\frac{3k^2}{2k-1}|W_1|_0^2+\frac{3q^2}{2k+1}|W_2|_0^2+\frac{3}{2k+1}|V_2|_0^2.$$

Now, choose k and ε so that

$$\frac{6(1+2\varepsilon^2)}{b(2k+1)}=\frac{1}{2}.$$

Then we obtain

$$2q\,|\mathrm{Im}\,W_1(1)\bar{W}_2(1)| \leqq C_1(k)\,|W_1|_0^2+\frac{q^2}{2}\,|W_2|_0^2+\frac{1}{2}\,|V_2|_0^2 + \frac{1}{2\varepsilon^2 b}\left(\frac{|V_1|_0^2}{q^2}+b^2\,|\beta|^2\right), \tag{2.9}$$

$$C_1(k)=\frac{(2k+1)k^2}{2(2k-1)}.$$

Hence, (2.8) and (2.9) yield

$$|\underline{W}'|_0^2+(q^2-C_1(k))\,|W_1|_0^2+\frac{q^2}{2}\,|W_2|_0^2 \leqq |\underline{V}_0|^2+\frac{1}{2}\,|V_2|_0^2+\frac{1}{2\varepsilon^2 b}\left(\frac{|V_1|_0^2}{q^2}+b^2\,|\beta|^2\right)$$

from where the assertion of the lemma follows immediately. □

Now we turn to the "evolution" equation (2.2′), where we choose $\mu=0$ and fix b in A and incorporate $(A(\mu, b)-A(0, b))\underline{w}$ into F. Clearly, A is a closed, densely defined linear operator in X. All terms in (2.2′) are well defined, if $\|\underline{w}\|_A<\hat{r}$ and $\hat{r}$ is sufficiently small. Given $\hat{r}$, then there are constants c_1, c_2 such that F satisfies

$$\|F(\mu, \underline{w})\| \leqq c_1\,|\mu|\,\|\underline{w}\|_A+c_2\,\|\underline{w}\|_A^2. \tag{2.10}$$

Moreover, F defines a C^k mapping from $\Lambda\times D(A)$ into X; Λ is some neighbourhood of 0 in $\mathbb{R}$.

The inequality given in Lemma 2.1 can be extended to the cone $|\mathrm{Re}\,z|\leqq \delta\,|\mathrm{Im}\,z|$ for some $\delta>0$. The spectrum ΣA_1 lies in the region $|\mathrm{Re}\,z|\geqq\hat{\beta}>0$. Therefore, to each $\beta'\in[0, \hat{\beta})$, there exists a $\gamma(\beta')$ such that the inequality

$$\|(\hat{A}_1-z)^{-1}\|_{X_1\to X_1}\leqq\frac{\gamma(\beta')}{1+|z|} \tag{2.11}$$

holds for all $z\in\mathbb{C}$ with $|\mathrm{Re}\,z|\leqq\beta'$.

Let us decompose $\underline{w}=\underline{w}_0\oplus\underline{w}_1$, $\underline{w}_j\in X_j\cap D(A)$, $j=0, 1$, $X=X_0\oplus X_1$, and similarly A and F. Then (2.2′) reads

$$\left.\begin{aligned}\partial_x\underline{w}_0&=A_0\underline{w}_0+F_0(\mu, \underline{w}_0+\underline{w}_1),\\ \partial_x\underline{w}_1&=A_1\underline{w}_1+F_1(\mu, \underline{w}_0+\underline{w}_1).\end{aligned}\right\} \tag{2.12}$$

There exist neighbourhoods of 0: $U_0'\subset X_0$, $U_2'\subset D(A)\cap X_1$, and Λ of $\mu=0$ in $\mathbb{R}$, such that

$$\underline{F}=(F_0, F_1)\in C_b^k(\Lambda\times U_0'\times U_2', X_0\times X_1) \tag{2.13}$$

holds, Furthermore, $\underline{F}(0, \underline{0})=0$, $\partial_{\underline{w}}\underline{F}(0, \underline{0})=0$.

THEOREM 2.2. *There exist neighbourhoods of zero* $U_0 \subset U_0' \subset X_0$, $U_2 \subset U_2' \subset X_1 \cap D(A)$, *a neighbourhood* $\Lambda_0 \subset \Lambda$ *of* $\mu = 0$ *and a function*

$$h = h(\mu, \underset{\sim}{w}_0) \in C_b^{k-1}(\Lambda_0 \times U_0, U_2)$$

with the following properties:

(i) *The set*

$$M_\lambda = \{(\underset{\sim}{w}_0, h(\mu, \underset{\sim}{w}_0)) \in X_0 \times (X_1 \cap D(A)) / \underset{\sim}{w}_0 \in U_0\}$$

is a local integral manifold of (2.11) *for each* $\mu \in \Lambda_0$.

(ii) *Every solution of* (2.12) *with* $\mu \in \Lambda_0$, $(\underset{\sim}{w}_0, \underset{\sim}{w}_1)(x) \in U_0 \times U_2$ *for all* $x \in \mathbb{R}$, *belongs to* M_λ.

(iii) *We have* $h(0, \underset{\sim}{0}) = \partial_{\underset{\sim}{w}_0} h(0, \underset{\sim}{0}) = \underset{\sim}{0}$.

(iv) *If* $R_j: X_j \to X_j$, $j = 0, 1$, *are linear isometries such that*

$$F_j(\mu, R_0\underset{\sim}{w}_0, R_1\underset{\sim}{w}_1) = -R_jF_j(\mu, \underset{\sim}{w}_0, \underset{\sim}{w}_1),$$
$$A_jR_j = -R_jA_j,$$

then

$$h(\mu, R_0\underset{\sim}{w}_0) = R_1 h(\mu, \underset{\sim}{w}_0)$$

holds.

The proof of this theorem follows immediately from [**20**]. Lemma 2.1, inequalities (2.10), (2.11) and property (2.13) verify its assumptions.

The reduction of system (2.12) for solutions, which stay in $U_0 \times U_2$ for all $x \in \mathbb{R}$, follows from the theorem above. They satisfy, for every $\mu \in \Lambda_0$,

$$\partial_x \underset{\sim}{w}_0 = A_0 \underset{\sim}{w}_0 + f_0(\mu, \underset{\sim}{w}_0), \tag{2.14}$$

where

$$f_0(\mu, \underset{\sim}{w}_0) = F_0(\mu, \underset{\sim}{w}_0 + h(\mu, \underset{\sim}{w}_0))$$

and h is given by Theorem 2.2. Moreover, (2.14) is reversible with respect to R_0, the action of R in X_0: $R = R_0 \oplus R_1$. In the subsequent sections, explicit expressions for (2.14) are derived and discussed.

3. Normal forms and their integration

Theorem 2.2 shows that, whenever a solution $\underset{\sim}{w}(x)$ of (2.2′) is sufficiently small in $D(A)$ for all x, its projection $\underset{\sim}{w}_0(x)$ onto the central part of the spectrum must satisfy (2.14). *Vice versa,* if there is a small bounded solution $\underset{\sim}{w}_0$ of (2.14), it corresponds to a solution $\underset{\sim}{w}$ of the full equation (2.2′) via the function h defining the centre manifold. Thus we have completely reduced the problem of finding small amplitude solutions of the original equation to the system of ordinary differential equations (2.14), which has the dimension $\dim \Sigma_0 =: m$, i.e. $\dim X_0 = m$, m being even. (2.14) is reversible, i.e. $A_0 + f_0$ anticommutes with R_0, where $R = R_0 \oplus R_1$. In the subsequent analysis we fix b and vary $\mu = \lambda - 1$ in a neighbourhood of 0. We also fix k in Theorem 2.2 to be some large positive integer, and thus prescribe the regularity of h ($k = \infty$ is not allowed).

solution for the full system (3.3). It will be given in the next section, although it is easy to find.

Similarly, we find, for positive and negative μ, a family of periodic solutions which can be parametrised by a phase-shift in ξ, and by their frequency ω, $\omega \in (0, 1)$. The leading term $\underline{\beta}^*$ can be found by (3.7) when setting $\mu = 0$, and the formulae remain valid if we replace the $\cosh^{-2}$-term by the lowest-order periodic solution. Sometimes, the periodic waves are required to have mean-value zero. This would make an adjustment of the mean-value of the free surface necessary and thus of the parameter λ. Galilean invariance can be used to do this and alter the constant in (2.1) which we have chosen to be $\lambda + \frac{1}{2}$. Similarly, we could proceed with the homoclinic orbit which envelopes the periodic solutions and satisfied $\alpha_0(\pm\infty) = 2\mu/3 + \mathcal{O}(\mu^2)$. Its asymptotic value would then be pushed to 0 and, thus, this orbit would coincide with the previously obtained solitary wave. We shall not pursue this idea further. We summarise in the following proposition:

PROPOSITION 3.1. *Given $b > \frac{1}{3}$ and a positive sufficiently small $\mu = \lambda - 1$, the NF-system* (3.5) *has a bounded solution which is unique up to shifts in x. It is given by* (3.9) *and corresponds to a solitary wave of depression, whose free-surface modulation is determined by* (3.10).

If μ is negative, then there exists a family of periodic solutions of (3.5) for each μ. They are parametrised by their frequencies ω which range in the interval $(0, (|\mu|/(b-\frac{1}{3}))^{\frac{1}{2}})$. The family is unique up to shifts in x.

3.2. Case $b < \frac{1}{3}$

The central part of the spectrum $\Sigma_0 A$ is $\{0, \pm iq\}$, where 0 has multiplicity two. The corresponding eigenvectors are

$$\underline{\varphi}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{\varphi}_1 = -\begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}, \quad \underline{\varphi}_+ = \begin{pmatrix} i \sinh q \\ -\cosh qy \\ i \sinh qy \end{pmatrix}, \quad \underline{\varphi}_- = \bar{\underline{\varphi}}_+ \quad (3.11)$$

and $q > 0$ satisfies $q \cosh q = (1 + bq^2) \sinh q$. We write $\underline{w}_0 = \alpha_0 \underline{\varphi}_0 + \alpha_1 \underline{\varphi}_1 + z\underline{\varphi}_+ + \bar{z}\underline{\varphi}_-$. In this basis, A_0 and R_0 read

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & iq & 0 \\ 0 & 0 & 0 & -iq \end{pmatrix}, \quad R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3.12)$$

As for the previous case, we may assume (2.14) in normal form (3.4), where $N = (N_0, N_1, N_+, N_-)$ satsifies (3.1). The differential operator D has the form

$$D := \alpha_0 \, \partial_{\alpha_1} - iqz \, \partial_z + iq\bar{z} \, \partial_{\bar{z}}.$$

We have to solve (3.1) in the ring of polynomials. The explicit form of this system is given below in (3.13). Observe that α_0, $|z|^2$, $\alpha_0 \ln z + iq\alpha_1$ are three independent integrals of $\ker D$. Thus a general function in $\ker D$ would depend on these quantities. Since we work in the space of polynomials, the transcendental integral drops out, as can be readily seen (cf. Appendix 6.3). Now (3.1) leads to

$$\begin{aligned} DN_0 &= 0, & DN_1 &= N_0, \\ DN_+ &= -iqN_+, & DN_- &= iqN_-. \end{aligned} \tag{3.13}$$

Hence $N_0 = N_0(\alpha_0, |z|^2)$ holds. In view of the reversibility, we have $N_0(\alpha_0, |z|^2) = -N_0(\alpha_0, |z|^2)$ and thus $N_0 = 0$. Therefore $DN_1 = 0$, which implies $N_1 = N_1(\alpha_0, |z|^2)$, without any simplification by reversibility. To determine N_+, set $M_+ = \bar{z}N_+$. Then M_+ satisfies $DM_+ = 0$ and thus $M_+ = M_+(\alpha_0, |z|^2)$ holds. Since $M_+ = 0$ for $z = 0$, we have $M_+ = |z|^2 M(\alpha_0, |z|^2)$ and thus $N_+ = zM(\alpha_0, |z|^2)$. Similarly, we obtain $N_- = \bar{z}P(\alpha_0, |z|^2)$. Moreover $\bar{P} = M$ holds and reversibility implies $M = -P$. Therefore $M = i\Psi$, Ψ real, and the normal form reads

$$\begin{aligned} \partial_x\alpha_0 &= \alpha_1, & \partial_x\alpha_1 &= \Phi(\mu, \alpha_0, |z|^2), \\ \partial_x z &= iz\Psi(\mu, \alpha_0, |z|^2), & \partial_x\bar{z} &= -i\bar{z}\Psi(\mu, \alpha_0, |z|^2), \end{aligned} \tag{3.14}$$

where Ψ and Φ are real polynomials of any given order s. System (3.9) has the integrals $|z|^2 = K$ and $\alpha_1^2 - \hat{\Phi}(\alpha_0, K) = H$, where $\partial_{\alpha_0}\hat{\Phi} = 2\Phi$. Hence, it is integrable.

We discuss the phase portrait for the physical problem under consideration. With the notation

$$\begin{aligned} \Phi(\mu, \alpha_0, |z|^2) &= c_1(\mu)\alpha_0 + c_2(\mu)\alpha_0^2 + d_2(\mu)\,|z|^2 + \dots, \\ \Psi(\mu, \alpha_0, |z|^2) &= \gamma_0(\mu) + \gamma_1(\mu)\alpha_0 + \dots, \end{aligned}$$

we have (cf. Appendix 6.1)

$$\begin{aligned} & c_1 = -\sigma\mu + \mathcal{O}(\mu^2), && c_2 = \tfrac{3}{2}\sigma + \mathcal{O}(\mu), \\ & \sigma d = d_2 = \sigma\Big(1 + \frac{1}{q}\sinh(2q)\Big) + \mathcal{O}(\mu), && \gamma_0 = q + \mathcal{O}(\mu), \\ & \gamma_1 = r(q)^{-1}(\sinh 2q + \mathcal{O}(\mu)), && \end{aligned} \tag{3.15}$$

where

$$\sigma = (\tfrac{1}{3} - b)^{-1}, \qquad r(q) = (b - q^{-2})\sinh^2 q + 1 + \mathcal{O}(\mu).$$

Observe that bounded solutions do not exist if $K > \mu^2 + \mathcal{O}(\mu^4)$. The qualitative behaviour of the set of bounded solutions is shown in Figure 3.1 ($\eta = \sigma/6d_2 + \mathcal{O}(\mu^2)$). Each point in Figure 3.1 corresponds to a periodic motion of radius $\sqrt{K}$ in the z-plane. Thus, each "equilibrium" in the (α_0, α_1)-plane is actually a periodic solution (of type 1, as we shall say later). There are other periodic solutions (of type 2), which live on a 2-torus with close-by quasiperiodic solutions. Which of the two cases arises depends on whether or not the frequencies of the α-orbit and of the z-orbit are rationally dependent.

Finally there are homoclinic solutions connecting a closed orbit with itself, eventually after a phase shift. They are solitary waves "with ripples" in the terminology of Beale **[5]** or "generalised solitary waves" after Sun **[21]**. The only true solitary wave appears in this diagram for $K = 0$, $\mu < 0$. Its form is determined

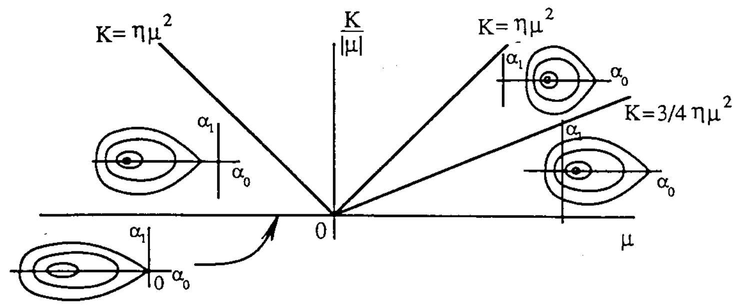

Figure 3.1

in lowest order of μ by

$$\alpha_0(x) = \mu \cosh^{-2}(\tfrac{1}{2}\sqrt{\sigma\,|\mu|}\,x), \qquad \alpha_1 = \alpha_0', \qquad z = 0. \tag{3.16}$$

According to (2.3), this describes a solitary wave of elevation, unique for every $\mu < 0$, if we neglect the free shift in ξ, which we shall do henceforth. Therefore: *in every order of our approximation of the full equation, solitary waves of elevation exist for* $\mu = \lambda - 1 < 0$. However, we are not able to prove their persistence under reversible perturbations to this day. Therefore the existence of solitary waves for $b < \frac{1}{3}$ is still open, when λ is close to 1. Along the curve III in Figure 2.1, solitary waves of elevation and of depression have been found by the authors **[14]**; for the persistence see **[16]**.

The RSW (solitary waves with ripples) are easily constructed. Set $z = r\exp(i\theta)$. From (3.14) and (3.15) we obtain via scaling

$$\begin{aligned} &\alpha_0 = \mu\beta_0(\nu x), \qquad \alpha_1 = \mu\nu\beta_1(\nu x), \\ &K = \mu^2 k, \qquad \theta = \vartheta(\nu x)/\nu, \qquad \nu = (|\mu|\,\sigma)^{\frac{1}{2}}, \end{aligned} \tag{3.17}$$

the following system ($\nu x = \xi$, $\partial_\xi = {}'$):

$$\begin{aligned} &\beta''_0 + (\operatorname{sgn}\mu)(\beta_0 - \tfrac{3}{2}\beta_0^2 - dk) = \mathcal{O}(\mu), \\ &r' = 0, \qquad \vartheta' = \Psi(\mu;\, \alpha_0,\, k^2) = q + \mathcal{O}(\mu). \end{aligned} \tag{3.18}$$

Denote by β_0^* the saddle of the two equilibria, and set $\beta = \beta_0^* + \gamma$. Then we have

$$\gamma_0'' - (1 - 6\,dk)^{\frac{1}{2}}\gamma_0 - (\operatorname{sgn}\mu)\tfrac{3}{2}\gamma_0^2 = \mathcal{O}(\mu).$$

Scaling again like

$$\gamma_0(\xi) = (1 - 6\,dk)^{\frac{1}{2}}\Gamma((1 - 6\,dk)^{\frac{1}{4}}\xi)$$

yields

$$\Gamma'' - \Gamma - (\operatorname{sgn}\mu)\tfrac{3}{2}\Gamma^2 = \mathcal{O}(\mu).$$

Therefore, $\Gamma = -(\operatorname{sgn}\mu)\cosh^{-2}((1 - 6\,dk)^{\frac{1}{4}}\xi)$ holds in lowest μ-order. Now, the RSW are given by (2.3) as

$$S = \left(-\alpha_0 + \frac{2}{q}\sinh q \cdot r\cos\theta\right)(1 + \mathcal{O}(\mu)),$$

where

$$\alpha_0(x) = \mu\beta_0^* - \mu(\operatorname{sgn}\mu)\rho^{\frac{1}{2}}\cosh^{-2}(\rho^{\frac{1}{4}}\nu x/2),$$
$$\alpha_0'(x) = \alpha_1(x), \qquad r(x) = |\mu|\, k^{\frac{1}{2}}, \tag{3.19}$$
$$\theta(x) = \theta_\infty + \theta_1(x)$$

and

$$3\beta_0^* = 1 + (\operatorname{sgn}\mu)\rho^{\frac{1}{2}}, \qquad \rho = 1 - 6\,dk,$$

$$\theta_\infty = \Psi(\alpha_0^*, k), \qquad \theta_1(x) = \theta_0 + \int_0^x (\Psi(\alpha_0(s), k) - \Psi(\alpha^*, k))\, ds.$$

θ_1 is a bounded function and $\theta_1(\infty) - \theta_1(-\infty)$ determines the asymptotic phase shifts; α_0^* denotes the largest root of Φ.

For a general overview of the solution behaviour in dependence of the parameters K and $H = \alpha_1^2 - \hat{\Phi}(\alpha_0, K)$ we introduce, in addition to the scaling (3.17),

$$H = (\operatorname{sgn}\mu)\mu^2\nu^2 h. \tag{3.20}$$

Then

$$\beta_1^2 = (\operatorname{sgn}\mu) f(\beta_0, k, h),$$

with

$$f = (\mu\nu)^{-2}(\hat{\Phi}(\mu\beta_0, \mu^2 k) + \mu^2\nu^2 h) = f_0(\beta_0, h, k) + \mathcal{O}(\mu),$$
$$f_0 = \beta_0^3 - \beta_0^2 + 2\,dk\beta_0 + h.$$

We need $\bar{f}_0 = (\operatorname{sgn}\mu)\, f_0 > 0$ for the existence of solutions β_0. The qualitative picture is summed up in Figure 3.2 for this cubic function in dependence on the parameters h and k.

For positive μ homoclinisation takes place on the curve EA, while for negative μ it occurs on OE. The interior of the triangular region $OAE(=\Delta)$ contains periodic solutions (β_0, β_1) not being constant. The latter occur along EA and OE and appear as saddle nodes or centres in Figure 3.1. Since $K > 0$, they correspond to nontrivial periodic solutions with oscillations in z. On OA we have $k = 0$ and therefore periodic solutions with $z = 0$.

For $(h, k) \in \Delta$, the frequency Ω_1 of β_0 is determined by

$$\frac{\pi}{\Omega_1} = (|\mu|\,\sigma)^{-\frac{1}{2}} \int_{\beta_m}^{\beta_M} |f(\beta_0, h, k)|^{-\frac{1}{2}}\, d\beta_0, \tag{3.21}$$

where β_m and β_M are consecutive roots of f such that $(\operatorname{sgn}\mu)f$ is positive in between. Set $T_1 = 2\pi/\Omega_1$ and define the frequency

$$\Omega_0 = \frac{1}{T_1}\int_0^{T_1} \Psi(\mu\beta_0(x), \mu^2 k)\, dx = q + \mathcal{O}(\mu)$$
$$= \pi^{-1}(|\mu|\,\sigma)^{-\frac{1}{2}}\Omega_1 \int_{\beta_m}^{\beta_M} \Psi(\mu\beta_0, \mu^2 k)\, |f(\beta_0, h, k)|^{-\frac{1}{2}}\, d\beta_0. \tag{3.22}$$

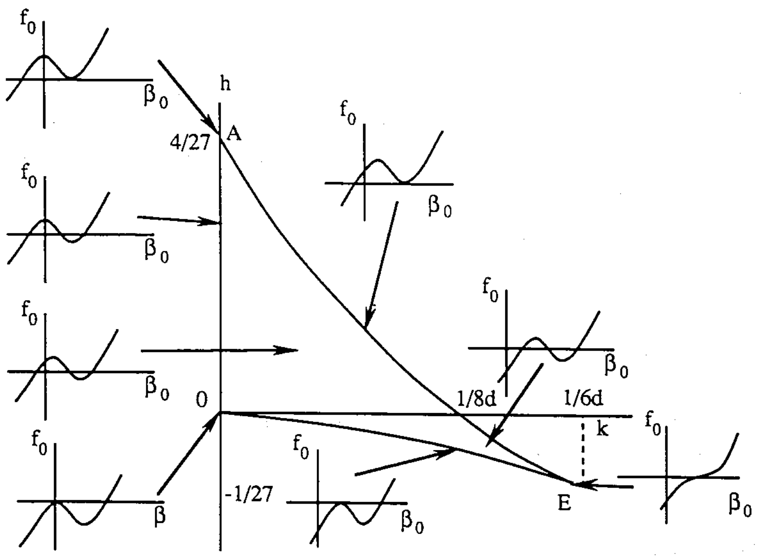

Figure 3.2. Shape of f_0 depending on (h, k).

The solution of (3.14) is expressed by

$$\alpha_0(x) = \mu\beta_0(x), \qquad z = (|\mu|\, k)^{\frac{1}{2}} e^{i(\Omega_0 x + g(x))}, \tag{3.23}$$

where g has period $2\pi\Omega_1^{-1}$ and 0 average. (3.23) defines a periodic solution of (3.14) if and only if $\Omega_0/\Omega_1 \in Q^+$. In all other cases the solution is quasiperiodic.

PROPOSITION 3.2. *In the case $b < \frac{1}{3}$, the NF-system* (3.14) *has periodic, quasiperiodic and homoclinic solutions*; *the latter connect periodic orbits* (*RSW*) *and are given by* (3.17). *Homoclinic solutions connecting* 0 *with itself* (*true solitary waves*) *exist for $\mu < 0$ and $K = 0$*; *see* (3.16). *For the full solution picture see Figure* 3.1.

4. Persistence for $K > 0$

Before we discuss the main topic of this section, namely the persistence of solutions of (3.14), i.e. the NF-system for $b < \frac{1}{3}$, we indicate the elementary arguments for the persistence of all small bounded solutions of (3.5), i.e. the solitary wave solution and the periodic solution for $b > \frac{1}{3}$. Of course, there is no K in this case. The full reduced system (2.14) corresponding to (3.5) reads

$$\partial_x \alpha_0 = \alpha_1 + r_0(\mu, \alpha_0, \alpha_1),$$
$$\partial_x \alpha_1 = \Phi(\mu, \alpha_0) + r_1(\mu, \alpha_0, \alpha_1).$$

In view of the reversibility with respect to R_0 in (3.3), we know that r_0 is odd in α_1, r_1 is even in α_1. Setting $r_0 = \alpha_1 \tilde{r}_0$ and replacing α_1 by $\alpha_1(1 + \tilde{r}_0)$, we see that we can assume $r_0 \equiv 0$, and $r := r_1$ to be an even function of α_1. Scaling as in (3.6) yields

$$\beta_0'' = \operatorname{sgn} \mu\beta_0 - \tfrac{3}{2}\beta_0^2 + R(\mu, \beta_0, \beta_1), \tag{4.0}$$

where R is even in $\beta_1 = \beta_0'$, bounded as a function of β_0, β_0', and of order $\mathcal{O}(\mu)$. We take β_0 as a C^1-function, bounded and even.

Now consider the case $\mu > 0$, and let $\underline{\beta}^* = (\beta_0^*, \beta_1^*)$ be the "reversible" homoclinic solution of (4.0). It decays like $\exp(-|\xi|)$ at infinity. Write $\underline{\beta} = \underline{\beta}^* + \hat{\underline{\beta}}$ and denote $R_{|\beta=\beta^*} =: R^*$. $2\bar{K}(\xi) = -\exp(-|\xi|)$ is the Green's function of $\hat{\beta}_0'' - \hat{\beta}_0$. The linearisation of (4.0) about $\hat{\beta}_0^*$ can be written

$$\begin{aligned}\hat{\beta}_0 &= -3\bar{K}\beta_0^* * \hat{\beta}_0 + \bar{K}D_{\beta_0}R^* * \hat{\beta}_0 - (\bar{K}D_{\beta_0'}R^*)_\xi * \hat{\beta}_0\\ &=: (\mathcal{L} + \bar{R}(\mu)) * \hat{\beta}_0,\end{aligned}$$

where $*$ denotes the convolution integral, $\mathcal{L} = -3\bar{K}\beta_0^*$. We work in $C_{b,e}(\mathbb{R})$, the space of bounded continuous, even functions. In view of the exponential decay of β_0^*, $\mathcal{L}$ is a compact operator in $C_{b,e}^0$. Since $\bar{R}(\mu)$ is $\mathcal{O}(\mu)$ and both operators map $C_{b,e}^0$ into itself, the applicability of the implicit function argument follows if 1 is not an eigenvalue of $\mathcal{L}$. If this were not the case, then

$$\beta'' - \beta + 3\beta_0^*\beta = 0$$

would have a nontrivial solution in $C_{b,e}^2$. Multiplication by $\beta_0^{*\prime}$ and integration between $-\infty$ and ξ yields $\beta'\beta_0^{*\prime} - \beta\beta_0^{*\prime\prime} = 0$, hence $\beta = c\beta_0^{*\prime}$ for some $c \neq 0$. But $\beta_0^{*\prime}$ is odd and thus the contradiction follows. Therefore the persistence of the homoclinic solution is proved for $b > \frac{1}{3}$.

For $\mu < 0$, we have to show persistence of periodic solutions of (4.0). However, this can be done quite similarly to (4.2), Lemma 4.4. We leave the details to the reader.

Now we turn to the main task of this section, the persistence of all solutions obtained for the NF-system (3.14) except for the quasiperiodic solutions, which are postponed to the next section. All of this is done under the general assumption that K, the integral of (3.14) describing the square of the amplitude of the z-oscillations, is strictly positive. This is a severe restriction, and we consider every result concerning $K = 0$ as highly nontrivial.

Persistence means the survival of a solution of (3.14) under reversible perturbations, where the reversibility is described by R_0 in (3.12). This can be done by restricting the analysis to *reversible solutions*, i.e. those which satisfy

$$R_0(\underline{\alpha}, z)(-x) = (\underline{\alpha}, z)(x). \tag{4.1}$$

If (4.1) holds with a minus-sign, $(\underline{\alpha}, z)$ is called *antireversible*. Actually, to assume (4.1) is no loss of generality. Let us take an arbitrary solution $(\underline{\alpha}, z)$ of (3.14). Then there is a $x_0 \in \mathbb{R}$, for which $\alpha_0'(x_0) = 0$. Shift x_0 to 0. Then α_0 must be even, since we could continue it from $x > 0$ as an even function of all of $\mathbb{R}$; α_1 is then odd; $r = K^{\frac{1}{2}}$ is even and Θ can always be chosen to be odd ($\Theta_0 = 0$). All these steps are compatible with the equation (3.14). In view of the unique solvability for given $\alpha_0'(0)$, $\alpha_1(0)$, $\Theta(0)$, K, (4.1) follows.

The proof of persistence will always be achieved by application of the implicit function theorem (IFT) in one form or another. Therefore, the only information supporting the proof is given by the linearisation of (3.14) about the NF-solution, which will be called $(\underline{\alpha}^*, z^*)$. Observe that the perturbation terms are of high order in μ and do not influence this linearisation. Granted that the linear part of (3.14) can be continuously inverted, we could use the IFT in two steps.

The first is for the system (3.14). $(\underset{\sim}{\alpha}^*, z^*)$ will solve it only in lowest μ-order. We can extend it to a solution of the full system (3.14), which we shall always assume to be done. The second step concerns the perturbed system, given below by (4.2). Here, the perturbations will be of some high algebraic order, $\mathcal{O}(\mu^{s+1})$ say. The error estimates obtained from the IFT should be of some order comparable to s. This is achieved by introducing the artificial parameter ε in (4.4a), which, *a posteriori*, will be identified with $|\mu|^{s'}$.

Now add a reversible perturbation $F=(F_0, \ldots, F_3)$ to (3.14), i.e. $R_0F=-FR_0$. This implies that F_0 and $\operatorname{Re} F_2$ are odd in (α_1, Θ), whereas F_1 and $\operatorname{Im} F_2$ are even in (α_1, Θ). Therefore, we may replace α_1+F_0 by α_1, maintaining the reversible structure. The perturbed system reads $(F_j=F_j(\underset{\sim}{\alpha}, r, \Theta))$:

$$\begin{aligned} &\partial_x\alpha_0=\alpha_1, && \partial_x\alpha_1=\Phi(\alpha_0, r^2)+F_1,\\ &\partial_x r=\operatorname{Re}(e^{-i\Theta}F_2), && \partial_x\Theta=\Psi(\alpha_0, r^2)+\frac{1}{r}\operatorname{Im}(e^{-i\Theta}F_2). \end{aligned} \tag{4.2}$$

If Φ, Ψ are polynomials of order s, then F_j is of order $s+1$ in $\underset{\sim}{\alpha}$ and r, and is 2π-periodic in Θ. Reversibility implies that (4.1) is compatible with (4.2).

To simplify the analysis, we shall use suitable scalings in μ. For negative μ, we define $(\delta_0>0)$:

$$\begin{aligned} &vx=\tau, \qquad v=(-\mu\sigma)^{\frac{1}{2}}, \qquad K=\mu^2k^2,\\ &\underset{\sim}{\alpha}(x)=(\alpha_0, \alpha_1)(x)=\mu(\beta_0(\tau), v\beta_1(\tau)),\\ &r(x)=|z(x)|=-\mu\rho(\tau), \qquad \Theta(x)=\frac{1}{v}\vartheta(\tau). \end{aligned} \tag{4.3}$$

Observe that all bounded solutions of (4.2) can be found in the $(\underset{\sim}{\beta}, \rho, \vartheta)$ phase space. In addition, the assumed symmetry properties are preserved as well. Now, (4.3) transforms (4.2) into

$$\begin{aligned} &\partial_\tau\beta_0=\beta_1, && \partial_\tau\beta_1=\Phi_1(\beta_0, \rho^2; \mu)+\varepsilon f_1,\\ &\partial_\tau\rho=\varepsilon g_1, && \partial_\tau\vartheta=q+\mu\Psi_1(\beta_0, \rho^2; \mu)+\frac{\varepsilon}{\rho}g_2, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \Phi_1&=\Phi/\mu v^2=\beta_0-\tfrac{3}{2}\beta_0^2+d\rho^2+\mathcal{O}(\mu),\\ \mu\Psi_1&=\Psi-q=(\gamma_0-q)/\mu+\gamma_1\beta_0+\mathcal{O}(\mu),\\ f_1&=-\frac{F_1}{|\mu|^{s'+1}v^2}=f_1\Big(\underset{\sim}{\beta}, \rho, \frac{\vartheta}{v}; \mu\Big)=\mathcal{O}(\mu^{s-1-s'}),\\ g_1&=\frac{\operatorname{Re}(\exp(-i\vartheta/v)F_2)}{|\mu|^{s'+1}v}=g_1\Big(\underset{\sim}{\beta}, \rho, \frac{\vartheta}{v}; \mu\Big)=\mathcal{O}(\mu^{s-\frac{1}{2}-s'}),\\ g_2&=\frac{\operatorname{Im}(\exp(-i\vartheta/v)F_2)}{|\mu|^{s'+1}}=g_2\Big(\underset{\sim}{\beta}, \rho, \frac{\vartheta}{v}; \mu\Big)=\mathcal{O}(\mu^{s-s'}). \end{aligned}$$

For technical reasons we have introduced the additional parameter ε. To obtain (4.2) from (4.4) and (4.3) we have to require

$$\varepsilon = |\mu|^{s'}. \tag{4.4a}$$

Observe that the functions f_1, g_1, g_2 are C^k, uniformly for $\nu \geqq 0$, as long as $s - s' > 1 + (k/2)$. Henceforth we assume this condition to be satisfied. Moreover, they are $2\pi\nu$-periodic in ϑ.

Let us define $\tilde{D} \subset \mathbb{R}^4$ as

$$\tilde{D} = \left\{(\underline{\beta}, \rho; \mu) \in \mathbb{R}^4 \Big/ \left|q + \mu\Psi_1 + \frac{\varepsilon}{\rho} g_2\right| > \frac{q}{2}\right\}.$$

In $\tilde{D}$ we can consider $\underline{\beta}$ and ρ as functions of ϑ under the symmetry requirement (4.1). Θ is odd in τ, and thus $\underline{\beta}$ and ρ satisfy

$$\hat{R}_0(\underline{\beta}, \rho)(-\vartheta) = (\underline{\beta}, \rho)(\vartheta), \tag{4.1'}$$

$$\hat{R}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

System (4.4) can now be written as follows ($\underline{\beta}' = \partial_\vartheta \underline{\beta}$):

$$\begin{aligned} &\beta_0' = \beta_1\left(1 + \mu\Psi_1 + \frac{\varepsilon}{\rho} g_2\right)^{-1}, \qquad \rho' = \varepsilon g_1\left(q + \mu\Psi_1 + \frac{\varepsilon}{\rho} g_2\right)^{-1}, \\ &\beta_1' = (\Phi_1(\beta_0, \rho^2; \mu) + \varepsilon f_1)\left(q + \mu\Psi_1 + \frac{\varepsilon}{\rho} g_2\right)^{-1}, \end{aligned} \tag{4.4'}$$

or in abbreviated form

$$\beta_0' = \beta_1' G, \qquad \beta_1' = \Phi_1 G, \qquad \rho' = \varepsilon g_1 G,$$

$$G = \left(q + \mu\Psi_1 + \frac{\varepsilon}{\rho} g_2\right)^{-1}, \qquad G|_{\varepsilon=0} = G(\beta_0, k, \mu).$$

The case μ positive is treated similarly with minor changes of the scaling. We leave these details to the reader.

4.1. Persistence of periodic solutions I

The periodic solutions of type I of (3.14) persist under reversible perturbations and, thus, they are continuable to periodic solutions of the full system (4.2). The first part of this section is devoted to a proof of this fact.

Assume that α_0^* is constant, thus $\alpha_1^* = 0$. We have $|z^*| = K^{\frac{1}{2}} > 0$. Consider the case $\mu < 0$; μ positive can be treated similarly. Scaling as in (4.3), we obtain $(\underline{\beta}^*, \rho^*)$ which, as a function of ϑ, satisfies (4.4') for $\varepsilon = 0$, $\underline{\beta}^*$ is a fixed point of (4.4') for $\varepsilon = 0$. Let us observe that

$$3\beta_0^* - 1 = \pm(1 - 6d\rho^{*2})^{\frac{1}{2}} + \mathcal{O}(\mu) \tag{4.5$_\pm$}$$

holds for small values of $|\mu|$. (4.5$_+$) corresponds to a centre, (4.5$_-$) to a saddle in Figure 3.1. Our aim is to apply the implicit function theorem (IFT) at with

$$\Gamma = (\underline{\gamma}_1 \underline{\gamma}_2 \underline{\gamma}_3), \qquad \Lambda = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write $f = \Gamma \underline{F}$ and $\gamma = \Gamma \underline{\delta}$. Observe that, since f is antireversible and γ reversible, we have to require F_1, δ_2, δ_3 even and F_2, F_3, δ_1 odd. Now, (4.7) reads

$$\delta_2 + \delta_1' = F_1, \qquad \delta_2' = F_2, \qquad \delta_3' = F_3,$$

which has a unique solution, if $[\delta_3]$ is prescribed. Moreover, $[\delta_2] = [F_1]$ holds. For given $[\delta_3]$, the inverse is uniformly bounded in bounded μ-intervals. Therefore we obtain

THEOREM 4.5. *Given μ_0 sufficiently small, let $(\underline{\alpha}^*, z^*)(x; \mu)$ be a nonconstant periodic solution of the normal-form system (3.9) for $|\mu| < \mu_0$. Then there exists a one-parameter family of solutions $(\underline{\alpha}, z)(x; \mu, \gamma)$ of the full system (4.2) with the same period. The free parameter is determined by the mean-value of $|z|$. This solution satisfies, if $[|z|]^2 = K^*$, $(\underline{a}, |z|) - (\underline{\alpha}^*, K^{*\frac{1}{2}}) = \mathcal{O}(\mu^{s-1})$ and $\partial_x(\Theta(x) - \Theta^*(x)) = \mathcal{O}(\mu^{s-\frac{1}{2}})$ uniformly for $x \in \mathbb{R}$.*

4.3. Persistence of homoclinic solutions

Assume that $(\underline{\alpha}^*, z^*)$ is one of the homoclinic solutions of (3.14). Again we treat the case $\mu < 0$ only. Its scaled version $(\underline{\beta}^*, \rho^*)(\vartheta)$ solves (4.4′) for $\varepsilon = 0$. ρ^* is constant and $\underline{\beta}^* = \underline{\beta}^\infty + \underline{\gamma}^*$ holds, where $\underline{\beta}^\infty = (\beta_0^\infty, 0)$ is constant and $\underline{\gamma}^*$ decays to 0 for $\vartheta \to \pm\infty$ as $\exp(-\omega_0|\vartheta|)$. Here, $\omega_0 = (\partial_{\beta_0}\Phi_1(\beta_0^\infty, \rho^{*2}; \mu))^{\frac{1}{2}} = (1 - 3\beta_0^\infty + \mathcal{O}(\mu))^{\frac{1}{2}}/q$ and (4.5)$_-$ holds. According to Proposition 4.2, $(\underline{\beta}^\infty, \rho^*)$ has, for given μ and prescribed $k = K/\mu^2$, a unique continuation as a $2\pi\nu$-periodic solution $(\underline{\bar{\beta}}, \bar{\rho})$ of (4.4′) for all sufficiently small $|\varepsilon|$. We write (4.4′) in the self-explanatory form

$$\begin{pmatrix} \underline{\beta} \\ \rho \end{pmatrix}' = N(\underline{\beta}, \rho; \mu) + \varepsilon F\left(\underline{\beta}, \rho, \frac{\vartheta}{\nu}; \mu\right). \tag{4.9}$$

It follows that $N(\underline{\beta}^\infty, \rho^*; \mu) = 0$ and

$$\begin{pmatrix} \underline{\gamma}^* \\ 0 \end{pmatrix}' = N(\underline{\beta}^*, \rho^*; \mu) = N^*. \tag{4.9a}$$

Similarly, we define $\tilde{N}$, when $(\underline{\bar{\beta}}, \bar{\rho})$ is the argument. We suppress μ and ϑ/ν in the notation. Set $(\underline{\beta}; \rho) = (\underline{\bar{\beta}} + \underline{\gamma}^* + \underline{\gamma}, \bar{\rho} + \delta)$ and obtain

$$\begin{pmatrix} \underline{\gamma} \\ \delta \end{pmatrix}' = N(\underline{\beta}^* + \underline{\gamma}, \rho^* + \delta) - N^* + \varepsilon\tilde{F}, \tag{4.10}$$

where

$$\begin{aligned} \varepsilon\tilde{F} = {} & N(\underline{\bar{\beta}} + \underline{\gamma}^* + \underline{\gamma}, \bar{\rho} + \delta) - N(\underline{\bar{\beta}}, \bar{\rho}) + N^* \\ & - N(\underline{\beta}^* + \underline{\gamma}, \rho^* + \delta) - \varepsilon F(\underline{\bar{\beta}}, \bar{\rho}) \\ & + \epsilon F(\underline{\bar{\beta}} + \underline{\gamma}^* + \underline{\gamma}, \bar{\rho} + \delta). \end{aligned}$$

Observe that $\tilde{\beta}+\gamma^*-\beta^*$, $\tilde{\rho}-\rho^*$ and $N(\tilde{\beta}, \tilde{\rho})$ are all of order $\mathcal{O}(\varepsilon)$. Therefore the factor ε is justified. Moreover, $\tilde{F}$ anticommutes with $\hat{R}_0$. Since $\gamma^*(\pm\infty)=0$, the requirement $(\gamma(\infty); \delta(\infty))=(0, 0)$ is compatible with (4.11).

We work in the following spaces ($m \leqq j$):

$$C^j_\omega = \Big\{(\beta, \rho) \in C^j(\mathbb{R}) / \sup_\vartheta \exp(\omega\,|\vartheta|)\,/(\beta, \rho)^{(m)}(\vartheta)| < \infty;$$

$$m=0, 1; (4.1') \text{ holds with the sign } (-1)^{j+1}\Big\}$$

for $j=0$, 1 and fixed positive $\omega < \omega_0$; the norms are denoted by $\|\cdot\|_{j,\omega}$. The functions $\tilde{\beta}$, $\tilde{\rho}$ belong to $C^0_b (= C^0_0$ in the present notation). F and N are continuously differentiable functions of their arguments as elements of C^0_ω and therefore

$$\|\tilde{F}\|_{0,\omega} \leqq c\,\|(\gamma, \delta)\|_{0,\omega}$$

holds, if the arguments $\tilde{\beta}$, $\tilde{\delta}$, β^*, etc. all belong to a bounded set $\Omega \subset C^0_0$; the constant c depends on Ω only. In particular, the estimate is uniform in $[-\mu_0, 0]$. Involving the IFT for the proof of persistence, it remains to show that

$$M = \frac{d}{d\vartheta} - DN^* \tag{4.11}$$

defines an isomorphism from $C^1_\omega \to C^0_\omega$. Define

$$\begin{gathered} DN^* = T_0 + T_1(\vartheta), \qquad T_0 = DN(\beta^\infty_0, \rho^*), \\ T_0 = \frac{1}{q}\begin{pmatrix} T_{00} & -2d\rho^* e_2 \\ 0 & 0 \end{pmatrix}, \qquad T_{00} = \begin{pmatrix} 0 & 1 \\ 1-3\beta^\infty_0 & 0 \end{pmatrix} + \mathcal{O}(\mu), \\ T_1(\vartheta) = \frac{1}{q}\begin{pmatrix} T_{10}(\vartheta) & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_{10} = \begin{pmatrix} 0 & 0 \\ -3\gamma^*_0 & 0 \end{pmatrix} + \mathcal{O}(\mu). \end{gathered} \tag{4.12}$$

T_{00} is a constant matrix with eigenvalues $\pm\omega_0$. Therefore, $(d/d\vartheta - T_{00})^{-1} = L$ exists and maps PC^0_ω into PC^1_ω, where $P(\gamma, \delta)=\gamma$.

We can write $M(\gamma, \delta) = (f, g)$ as

$$\begin{aligned} (1 - LT_{10})\gamma &= L(f + 2d\rho^*\,\delta e_2), \\ \delta(\vartheta) &= -\int_\vartheta^\infty g\, d\vartheta. \end{aligned} \tag{4.13}$$

It remains to be shown that $1 - LT_{10}$ has a continuous inverse in PC^0_ω. However, LT_{10} is compact. This follows from the fact that $\gamma^*_0 \in PC^0_\omega$. Therefore, a bounded set in PC^0_ω is mapped by LT_{10} into a bounded set in PC^1_ω with elements having uniform exponential decay at infinity. Hence, Arzela-Ascoli applies. Therefore, to finish the proof of the applicability of the IFT, we need that 1 is not eigenvalue of LT_{10}. But $LT_{10}\gamma=\gamma$ implies $\gamma' - (T_{00} + T_{10}(\vartheta))\gamma = 0$, which is equivalent to

$$\gamma''_0 - D_{\beta_0}\Phi_1(\beta^*_0, \rho^{*2})\gamma_0 = 0.$$

$\beta_0^{*\prime} = \gamma_0^{*\prime}$ solves the equation, but it violates (4.1′). Multiplying by $\beta_0^{*\prime}$ and integrating over $(-\infty, \vartheta]$ yields

$$(\gamma_0'\beta_0^{*\prime} - \gamma_0\beta_0^{*\prime\prime})(\vartheta) = 0;$$

$\gamma_0 = 0$ is the only even solution.

Finally, observe that all estimates are uniform in μ and $\nu = (|\mu|\,\sigma)^{\frac{1}{2}}$ for $\mu \in [-\mu_0, 0]$ and $\mu_0 > 0$ sufficiently small. We summarise: the operator

$$M\colon C^1_\omega \cap \{\rho(0) = 0\} \to C^0_\omega$$

defined in (4.10) defines an isomorphism for every $\mu \in [-\mu_0, 0]$.

THEOREM 4.6. *Given a family of homoclinic solutions* $(\underline{\alpha}^*, z^*)(\mu)$ *of the* NF-*system* (3.14), $\mu \in (0; \mu_0)$, *we write* $(\underline{\alpha}, z^*) = (\alpha_\infty, z_\infty) + (\gamma^*, 0)$, *where* $\underline{\alpha}_\infty$, $z_\infty = |z^*| \exp(\Theta_\infty x)$ *is a periodic solution of* (3.14), *and* $\underline{\gamma}^*(\pm\infty) = 0$. *Denote by* $(\tilde{\underline{\alpha}}, \tilde{z})$ *the continuation of* $(\underline{\alpha}_\infty, z_\infty)$ *according to Theorem* 4.3. *For some* $0 < \mu_0' \leqq \mu_0$, *there exists a family of homoclinic solutions* $(\underline{\alpha}, z)(\mu)$, $\mu \in (0, \mu_0')$, *of the full system* (4.2). *Moreover, if we write* $(\underline{\alpha}, z) = (\tilde{\underline{\alpha}}, \tilde{z}) + (\gamma, \zeta)$, *where* $\zeta = \delta \exp(i\Theta)$, *the following estimates are valid, for every* ω, $0 < \omega < \omega_0 = (1 - 3\beta_0^\infty + \mathcal{O}(\mu))^{\frac{1}{2}}$ $(= \rho^{\frac{1}{2}}$ *in the notation of* (3.17)):

$$\left.\begin{aligned} |(\tilde{\underline{\alpha}}_\infty, \tilde{z}) - (\underline{\alpha}^*_\infty, z^*)| &\leqq c_0\,|\mu|^{s'}, \\ |(\underline{\gamma}, \zeta) - (\underline{\gamma}^*, 0)| &\leqq c_1\,|\mu|^{s'}\,e^{-\omega\nu|x|}. \end{aligned}\right\} \tag{4.14}$$

The constants depend on μ_0', s' *and* c_1, *also on the choice of* ω.

The proof is an immediate consequence of the IFT and the fact that all bounds of our previous considerations are uniform in $\mu \in (0, \mu_0)$. Also we have to apply Theorem 4.3.

The second estimate in (4.1) is not quite satisfactory, since it does not decay to 0 as fast as $\underline{\gamma}^*$ does (cf. (3.16)). Thus, the estimate is not uniform in x relative to $\underline{\gamma}^*$, a facet which appears also in **[5]**. It stems from the fact $L = (d/d\vartheta - T_{00})^{-1}$ respects a decay $\exp(-\omega/|\vartheta|)$ only if $\omega < \omega_0$.

5. Persistence of quasi-periodic solutions for the full system

To prove that most of the quasi-periodic solutions of the normal form system (3.14) persist under the reversible perturbation given by higher order terms, we use the same approach as in **[15]** for a different reversible vector field. We show below how our analysis fits into the frame of **[15]**.

In what follows, we use the notations $X = (k, h)$, where k, h are defined in (3.17) and (3.20). We consider $X \in \Delta$, the triangular area defined in Figure 3.2. Now we introduce two angles θ_0 and θ_1. First we define

$$\theta_1 = \operatorname{sgn}(\beta_1)\frac{\Omega_1}{\nu}\int_{\beta_m}^{\beta_0} \frac{ds}{|f(s, X)|^{\frac{1}{2}}} \bmod (2\pi) \tag{5.1}$$

where by convention $\theta_1 = \pm\pi$ for β_0 and β_M, and where X defines a closed orbit in the (β_0, β_1) plane. Now let us set

$$\theta_0 = \frac{\Omega_0}{\Omega_1}\theta_1, \tag{5.2}$$

where Ω_0 and Ω_1 are functions of X, given in (3.21), (3.22). It is clear that the variables (X, θ_0, θ_1) describe diffeomorphically the region of the four-dimensional space filled with the quasi-periodic solutions of (3.14), and that we have for this system

$$\begin{aligned} \frac{dX}{dx} &= 0 \quad \text{in} \quad \mathbb{R}^2, \\ \frac{d\Theta}{dx} &= \Omega(\mu, X) \quad \text{in} \quad T^2, \end{aligned} \tag{5.3}$$

where

$$\Omega = (\Omega_0, \Omega_1), \qquad \Theta = (\theta_0, \theta_1),$$

$$\Omega_0 = q + \mu\omega_0(X) + \mathcal{O}(\mu^2), \qquad \Omega_1 = |\mu|^{\frac{1}{2}}\,\omega_1(X) + \mathcal{O}(|\mu|^{\frac{3}{2}}),$$

$$\omega_1^{-1} = \frac{1}{\pi\sigma^{\frac{1}{2}}}\int_{\beta_{0m}}^{\beta_{0M}} |f_0(\beta_0, X)|^{-\frac{1}{2}}\, d\beta_0,$$

$$\omega_2 = \frac{\omega_1}{\pi\sigma^{\frac{1}{2}}}\int_{\beta_{0m}}^{\beta_{0M}} \Psi_0(\beta_0)\,|f_0(\beta_0, X)|^{-\frac{1}{2}}\, d\beta_0,$$

β_{0m} and β_{0M} are consecutive zeros of $f_0(\beta, X)$, and $\Psi_0(\beta_0)$ is defined by

$$\Psi(\mu\beta_0, \mu^2 K) = \mu\Psi_0(\beta_0) + \mathcal{O}(\mu^2).$$

The action of the reversibility operator R_0 is just $\Theta \to -\Theta$, X being unchanged. Hence we see that we enter exactly into the frame of the study of **[15]** (cf. section 5) for the full system (4.2). Notice that for the persistence property we cannot reduce to the centre manifold, because of the loss of regularity, so we stay in the full space $D(A)$ and keep the hyperbolic part of the linear operator A. In **[15]** it is shown how to obtain the normal form without reducing to the centre manifold, by using a suitable Green's kernel of the hyperbolic part of A. Now the technique used for proving the persistence rests upon the use of the Nash–Moser implicit function theorem in C^∞-Fréchet-spaces, as developed by Hamilton **[10]**. For the proof of the following result, see **[15**, theorem 5].

THEOREM 5.1. *Let ρ by any diophantine number, i.e.*

$$|p + \rho q| \geqq \frac{c}{|q|^{1+\delta}}$$

holds for some positive c and δ and all $p/q \in Q$. We assume that, for a given $\mu \neq 0$, there exists $\bar{X}_0 \in \Delta$ (Figure 3.2) satisfying

$$\rho = \frac{\Omega_1(\mu, \bar{X}_0)}{\Omega_0(\mu; \bar{X}_0)} \quad \textit{and} \quad \operatorname{Det}(D_X\Omega)|_{X=\bar{X}_0} \neq 0.$$

Then, for any $X_0 \in C_\rho \subset \Delta$, where C_ρ is a curve close to $\bar{X}_0$, and for small $|\mu|$, system (4.2) admits a quasi-periodic solution whose principal part is given by the solution of the normal form equation, and which is parametrised by $\Theta \in T^2$. It has rotation number ρ. Moreover, these quasi-periodic solutions are reversible in the sense of (4.1).

Remark 5.2. The result can be expressed in saying that "most of" the quasi-periodic solutions we found with the normal form (3.14) persist, when adding the "flat terms". The persistence set in Δ can be locally represented by a product of a curve with a Cantor set.

Remark 5.3. It is well-known that the diophantine numbers form a set of full Lebesgue-measure in $(0, 1)$. The rotation number is the ratio between the two independent frequencies. It is known that, when it is diophantine, the flow can be linearised on the torus T^2 by a suitable diffeormorphism.

6. Appendix

6.1. Identification of coefficients

In this section, we describe a method of calculating the coefficients in the normal form (3.14). We treat the more interesting case $b < \frac{1}{3}$. Starting with the reduced system (2.14), one observes that $h(\lambda, \underline{w}_0) = \mathcal{O}(\mu\, |\underline{w}_0| + |\underline{w}_0|^2)$. Therefore

$$f_0 = F_0(\lambda, \underline{w}_0) + \mathcal{O}(\mu\, |\underline{w}_0|^2 + |\underline{w}_0|^3)$$

holds, and the coefficient of order $\mu \rfloor w|_0$ and $\rfloor w|_0^2$ can be calculated by simply setting $h = 0$. For higher order calculations, h has to be determined to the appropriate order. For a general algorithm, see **[13]**.

According to the decomposition $X = X_0 \oplus X_1$, in Section 2, we have to project F onto F_0. While X_0 is spanned by φ_0, φ_1, $\varphi_\pm$ as given in (3.11) and A_0 in (3.12), we construct the projection by using the adjoint A^* of A. In X we introduce the scalar product

$$(\underline{w}_1, \underline{w}_2) = \beta_1\bar{\beta}_2 + [W_1\bar{W}_1 + W_2\bar{W}_2].$$

With respect to this product, A^* is given by

$$A^*\underline{w} = \begin{pmatrix} -W_1(1) \\ -W_2' + W_2(1) \\ W_1' \end{pmatrix},$$

$$D(A^*) = \mathbb{C} \times H^1(0, 1) \times H^1(0, 1) \cap \{\beta = -bW_2(1),\ W_2(0) = 0\}.$$

We have $\Sigma A^* = \Sigma A$ and the generalised eigenfunction $\underline{\psi}^j$ to the central part of ΣA^* reads

$$(b - \tfrac{1}{3})\underline{\psi}^0 = \begin{pmatrix} 0 \\ b + \frac{1}{2}(y^2 - 1) \\ 0 \end{pmatrix}, \qquad (b - \tfrac{1}{3})\underline{\psi}^1 = \begin{pmatrix} -b \\ 0 \\ y \end{pmatrix},$$

$$r(q)\underline{\psi}^+ = \begin{matrix} ib \sinh q \\ -\cosh qy + \dfrac{\sinh q}{q} \\ -i \sinh qy \end{matrix},$$

$$r(q) = \left(b - \frac{1}{q^2}\right)\sinh^2 q + 1 \qquad \underline{\psi}^- = \bar{\underline{\psi}}^+.$$

The conditions $(\underline{w}, \underline{\psi}^\pm) = (\underline{w}, \underline{\psi}^j) = 0$, $j = 0, 1$ determine X_1.

Henceforth we neglect all terms of order $\mathcal{O}(\mu\,|\underline{w}_0|^2+|\underline{w}_0|^3)$, since we observe that the coefficients we need in the normal form (3.14) are of order $\mu\,|\underline{w}_0|+|\underline{w}_0|^2$. Equation (2.14), expressed in the basis (φ_j), then reads

$$\dot{\underline{a}} = A_0\underline{a} + \mu F_{11}(\underline{a}) + F_{02}(\underline{a}), \tag{6.1}$$

where $\underline{a}=(a_0, a_1, a_+, a_-)^t$, A_0 as in (3.12); F_{11} is linear, F_{02} quadratic in $\underline{a}$. They have the components (F, ψ^j), $(F, \psi^{\pm})$,

$$F = -\frac{\mu}{b}[W_1]\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}(3[W_1^2]+[W_2^2])(2b)^{-1}\\ W_2\,\partial_y W_1 - 3W_1\,\partial_y W_2\\ -W_1\,\partial_y W_1 + W_2\,\partial_y W_2\end{pmatrix},$$

which can be deduced from (2.2). We have to insert

$$W_1 = a_0 - (a_+ + a_-)\cosh qy,$$
$$W_2 = -a_1 y + i(a_+ + a_-)\sinh qy$$

and obtain the explicit formula of F_{11} and F_{02}.

The transformation to normal form is achieved via

$$\underline{a} = \underline{x} + \mu\Phi_{11}(\underline{x}) + \Phi_{02}(\underline{x}), \qquad \underline{x} = (\underline{\alpha}, z, \bar{z}),$$

to obtain

$$\dot{\underline{x}} = A_0\underline{x} + \mu N_{11}\underline{x} + N_{02}(\underline{x}). \tag{6.2}$$

Comparision with (6.1) yields

$$D_x\Phi(x)A_0x - A_0\Phi(x) = F(x) - N(x),$$

where Φ is $\mu\Phi_{11}$ (respectively Φ_{02}) and similarly for F and N. The solvability is guaranteed by **[9]** and N is known by (3.14) up to the coefficients to be determined.

Elementary calculations yield for $\Phi=\Phi_{11}$

$$c_1(\mu)(b-\tfrac{1}{3}) = \mu, \qquad \gamma_1(\mu)qr(q) = -\sinh^2 q$$

and for $\Phi=\Phi_{02}$

$$c_2(b-\tfrac{1}{3}) = -\tfrac{3}{2}, \qquad d_2(b-\tfrac{1}{3}) = 1 + \frac{2}{q}\cosh q\sinh q,$$

which are the values used in Sections 3 and 4.

6.2. Existence of a *k*-parametrised family of periodic solutions of type II

For the completion of the proof of persistence of periodic solutions of type II, we have to embed a given solution $(\underline{\beta}^*, \rho^*)$ of period $p=2\pi\nu l$ of the normal form scaled system (4.4′) for $\varepsilon=0$ into a k-family $(\underline{\beta}, \rho)$ of solutions having the same period. In this section we shall show that this is possible.

For this reason, we define

$$\varphi(h,k) = \int_{\beta_m}^{\beta_M} \Psi(\beta_0, k; \mu)(\hat{\Phi}(\beta_0, k; \mu)+h)^{-\frac{1}{2}}\,d\beta_0,$$

where $\Psi = q + \mu\Psi_1$, $\partial_{\beta_0}\hat{\Phi} = 2\Phi_1$ and Ψ_1, Φ_1 are given in (4.4). β_m, β_M are consecutive zeros of $\beta_1 = (\hat{\Phi} + h)^{\frac{1}{2}}$. Obviously, the assertion is true, if we can find for every k^* a constant h^* such that

$$\varphi(h^*, k^*) = \pi\nu l, \qquad \partial_h\varphi(h^*, k^*) \neq 0 \tag{6.3}$$

holds for some suitable $l \in \mathbb{N}$. Henceforth, we fix k^* and suppress its explicit notation mostly, similarly for μ. We denote by $\tilde{\gamma}_0$ (respectively γ_0^*) the values of β_0, where $\hat{\Phi}$ assumes its minimum (respectively maximum). Remember that we have

$$\hat{\Phi} = \beta_0^2 - \beta_0^3 - 2\,dk^*\beta_0 + \mathcal{O}(\mu).$$

Define h_0 (respectively h_1) by $\hat{\Phi}(\tilde{\gamma}_0) + h_1 = 0$ (respectively $\Phi(\gamma_0^*) + h_0 = 0$); then $h_0 < h_1$ and $\varphi(h, k^*) \to +\infty$ as $h \to h_1 - 0$. The constant h varies in the bounded interval $[h_0, h_1]$.

First we take $|\beta_0 - \gamma_0^*|$ and $|\mu|$ small. All the arguments will be robust with respect to μ. Hence we set $\mu = 0$ throughout the proof and leave the perturbation argument to the reader. We have

$$\hat{\Phi}(\beta_0) = a_0(h) - \rho(\beta_0)\gamma^2,$$

where h is close to h_0, $\tilde{\gamma}_0 + \gamma_0^* = \frac{2}{3}$, and

$$\gamma = \beta_0 - \gamma_0^*, \qquad a_0(h) = \hat{\Phi}(\gamma_0^*, k^*) + h,$$
$$\rho(\beta_0) = \alpha_0 + \gamma, \qquad \alpha_0 = 3\gamma_0^* - 1 = (1 - 6\,dk^*)^{\frac{1}{2}}.$$

Observe that $\rho \geqq \alpha_0/3 > 0$. Define the transformation

$$\xi = \gamma\left(\frac{\rho(\beta_0)}{a_0(h)}\right)^{\frac{1}{2}}. \tag{6.4}$$

Then $2\xi'(\beta_0) = 3(\beta_0 - \tilde{\gamma}_0)(\rho a_0)^{-\frac{1}{2}} > 0$ holds and thus (6.4) is invertible in every compact subinterval of $(h_0, h_1]$, and $\beta_0 = \beta_0(\xi, h)$. We obtain

$$\varphi(h, k^*) = q a_0(h)^{-\frac{1}{2}} \int_{-1}^{1} \beta_0'(\xi, h)(1 - \xi^2)^{-\frac{1}{2}}\, d\xi.$$

Define $b = \partial_{\beta_0}(\gamma\sqrt{\rho}) = \rho^{\frac{1}{2}} + \gamma\rho^{-\frac{1}{2}}/2$. Then we have

$$b\beta_0' = a_0^{\frac{1}{2}}, \qquad b\,\partial_h\beta_0' + (\partial_h b)\beta_0' = a_0^{-\frac{1}{2}}/2, \qquad 2b\rho^{\frac{1}{2}} = 2\alpha_0 + 3\gamma = 3(\beta_0 - \tilde{\gamma}_0) > 0,$$

whence it follows that $(\partial_h b = b_h)$:

$$\partial_h\varphi(h, k^*) = -q a_0(h)^{-1} \int_{-1}^{1} b_h\beta_0'^2(1 - \xi^2)^{-\frac{1}{2}}\, d\xi. \tag{6.5}$$

We shall show that the even part b_h^e of b_h is negative if $h - h_0$ and thus $|\gamma|$ is small. It is easy to verify that

$$b_h = \frac{1}{4\rho a_0^{\frac{1}{2}}}\left(1 + \frac{2\alpha_0}{3(\beta_0 - \tilde{\gamma}_0)}\right)\xi = \frac{\xi}{4\alpha_0^{\frac{1}{2}}}\frac{4\alpha_0 + 3\gamma}{(2\alpha_0 + 3\gamma)(\alpha_0 + \gamma)}.$$

Set $h = h_0 + \delta,\ \delta > 0$ and obtain $a_0 = \delta$,

$$\beta_1 = q\beta_0' = (\hat{\Phi}(\gamma_0^* + \gamma, k^*) + h_0 + \delta)^{\frac{1}{2}} = \delta^{\frac{1}{2}}\Big(1 - \frac{\alpha_0}{\delta}\gamma^2 - \frac{\gamma^3}{\delta}\Big)^{\frac{1}{2}},$$

which implies

$$q\gamma' = \delta^{\frac{1}{2}}\Big(1 - \frac{\alpha_0}{\delta}\gamma^2 - \frac{\gamma^3}{\delta}\Big)^{\frac{1}{2}}$$

and thus $\gamma = \mathcal{O}(\delta^{\frac{1}{2}})$. Finally, it follows from (6.5) that

$$b_h = \frac{\xi}{2\alpha_0} - \frac{7}{8\alpha_0^{\frac{5}{2}}}\xi^2 + 0(\delta^{\frac{1}{2}}\xi^3),$$

implying $b_h^e < 0$ for sufficiently small δ. Since this result is robust with respect to μ, we have shown the solvability of (6.3), whenever $0 < h^* - h_0 < \delta_0$, and δ_0 is small and independent of μ.

Finally, we show that $\partial_h \varphi$ stays positive for all $h \in (h_0, h_1)$. From the previous argument we know it for $(h_0, h_0 + \delta_0) =: I_\delta$. Rewrite (6.5) as

$$\partial_h \varphi = -q\int_{-1}^{1} b_h b^{-2}(1 - \xi^2)^{-\frac{1}{2}}\, d\xi$$

and define

$$\Gamma := a_0^{\frac{1}{2}} b_h b^{-2} = \frac{4\alpha_0 + 3\gamma}{(2\alpha_0 + 3\gamma)^3}\xi, \qquad P = b_h b^{-2}.$$

Observe that $\gamma_h = \xi/(2b\sqrt{a_0})$ and thus

$$\Gamma_h = -\frac{3\xi^2}{ba_0^{\frac{1}{2}}}\Big(1 + \frac{3\alpha_0}{(2\alpha_0 + 3\gamma)^4}\Big) < 0.$$

Therefore

$$P_h + P/2a_0 < 0$$

holds, also for $P^e = (P(\xi) + P(-\xi))/2$. Therefore $P^e = Q\exp(-\int_{h_0'}^{h}(2a_0)^{-1}) < 0$ for $h \in (h_0', h_1)$, since $Q_h < 0$ and $Q(h_0') < 0$ if $h_0' \in I_\delta$. Hence, $\partial_h \varphi$ is positive for $h \in (h_0, h_1)$. We have shown that every periodic solution of type II can be embedded into a smooth family of periodic solutions parametrised by k, which have a constant period. The family parameter $h = h(k)$ varies over the whole interval (h_0, h_1).

6.3. Addendum to the normal form for $b > \frac{1}{3}$

While constructing the normal form in Case 3.2, i.e. $b < \frac{1}{3}$, λ close to 1, we have left the following proposition without proof: given a polynomial $N(\alpha_0, \alpha_1, z, \bar{z})$ on $\mathbb{R}^4$, $z \in \mathbb{C}$ and $\bar{z}$ its complex conjugate, which satisfies for some $q > 0$

$$DN := \alpha_0 N_{\alpha_1} - iqzN_z + iq\bar{z}N_{\bar{z}} = 0, \tag{6.6}$$

then $N = \varphi(\alpha_0, |z|^2)$ for some polynomial φ.

For the proof, observe that

$$I := \alpha_0, \qquad K = |z|^2, \qquad L := \alpha_0 \ln z + iq\alpha_1$$

are three independent integrals of (6.6) for $\alpha_0 \neq 0$. (We can check that the Jacobian matrix of the map $(\alpha_0, \alpha_1, z, \bar{z}) \mapsto (I, K, L, M = \alpha_1)$ is $-\alpha_0$.) It is then possible to work locally and set

$$N(\alpha_0, \alpha_1, z, \bar{z}) = \varphi_1(I, K, L, M).$$

$DN = 0$ is now equivalent to $\partial\varphi_1/\partial M = 0$ (since I, K, L are constants along the characteristics). Hence, in fact, one has

$$N(\alpha_0, \alpha_1, z, \bar{z}) = \varphi(I, K, L).$$

Now, we obtain

$$\partial_I \varphi = N_{\alpha_0} + \frac{i}{q} \ln z N_{\alpha_1},$$

$$\partial_K \varphi = \frac{1}{z} N_{\bar{z}},$$

$$\partial_L \varphi = \frac{1}{iq} N_{\alpha_1}.$$

Therefore, N being a polynomial, all derivatives of φ of sufficiently high order must vanish. Thus φ itself is a polynomial of its arguments. But this is impossible, if φ depends genuinely on L. To see this, set $\arg z = 0 = \alpha_1$, $\alpha_0 = 1$. Then

$$\begin{aligned} N &= a_s |z|^s + a_{s-1} |z|^{s-1} + \ldots + a_0 \\ &= b_\sigma |z|^\sigma P_\sigma(\ln |z|) + \ldots + b_0 = \varphi, \end{aligned}$$

where P_σ is a polynomial in $\ln |z|$. Observe that $\sigma = s$ must hold. Divide by $|z|^s$ and consider $|z| \to +\infty$; then

$$a_s = \lim_{|z| \to \infty} b_s P_s(\ln |z|).$$

This limit exists only if P_s is independent of $\ln |z|$. Proceed to a_{s-1}, etc., and conclude that φ must be independent of L. □

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