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Perturbed Homoclinic Solutions in Reversible 1:1 Resonance Vector Fields

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I. INTRODUCTION

Let us consider a smooth vector field in $\mathbb{R}^4$ such that the origin is a fixed point. One says that we have a 1:1 *resonance* if the linear operator, given by the differential at the origin, has two double pure imaginary eigenvalues $\pm i\omega_0$, such that there are two complex conjugated couples of eigenvectors and generalized eigenvectors. Reversible vector fields are those which anti-commute with some symmetry S. In particular, in such cases the spectrum of the linear operator is symmetric with respect to both the real and imaginary axis. This shows that, at the linear level, reversible perturbations of such vector fields are codimension 2, since one can play on the imaginary part (detuning) and on the real part of the four symmetric eigenvalues of the perturbed problem. If we do not specialize the precise point on the imaginary axis where eigenvalues meet at criticality, then one only *needs a single parameter* $\mu \in \mathbb{R}$. The ordinary case without reversibility is of codimension 3; it was studied in some detail by S. Van Gils *et al.* [1]. This previous work did not consider our present situation which would correspond to a very degenerated case.

In fact, there are many classical mechanical two degrees of freedom problems ruled by such vector fields, especially the (simplified) problem of the flutter of a wing; however, we want to emphasize two physically important problems where the situation we study in this work is the revelant one.

First, in the analysis of nonlinearly resonant surface waves, K. Kirchgässner [2] consider the parameter plane (b, λ), where surface tension occurs in the Bond number b, and where gravity occurs in λ which is the square of the inverse of the Froude number. In this plane, there is a curve where a 1:1

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resonance takes place, in a reversible frame (invariance under the reflection $x \to -x$), which rules bifurcating travelling waves, for $b < 1/3$ and $\lambda > 1$. The limit case $b = 1/3$, $\lambda = 1$ corresponds to a quadruple 0 eigenvalue with a 4×4 Jordan block. The study in the full neighborhood of this codimension 2 case (reversible) is still an open problem, despite of the work [16].

Second, in the study of steady bifurcating solutions in hydrodynamic instability problems taking place in infinitely long cylinders (with the reflection symmetry $x \to -x$), it is shown in [3] that the solutions are ruled again by a reversible 1:1 resonant system.

In both these cases the evolution variable is the extended spatial coordinate x, the solutions being time independent in the second case, while they are of a travelling wave form in the first case, linking time and space variables. However, for traditional reasons, in what follows we denote by t the evolution variable.

In this work, we are interested in showing the existence of *homoclinic type solutions*. The big advantage of our problem is that the vector field can be approximated as closely as we wish, near the origin, by an *integrable vector field*. On this integrable field in $\mathbb{R}^4$ we can find, in an elementary way, all existing solutions. It then remains to prove for the full system, the persistence of the solutions which interest us. Other types of solutions (quasi-periodic) were already examined in [4]. We can show that there are, in the *supercritical case*, solutions connecting a periodic solution to itself with a phase shift. Moreover at least two of these solutions have their main amplitude component which cancels in the middle of the orbit. In the *subcritical case* we find a homoclinic solution tending towards 0 at infinity (like a solitary wave). All these solutions give a new insight for the mentioned physical problems. For instance, this proves in hydrodynamical stability problems (in the supercritical case) the existence of a stationary regime looking like the basic flow in the middle of the cylindrical domain, and looking like the periodic classically bifurcating flow at both infinities (with a phase shift) of the domain (see [3]). This also gives, for illustrating the subcritical case, a new type of solitary wave which was announced by [5], and in hydrodynamical stability problems which might be subcritical [for instance the Couette–Taylor problem for a small enough gap, and slightly counter-rotating cylinders (see [6])], this gives the existence of a stationary regime looking like the Couette flow at the top and bottom, and looking like Taylor vortices at the middle of the cylinders. In addition, the technique we develop is applicable to other reversible problems, like the one studied in the above-mentioned water wave problem (see [7]) for $b < 1/3$ and λ close to 1, where the critical eigenvalues are 0 (which is double), and a pair of pure imaginary simple ones. This leads also to an integrable normal form and analogous results hold, giving a new proof of the results shown in [7].

II. The Reversible 1:1Resonance Normal Form

II.1. *The General Frame*

Let us consider a one parameter family of regular 4-dimensional vector fields in $\mathbb{R}^4$, with a fixed point at the origin, and such that the linearized operator for $\mu = 0$, has a 1:1 resonance singularity. This means that there is one pair of double eigenvalues $\pm i\omega_0$ with two dimensional Jordan blocks. More precisely we have a differential equation of the following form in $\mathbb{C}^2$,

$$\begin{aligned}\frac{dA}{dt} &= i\omega_0 A + B + f(\mu, A, \bar{A}, B, \bar{B}) \\ \frac{dB}{dt} &= i\omega_0 B + g(\mu, A, \bar{A}, B, \bar{B}),\end{aligned} \tag{1}$$

where $f, g = O(|\mu| (|A| + |B|) + |A|^2 + |B|^2)$. In additon we assume that our system is *reversible*, i.e., we have the symmetry S defined by: $S(A, B) = (\bar{A}, -\bar{B})$ which *anticommutes* with the vector field. This means that we have the following properties:

$$\begin{aligned}f(\mu, \bar{A}, A, -\bar{B}, -B) &= -\overline{f(\mu, A, \bar{A}, B, \bar{B})} \\ g(\mu, \bar{A}, A, -\bar{B}, -B) &= \overline{g(\mu, A, \bar{A}, B, \bar{B})}.\end{aligned} \tag{2}$$

Now, to simplify the form of (1), we can put it into *normal form*. This, of course, can only arrange coefficients up to a given order, but this greatly simplifies the further analysis. It is shown in Elphick *et al.* [8] that a good choice of normal from associated with the critical linear operator of the vector field (1) is

$$\begin{aligned}\frac{dA}{dt} &= i\omega_0 A + B + A\varphi_0\left[\mu; |A|^2; \frac{i}{2}(A\bar{B} - \bar{A}B)\right], \\ \frac{dB}{dt} &= i\omega_0 B + B\varphi_0\left[\mu; |A|^2; \frac{i}{2}(A\bar{B} - \bar{A}B)\right] + A\varphi_1\left[\mu; |A|^2; \frac{i}{2}(A\bar{B} - \bar{A}B)\right],\end{aligned} \tag{3}$$

where φ_0 and φ_1 are two polynomials in their last two arguments. Moreover, this normalization process preserves reversibility. This results easily from a proof similar to that for usual symmetries (see [8]) since S in unitary. We notice that $|A|^2$ and $(i/2)(A\bar{B} - \bar{A}B)$ are invariant under S; hence property (2) gives a pure imaginary function φ_0 and a real function φ_1. Finally, the system in now written as follows, up to order $O(|A|+|B|)^N$, with arbitrary N:

$$\frac{dA}{dt} = i\omega_0 A + B + iAP\left[\mu; |A|^2; \frac{i}{2}(A\bar{B} - \bar{A}B)\right],$$
$$\frac{dB}{dt} = i\omega_0 B + iBP\left[\mu; |A|^2; \frac{i}{2}(A\bar{B} - \bar{A}B)\right] + AQ\left[\mu; |A|^2; \frac{i}{2}(A\bar{B} - \bar{A}B)\right]. \tag{4}$$

Here P and Q are real polynomials in their two last arguments, with μ dependent coefficients and they are such that $P(0, 0, 0) = Q(0, 0, 0) = 0$. Let us observe that this normal form is rotationally invariant. Indeed, the vector field commutes with the operator $R_\varphi: (A, B) \to (Ae^{i\varphi}, Be^{i\varphi})$ for any real φ. We now check that

$$R_\varphi S = SR_{-\varphi} \tag{5}$$

which means that we have an $O(2)$ group acting (noncommuting) on (4).

As noticed in [3], system (4) is not hamiltonian in general, but it is *integrable*. Indeed, $(A\bar{B} - \bar{A}B)$ is an integral. To give the expression of the other integral, we define

$$G(\mu, u, v) = \int_0^u Q(\mu, s, v)\, ds; \tag{6}$$

then, as could be checked easily, the following function is an integral

$$H[\mu, |A|^2, |B|^2, v] \equiv |B|^2 - G[\mu, |A|^2, v], \tag{7}$$

where $v = (i/2)(A\bar{B} - \bar{A}B)$ is the basic first integral.

To solve this problem, let us change variables:

$$A = r_0 e^{i(\omega_0 t + \psi_0)}, \qquad B = r_1 e^{i(\omega_0 t + \psi_1)}. \tag{8}$$

Then, system (4) becomes

$$\frac{dr_0}{dt} = r_1 \cos(\psi_1 - \psi_0),$$
$$\frac{dr_1}{dt} = r_0 \cos(\psi_1 - \psi_0)\, Q(\mu, r_0^2, K),$$
$$\frac{d(\psi_1 - \psi_0)}{dt} = -\frac{\sin(\psi_1 - \psi_0)}{r_0 r_1} [r_1^2 + r_0^2\, Q(\mu, r_0^2, K)], \tag{9}$$
$$\frac{d\psi_0}{dt} = (r_1/r_0) \sin(\psi_1 - \psi_0) + P(\mu, r_0^2, K),$$

where the two integrals are now:

$$r_0 r_1 \sin(\psi_1 - \psi_0) = K$$
$$r_1^2 - G(\mu, r_0^2, K) = H. \tag{10}$$

When it is nonvoid, the intersection, in the 4-dimensional space $\mathbb{C}^4$, of the two hypersurfaces $|B|^2 - G[\mu, |A|^2, K] = H$, and $(i/2)(A\bar{B} - \bar{A}B) = K$, is locally a 2-dimensional "tubular" invariant manifold, which is invariant under the group of rotations: $(A, B) \to (Ae^{i\varphi}, Be^{i\varphi})$. All orbits, solutions of (4), are contained in such "tubular" manifolds. Let us now give precisions on the dynamics on these manifolds.

If we set $u_0 = r_0^2$, $u_1 = r_1^2$, then taking account of (10), system (9) reduces to

$$\left(\frac{du_0}{dt}\right)^2 = 4\{u_0[G(\mu, u_0, K) + H] - K^2\},$$
$$\frac{d(\psi_1 - \psi_0)}{dt} = -K(u_0 u_1)^{-1}[u_0 Q(\mu, u_0, K) + G(\mu, \mu_0, K) + H], \tag{11}$$

where we observe that $u_0 Q(\mu, u_0, K) + G(\mu, u_0, K) + H$ is the derivative of $u_0[G(\mu, u_0, K) + H]$ with respect to u_0.

To study more precisely the behaviour of the solutions, let us define the principal parts of P and Q:

$$P(\mu, u, v) = p_1 \mu + p_2 u + p_3 v + O(|\mu| + |u| + |v|)^2,$$
$$Q(\mu, u, v) = -q_1 \mu + q_2 u + q_3 v + O(|\mu| + |u| + |v|)^2. \tag{12}$$

Coefficients p_j and q_j have a physical meaning related to the original problem. For the linear operator occurring in (4), the eigenvalues are

$$i[\omega_0 + P(\mu, 0, 0)] \pm \sqrt{Q(\mu, 0, 0)}, \qquad \text{and the complex conjugate.} \tag{13}$$

Let us take, by convention, that for $\mu > 0$, they correspond to four pure imaginary eigenvalues (see Fig. 1). This shows that $Q(\mu, 0, 0)$ is < 0 for $\mu > 0$. The generic situation is then when

$$q_1 > 0. \tag{14}$$

II.2. *Generic Supercritical or Subcritical Cases*

Because of the form (8) of A and B, let us now consider periodic solutions of (4) which correspond to steady solutions of (11), the phase ψ_0

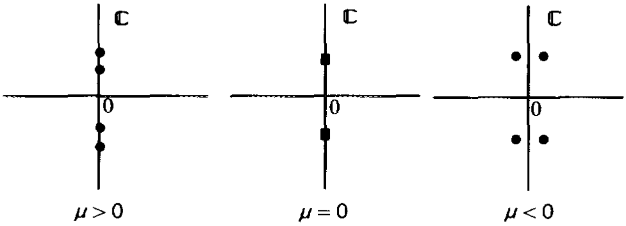

FIG. 1. Eigenvalues of the differential at the origin.

being affine in t. We observe that these solutions are given by double roots of the polynomial

$$f(u_0) = u_0[G(\mu, u_0, K) + H] - K^2 \tag{15}$$

The occurrence of double roots leads to a relationship between H and K. More precisely, we have

$$\begin{aligned} u_0[G(\mu, u_0, K) + H] - K^2 &= 0, \\ G(\mu, u_0, K) + H + u_0 Q(\mu, u_0, K) &= 0. \end{aligned} \tag{16}$$

Let us define α by

$$\alpha = P(\mu, u_0, K) + K/u_0, \tag{17}$$

in such a way that the frequency of a periodic solution is $\omega = \omega_0 + \alpha$.

From system (16), we deduce the new equation:

$$[\alpha - P(\mu, u_0, K)]^2 + Q(\mu, u_0, K) = 0. \tag{18}$$

We can then solve, by the implicit function theorem, with respect to (u_0, K), the system formed with (17) and (18) to obtain

$$\begin{aligned} u_0 &= \frac{q_1}{q_2}\mu - \frac{\alpha^2}{q_2} + \frac{2(p_1 q_2 + p_2 q_1) - q_1 q_3}{q_2^2}\alpha\mu + \frac{(q_3 - 2p_2)}{q_2^2}\alpha^3 + \cdots \\ &= \frac{q_1}{q_2}[\mu - \mu_c(\omega)]\left[1 + \frac{2(p_1 q_2 + p_2 q_1) - q_1 q_3}{q_1 q_2}\alpha + \cdots\right] \end{aligned} \tag{19}$$

$$K = \frac{q_1}{q_2}[\mu - \mu_c(\omega)]\left(\alpha - \frac{p_1 q_2 + p_2 q_1}{q_2}\mu + \cdots\right), \tag{20}$$

where $u_0 = 0$ if $\mu = \mu_c(\omega)$, with

$$\mu_c(\omega) = \frac{1}{q_1}\alpha^2 - \frac{2p_1}{q_1^2}\alpha^3 + O(\alpha^4), \qquad \omega = \omega_0 + \alpha. \tag{21}$$

In the (ω, μ) plane, the curve given by $\mu = \mu_c(\omega)$ is the "neutral curve" where bifurcation of periodic solutions takes place (see Fig. 2). We see in (19) that this bifurcation is *supercritical* $(\mu > \mu_c(\omega))$ if $q_2 > 0$, while it is *subcritical* $(\mu < \mu_c(\omega))$ if $q_2 < 0$. In addition, we observe that the nontrivial periodic solution for $K = 0$, corresponds, in the (ω, μ) plane, to a curve tangent to the following line crossing the "neutral curve" at the point $(\omega_0, 0)$:

$$\mu_0(\omega) = \frac{q_2}{p_1 q_2 + p_2 q_1} \alpha, \qquad \omega = \omega_0 + \alpha. \tag{22}$$

The relationship between H and K is obtained in a parametric form by using $(16)_2$ and (20):

$$H = \frac{q_1}{2q_2} [\mu - \mu_c(\omega)][q_1 \mu + 3\alpha^2 + \cdots]. \tag{23}$$

For a fixed μ, we can introduce scales and the following new parameters:

$$\begin{gathered} K = |\mu|^{3/2} \kappa, \qquad H = \mu^2 h, \qquad \alpha = |q_1 \mu|^{1/2} \beta, \\ h_m = \frac{q_1^2}{2q_2}, \qquad \kappa_E = \left(\frac{4q_1^3}{27 q_2^2} \right)^{1/2}. \end{gathered} \tag{24}$$

Then, the principal part of the set in the (h, κ) plane where periodic solutions of (4) occur is given by

$$\begin{gathered} h = h_m [1 - \operatorname{sgn}(\mu) \beta^2][1 + 3 \operatorname{sgn}(\mu) \beta^2], \\ \kappa = \frac{3\sqrt{3}}{2} \kappa_E [\operatorname{sgn}(\mu) - \beta^2] \beta, \end{gathered} \tag{25}$$

where $\operatorname{sgn}(\mu)$ means ± 1 depending on the sign of μ. If $q_2 > 0$, then $\beta \in [-1, 1]$, while if $q_2 < 0$ then $\beta^2 > \operatorname{sgn}(\mu)$.

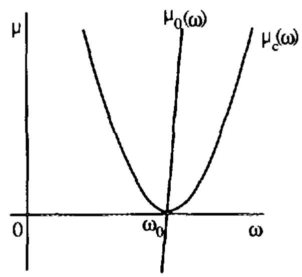

FIG. 2. Neutral curve and curve where $K = 0$.

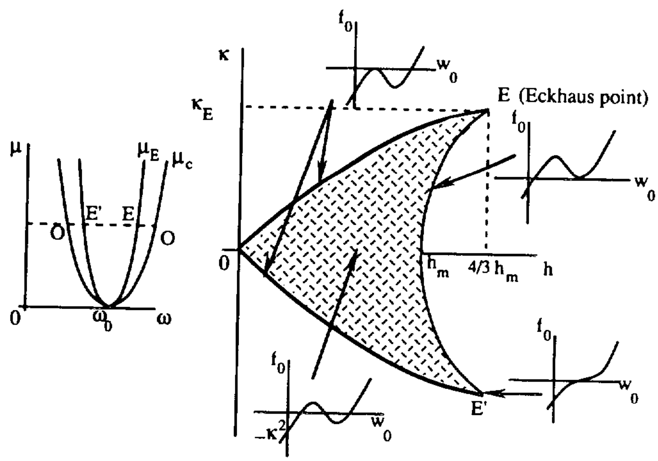

FIG. 3. Different shapes of f_0 for $\mu > 0$, $q_2 > 0$ (supercritical bifurcation).

We now introduce $f(|\mu|\, w_0) = |\mu|^3 f_0(w_0) + O(|\mu|^{7/2})$, where f_0 is the following cubic polynomial:

$$f_0(w_0) \equiv \frac{q_2}{2} w_0^3 - \mathrm{sgn}(\mu)\, q_1 w_0^2 + h w_0 - \kappa^2. \tag{26}$$

We represent in Figs. 3, 4, 5 the curves which form the set (25), and the form of the graph of f_0. We recall that double roots correspond to steady solutions of (11), and hence to periodic solutions of (4).

We notice that, in the case when $q_2 > 0$, there are two singularities E, E′

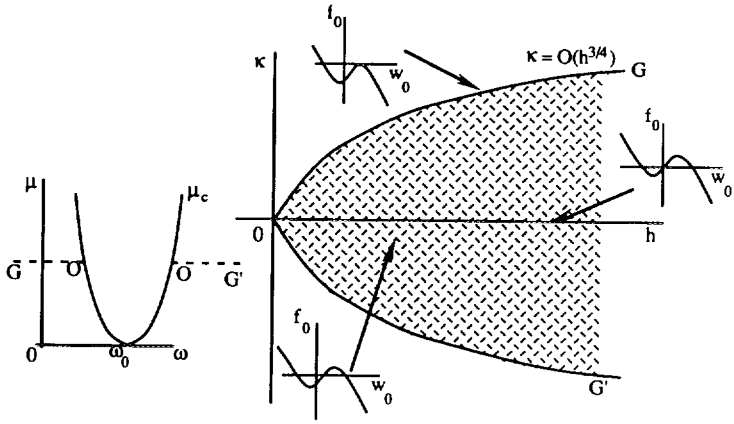

FIG. 4. Different shapes of f_0 for $\mu > 0$, $q_2 < 0$ (subcritical bifurcation).

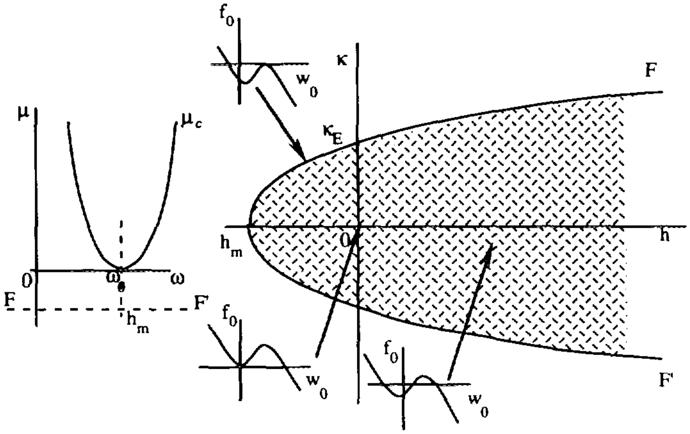

FIG. 5. Different shapes of f_0 for $\mu<0$, $q_2<0$ (subcritical bifurcation).

on the curve (25), corresponding to $\beta=\pm 1/\sqrt{3}$ ($\mu>0$). This corresponds in the (ω, μ) plane to a parabola $\mu_E=3\alpha^2/q_1$, which is the famous "Eckhaus instability curve." Equation (11) shows that on the arc EE' in Fig. 3, we have a *one parameter family of homoclinic solutions.* For the vector field (4), these orbits are homoclinic to the same periodic solution, but with a phase shift between $t=\pm\infty$. This phase shift is given by

$$2\delta_K=\int_{-\infty}^{+\infty}\left[P(\mu, u_0(t), K)+\frac{K}{u_0(t)}-\alpha\right]dt, \tag{27}$$

where α is the value of $P(\mu, u_0, K)+K/u_0$ for the double root u_0 of $f(u_0)=0$.

The hatched regions on Figs. 3, 4, 5 give periodic solutions of (11), i.e., quasi-periodic solutions of (4). We show in [4] that most of these solutions persist when one considers the full system (1). Since we are here interested in the homoclinic solutions, we see that in addition to the family obtained in the above case when $q_2>0$, we also have another homoclinic orbit for $q_2<0$ and $\mu<0$, for $H=K=0$. This last orbit is *homoclinic to the origin.*

Similarly, in the hamiltonian case (which is a special case of ours), the existence of periodic solutions for the normal form has been proved in [9] (see Figs. 4.18 and 4.19, p. 79).

II.3. *Geometrical Structure of the Homoclinic Orbits*

Supercritical Case

Let us observe that, for a fixed μ, the periodic solutions we found by $(10)_2$, (19), (20), (23), are such that $\cos(\psi_1-\psi_0)=0$. Let us denote by X

the 4 component vector $(A, B, \bar{A}, \bar{B})$ (which is real in the complex Jordan basis). We can write the periodic solutions under the form

$$\begin{aligned} X(t, \varphi_0) &= R_{\Omega_K t + \varphi_0} X_K \\ X_K &= (\rho_0(K), \operatorname{sgn}(K)\, i\rho_1(K), \rho_0(K), -\operatorname{sgn}(K)\, i\rho_1(K)), \end{aligned} \tag{28}$$

where R_φ is defined before (5) and where we observe that $\rho_1(0) = 0$. In (28) we defined $\Omega_K = \omega_0 + \alpha_K$, where we use, for convenience, parameterization by K on the set corresponding to the arc EE′ of Fig. 3. We notice that there are two values of the phase φ_0 such that the periodic orbit is *reversible*. For these orbits, this means that we have

$$SX(t, \varphi_0) = X(-t, \varphi_0). \tag{29}$$

Indeed, we observe that $SX_K = X_K$, and (29) with (5) leads to $R_{\varphi_0} X_K = R_{-\varphi_0} X_K$, which is true only for $\varphi_0 = 0$ or π.

Let us now consider the homoclinic orbits. They have the form

$$X(t, \varphi_0, \varphi_1) = R_{\Omega_K t + \varphi_0} \mathscr{H}_K(t + \varphi_1) \tag{30}$$

where φ_0 and φ_1 are arbitrary phases. For $K \neq 0$, $\mathscr{H}_K(t)$ may be written as

$$\begin{aligned} \mathscr{H}_K(t) = (&r_0(t, K)\, e^{i\psi_0(t, K)}, (\operatorname{sgn} K)\, ir_1(t, K)\, e^{i\psi_1(t, K)}, r_0(t, K)\, e^{-i\psi_0(t, K)}, \\ &-(\operatorname{sgn} K)\, ir_1(t, K)\, e^{-i\psi_1(t, K)}), \end{aligned} \tag{31}$$

while for $K = 0$, we have

$$\mathscr{H}_0(t) = (ir_0(t)\, e^{i\psi(t)}, ir_1(t)\, e^{i\psi(t)}, -ir_0(t)\, e^{-i\psi(t)}, -ir_1(t)\, e^{-i\psi(t)}), \tag{32}$$

where $\psi_0(\cdot, K)$, $\psi_1(\cdot, K)$, $\psi(\cdot)$, and $r_0(\cdot)$ are odd functions, while $r_0(\cdot, K)$, $r_1(\cdot, K)$, $r_1(\cdot)$ are even.

Moreover, when $t \to \pm\infty$, $[r_0(t, K), r_1(t, K), r_0(t), r_1(t), \psi_0(t, K), \psi_1(t, K), \psi(t)]$ tends exponentially towards $[\rho_0(K), \rho_1(K), \pm\rho_0(0), 0, \pm\delta_K, \pm\delta_K, \pm\delta]$. This shows that

$$\mathscr{H}_K(t) \to R_{\pm\delta_K} X_K \qquad \text{when} \quad t \to \pm\infty, \tag{33}$$

and this is valid also for $K = 0$, with δ and δ_0 defined by:

$$\delta_0 = \delta + \pi/2, \qquad 2\delta = \int_{-\infty}^{+\infty} [P(\mu, r_0^2(t), 0) - \alpha_0]\, dt. \tag{34}$$

We again observe that for $\varphi_1 = 0$, and $\varphi_0 = 0$ or π we have *reversible* homoclinic orbits.

If we denote by $H = H(K)$ the equation of the curve EE′ in Fig. 3, we may express in $\mathbb{R}^4$ the 3-dimensional manifold which contains all periodic and homoclinic orbits, which are of interest for us. This is given by the equation

$$B\bar{B} - G\left(\mu, A\bar{A}, \frac{i}{2}(A\bar{B} - \bar{A}B)\right) - H\left(\frac{i}{2}(A\bar{B} - \bar{A}B)\right) = 0, \tag{35}$$

and the solutions (28) and (30) satisfy (35), a fixed value of K giving the intersection of the manifold (35) with the manifold $(i/2)(A\bar{B} - \bar{A}B) = K$. This 2-dimensional intersection which is rotationally invariant (with a tubular form) then contains at the same time the periodic and the homoclinic orbits corresponding to K. Another interesting fact is that the *singularities of the manifold* (35) *are precisely located all along the 2-dimensional submanifold formed with the family of periodic orbits.* Indeed, we verify that the differential of (35) is 0 if and only if

$$\frac{\partial G}{\partial v} + H' = \pm 2\sqrt{-Q}, \qquad \frac{i}{2}(A\bar{B} - \bar{A}B) = \pm A\bar{A}\sqrt{-Q}, \qquad B\bar{B} = -A\bar{A}Q,$$

which means precisely that $A\bar{A} = u_0$ is a double root of $f(u_0) = 0$ (see (16)), i.e., corresponding to a periodic solution of (4).

Let us now rewrite system (4) under a more global form

$$\frac{dX}{dt} = F(\mu, X) \tag{36}$$

and the 4-dimensional vector field F satisfies:

$$R_\varphi F(\mu, X) = F(\mu, R_\varphi X), \qquad SF(\mu, X) = -F(\mu, SX). \tag{37}$$

If we denote by L the infinitesimal generator of the group R_φ, we then have $L\colon (A, B) \to (iA, iB)$. We now set

$$X = R_{\Omega_K t} Y, \tag{38}$$

so that Y satisfies the new system

$$\frac{dY}{dt} = F(\mu, Y) - \Omega_K LY. \tag{39}$$

Due to the forms (28) and (30) of periodic and homoclinic solutions, we know that there is a *circle of fixed points* for (39) given by $R_{\varphi 0} X_K$, and a *circle of heteroclinic orbits* $R_{\varphi 0}\mathscr{H}_K(\cdot)$ connecting two (in general) different fixed points of the above circle. The change of variable (38) allows us to look for eigenvalues of the linear operator

$$\mathscr{L}_k = D_X F(\mu, X_K) - \Omega_K L \tag{40}$$

instead of Floquet exponents of the linearized vector field near the periodic orbit. Notice, in addition, that the eigenvalues of operator $\mathscr{L}_k$ do not change if we replace X_K by $R_{\varphi_0} X_K$.

A first remark is that we already have a *double zero eigenvalue* (nonsemisimple). Indeed we easily verify, by differentiating the identity $F(\mu, R_{\varphi_0} X_K) - \Omega_K L R_{\varphi_0} X_K = 0$, with respect to φ_0 and K:

$$\mathscr{L}_k(LX_K) = 0, \tag{41}$$

$$\mathscr{L}_k\left(\frac{dX_K}{dK}\right) = \frac{d\Omega_K}{dK} LX_K. \tag{42}$$

We may notice, using (20), that $d\Omega_K/dK$ is nonzero, and that it goes to infinity near Eckhaus points E and E′ of Fig. 3. Now, for computing the two last eigenvalues, we do not intend to compute the 4×4 matrix of the linear operator (40). We can reach them by studying the solution given by the heteroclinic orbit, for t close to infinity. Indeed. let us define the second-order derivate of the function $f(u_0)$ at the double root as

$$f''[\rho_0^2(K)] = \tfrac{1}{2} \mu \lambda_K^2 \tag{43}$$

which is positive by construction (see Fig. 3). Then we have, by $(11)_1$, near $u_0 = \rho_0^2(K)$

$$\frac{dr_0}{dt} \approx \pm \mu^{1/2} \lambda_K [r_0 - \rho_0(K)], \tag{44}$$

where signs + or − correspond respectively to t near $-\infty$ or $+\infty$. This shows that the two remaining eigenvalues are equal to $\pm \mu^{1/2} \lambda_K$. Notice that the fact that these eigenvalues are opposite, is due to reversibility, since we verify that $\mathscr{L}_k S = -S \mathscr{L}_k$. Hence eigenvectors $\xi_\pm$ belonging to $\pm \mu^{1/2} \lambda_K$ are exchanged by symmetry S. In fact, for the reversible homoclinic solutions of (4) and (39), we have the behaviour (33) at infinity. The tangent for $t \to -\infty$, at the "starting" fixed point $R_{-\delta_K} X_K$ of (39), is $R_{-\delta_K} \xi_+$, while for $t \to +\infty$ the tangent at the "arrival" fixed point $R_{\delta_K} X_K$ is $R_{\delta_K} \xi_-$. Moreover, we again verify the property:

$$S R_{\delta_K} \xi_- = R_{-\delta_K} \xi_+. \tag{45}$$

Let us finally notice that on arc OE or OE′ of Fig. 3 (as well as on the boundaries of the hatched regions of Figs. 4 and 5) we have periodic solutions of a different type than those on arc EE′. Formula (43) gives a negative number, i.e., the two eigenvalues are opposite and pure imaginary, of order $|\mu|^{1/2}$. The periodic orbits are of elliptic structure instead of hyperbolic as on EE′.

Subcritical Case

Let us end this section by saying a few words on the subcritical case ($q_2 < 0$) for $\mu < 0$ (see Fig. 5). For periodic orbits the structure is the same as for the supercritical case on arc OE or OE′. For quasi-periodic orbits it is always the same study. But now, in the present case, there is a homoclinic orbit which takes the form

$$X(t, \varphi_0, \varphi_1) = R_{\Omega'_0 t + \varphi_0} \mathscr{H}'_0(t + \varphi_1), \tag{46}$$

where $\Omega'_0 = \omega_0 + P(\mu, 0, 0)$ and $\mathscr{H}'_0(t) = (r_0(t)\, e^{i\psi(t)}, r_1(t)\, e^{i\psi(t)}, r_0(t)\, e^{-i\psi(t)}, r_1(t)\, e^{-i\psi(t)})$, with r_0 even, and r_1, ψ odd functions of t. Moreover, as $t \to \pm\infty$ $(r_0(t), r_1(t), \psi(t)) \to (0, 0, \pm\delta)$ with

$$2\delta = \int_{-\infty}^{+\infty} [P(\mu, r_0^2(t), 0) - P(\mu, 0, 0)]\, dt,$$

and $\mathscr{H}'_0(0) = (r_0(0), 0, r_0(0), 0)$. Reversible orbits are obtained for $\varphi_1 = 0$, and $\varphi_0 = 0$ or π (as for (30)). Finally, the local study near the origin is already known: we see that these homoclinic orbits spiral out from the origin (t near $-\infty$) in the 2-dimensional invariant space belonging to the two eigenvalues $\pm i\Omega'_0 + \sqrt{Q(\mu, 0, 0)}$, while for t near $+\infty$, these orbits spiral towards the origin in the 2-dimensional invariant space (symmetric of the previous one) belonging to the two eigenvalues $\pm i\Omega'_0 - \sqrt{Q(\mu, 0, 0)}$.

III. Solutions Homoclinic to a Periodic Solution for a Reversible System

In this section we consider the general situation of a 4-dimensional reversible vector field obtained by perturbation of a vector field having orbits homoclinic to some periodic solution. In Section IV, we shall use a refinement of the proof of this section for our specific problem, where the unperturbed field is given by the normal form (4). More precisely, let us make the following assumptions: our system (supposed to be regular enough) has the form

$$\frac{dX}{dt} = F(X) + \mathscr{R}(X, \varepsilon), \qquad \mathscr{R}(X, 0) = 0, \tag{47}$$

and the 4-dimensional vector fields F and $\mathscr{R}$ satisfy

$$R_\varphi F(X) = F(R_\varphi X), \qquad SF(X) = -F(SX), \qquad S\mathscr{R}(X, \varepsilon) = -\mathscr{R}(SX, \varepsilon), \tag{48}$$

where S is the symmetry operator ($S^2 = \mathrm{Id}$) of reversibility, and R_φ denotes a representation of the circle rotation group action, and we denote by L its infinitesimal generator. Moreover

$$R_\varphi S = SR_{-\varphi} \qquad \text{and} \qquad SL = -LS. \tag{49}$$

We assume that, when ε is equal to 0, this system admits a two-parameter family of periodic solutions which one can write as

$$X(t, \varphi_0, K) = R_{\Omega_K t + \varphi_0} X_K, \qquad SX_K = X_K, \tag{50}$$

as well as a two-parameter family of homoclinic solutions:

$$\begin{aligned} \mathscr{H}(t, \varphi_0, K) &= R_{\Omega_K t + \varphi_0} \mathscr{H}_K(t), \qquad & \mathscr{H}_K(t) &\to R_{\delta_K} X_K \text{ as } t \text{ goes to } +\infty, \\ S\mathscr{H}_K(0) &= \mathscr{H}_K(0), \qquad & \mathscr{H}_K(t) &\to R_{-\delta_K} X_K \text{ as } t \text{ goes to } -\infty. \end{aligned} \tag{51}$$

One can check that both X_K and $\mathscr{H}_K(t)$ are solutions of:

$$\frac{dX}{dt} = F(X) - \Omega_K LX. \tag{52}$$

Among orbits (50) and (51) only four are *reversible* ($SX(-t) = X(t)$), namely:

> $R_{\Omega_K t} X_K$ and $R_{\Omega_K t + \pi} X_K$ for the periodic orbits, and $R_{\Omega_K t} \mathscr{H}_K(t)$ and $R_{\Omega_K t + \pi} \mathscr{H}_K(t)$ for the homoclinic ones.

III.1. *Persistence of Periodic Solutions*

We first wonder which conditions have to be satisfied for the entire system (when $\varepsilon \neq 0$) to admit periodic solutions. To achieve such a study, one needs information about the linearized system around the periodic orbit $R_{\Omega_K t} X_K$, which can be written:

$$\frac{dY}{dt} = \mathscr{A}_K(t)\, Y, \qquad \mathscr{A}_K(t) =: R_{\Omega_K t} DF(X_K)\, R_{-\Omega_K t} = DF(R_{\Omega_K t} X_K). \tag{53}$$

Then, in the same way we obtained (41), (42), we now have

$$\left(\frac{d}{dt} - \mathscr{A}_K(t)\right) R_{\Omega_K t} LX_K = 0, \qquad \left(\frac{d}{dt} - \mathscr{A}_K(t)\right) R_{\Omega_K t} \frac{dX_K}{dK} = -\frac{d\Omega_K}{dK} R_{\Omega_K t} LX_K,$$

i.e., $d/dt - \mathscr{A}_K(t)$ has nonsemisimple eigenvalue 0 in the space of $2\pi/\Omega_K$-periodic functions (0 is a Floquet exponent of $\mathscr{A}_K(t)$). We make the assumption that the other Floquet exponents of $\mathscr{A}_K(t)$ are $\pm\lambda$ with eigenvectors $R_{\Omega_K t}\xi_+$ and $R_{\Omega_K t}\xi_- = SR_{-\Omega_K t}\xi_+$.

This means that a system of independent solutions of (53) is

$$R_{\Omega_K t} LX_K, \qquad R_{\Omega_K t}\left(\frac{dX_K}{dK} + \frac{d\Omega_K}{dK} t \cdot LX_K\right),$$

$$e^{\lambda t} R_{\Omega_K t}\xi_+, \qquad e^{-\lambda t} R_{\Omega_K t}\xi_- ;$$

this is due to the fact that $Y(t)$ is a solution of (53) iff $Y(t) = R_{\Omega_K t}\tilde{Y}(t)$ with $\tilde{Y}(t)$ a solution of:

$$\frac{d\tilde{Y}}{dt} = \mathscr{L}_K \tilde{Y}.$$

This completes the study of the linear operator around the periodic orbit $R_{\Omega_K t} X_K$. We now apply these results to the research of reversible periodic solutions of (47). We need the following:[1]

PROPOSITION. *An orbit $X(t)$ of an autonomous reversible system is periodic and reversible iff there exist T_1 and $T_2 \neq T_1$ such that $SX(T_1) = X(T_1)$ and $SX(T_2) = X(T_2)$.*

Proof. It is clear that, if $X(t)$ is a reversible and T-periodic solution of the reversible system $dX/dt = F(X)$, we have $SX(0) = X(0)$ and $SX(T/2) = X(T/2)$.

Conversely, let $X(t)$ be a solution and suppose that there exist T_1 and T_2 such that: $SX(T_1) = X(T_1)$ and $SX(T_2) = X(T_2)$. Then, $\tilde{X}(t) =: SX(-t + 2T_1)$ is another solution of the same equation which satisfies: $\tilde{X}(T_1) = X(T_1)$. By uniqueness we have $\tilde{X}(t) \equiv X(t)$. It follows that, up to a translation on t, $X(t)$ is *reversible* and that, applying the same argument to T_2:

$$X(t) = SX(-t + 2T_1) = SX(-t + 2T_2).$$

Hence

$$X(t + 2(T_2 - T_1)) = SX(-t + 2T_1) = X(t);$$

i.e., *$X(t)$ is periodic and a period of $X(t)$ is $2|T_2 - T_1|$.*

So, in order to find reversible periodic solutions of (47) it is sufficient to

[1] We thank A. Vanderbauwhede for indicating this characterization to us.

look for two symmetric points which belong to the same orbit of (47). More precisely, let us consider X_K and $R_\pi X_K$, the two symmetric points of the orbit $R_{\Omega_K t} X_K$ and, to each of these points, let us associate the hyperplanes H_1 and H_2 defined as follows:

$$H_1 \text{ spanned by } \frac{dX_K}{dK},\ \xi_+ + \xi_-,\ \xi_+ - \xi_-, \text{ based at } X_K,$$

$$H_2 \text{ spanned by } R_\pi \frac{dX_K}{dK},\ R_\pi(\xi_+ + \xi_-),\ R_\pi(\xi_+ - \xi_-), \text{ based at } R_\pi X_K.$$

Then we can define coordinates (x, y, z) for $X_1 \in H_1$ by

$$X_1 = X_K + x\frac{dX_K}{dK} + y(\xi_+ + \xi -) + z(\xi_+ - \xi_-),$$

and in the same way, coordinates (x', y', z') for $X_2 \in H_2$.

Both H_1 and H_2 intersect transversally the orbit $R_{\Omega_K t} X_K$: $[\partial R_{\Omega_K t} X_K/\partial t|_{t=0} = \Omega_K L X_K$ and LX_K, dX_K/dK, ξ_+, ξ_- are linearly independent vectors. Hence, for the perturbed system (47), the first return map $\Pi\colon H_1 \to H_2$ maps the 2-dimensional affine subspace $\{(x, y, 0)/|x|$ and $|y|$ small enough$\}$ into a 2-dimensional surface which can be written:

$$\{(x', y', z')/x' = f(x, y, \varepsilon),\ y' = g(x, y, \varepsilon),\ z' = h(x, y, \varepsilon)\}.$$

So, the set of points near X_K, invariant under S, which is mapped into points near $R_\pi X_K$, invariant under S, is given by:

$$\{(x, y, 0)/h(x, y, \varepsilon) = 0\}.$$

From the properties of the linearized flow, and from the form of the independent solutions of (53), we find that for the linearized map when $\varepsilon = 0$:

$$D\Pi(\xi_+) = \exp\left(\frac{\lambda\pi}{\Omega_K}\right) R_\pi \xi_+, \qquad D\Pi(\xi_-) = \exp\left(-\frac{\lambda\pi}{\Omega_K}\right) R_\pi \xi_-.$$

Hence, we can deduce that:

$$\begin{aligned}
f(x, y, \varepsilon) &= x + O([|x| + |y|]^2 + |\varepsilon|),\\
g(x, y, \varepsilon) &= y\cosh(\lambda\pi/\Omega_K) + O([|x| + |y|]^2 + |\varepsilon|),\\
h(x, y, \varepsilon) &= y\sinh(\lambda\pi/\Omega_K) + O([|x| + |y|]^2 + |\varepsilon|).
\end{aligned}$$

Since, in the hyperbolic case, λ is not 0, this makes it possible to solve the equation $h(x, y, \varepsilon) = 0$, with respect to y as a function of (x, ε), by means of the implicit function theorem. Then, for any fixed K, there is a 1-parameter family of reversible periodic solutions, parametrized by x. Here K is arbitrary, which seems to give an additional free parameter and this is in contradiction with the result on the uniqueness of our x-family. In fact x corresponds roughly to a shift on K (see the definition). As a conclusion, we can say that:

PROPOSITION. *For each value of ε, close enough to 0, the system admits a continuous family of periodic reversible solutions close to the orbit $R_{\Omega_K t}X_K$, say $X_K(t, \varepsilon)$, where $X_K(t, 0) = R_{\Omega_K t}X_K$.*

Remark for Periodic Solutions of Elliptic Structure. We noticed in Section II.3 that there are periodic solutions of the normal form (4) which are such that, in addition to the double 0 Floquet exponent, we have two pure imaginary Floquet exponents $\pm i\lambda'$. The same analysis as above applies in choosing the basis vector $-i(\xi_+ - \xi\ \ )$ instead of $\xi_+ - \xi\ \ $. All calculations are the same, except that we now have to replace $\sinh(\lambda\pi/\Omega_K)$ by $-\sin(\lambda\pi/\Omega_K)$. This quantity is $\neq 0$ provided that λ is not a multiple of Ω_K/π; in this case the persistence result holds again.

III.2. *Persistence of Two Different Reversible Homoclinic Solutions*

We now look for the eventual persistence of homoclinic solutions when $\varepsilon \neq 0$. In fact we shall only deal with reversible homoclinic solutions. Let us consider the plane P_1 of vectors invariant under S. We know that the 2-dimensional stable manifold $W_s(K)$, of the periodic orbit $R_{\Omega_K t}X_K$, intersects P_1 at $\mathscr{H}_K(0)$ and we can see that this intersection is transverse since we have

$$W_s(K) = \{R_{\Omega_K t+\varphi}\mathscr{H}_K(t);\ t, \varphi \text{ real}\},$$

and the tangent space to $W_s(K)$ at $\mathscr{H}_K(0)$ is spanned by $L\mathscr{H}_K(0)$ and $(d\mathscr{H}_K/dt)(0)$.

Due to the reversibility of the orbit, it is easy to check that:

$$SL\mathscr{H}_K(0) = -L\mathscr{H}_K(0) \qquad \text{and} \qquad S\frac{d\mathscr{H}_K}{dt}(0) = -\frac{d\mathscr{H}_K}{dt}(0).$$

Hence, these antisymmetric vectors do not belong to P_1.

Now, let $W_s(K, \varepsilon)$ be the stable manifold of the perturbed periodic orbit $X_K(t, \varepsilon)$. Then $W_s(K, \varepsilon)$ intersects transversally P_1 at a unique point, say $\mathscr{H}_K(0, \varepsilon)$, which is symmetric under S and belongs to an orbit $\mathscr{H}_K(t, \varepsilon)$ in

$W_s(K, \varepsilon)$. Since the existence of one symmetric point on an orbit implies that this orbit is reversible, we have

$$\mathscr{H}_K(t, \varepsilon) = S\mathscr{H}_K(-t, \varepsilon),$$

and it follows that this orbit also belongs to the unstable manifold $W_u(K, \varepsilon) = SW_s(K, \varepsilon)$ of $X_K(t, \varepsilon)$. If we let $R_\pi X_K$ play the same role as X_K above we can state:

PROPOSITION. *For each periodic reversible solution of* (47), *there exist two reversible and homoclinic solutions. These homoclinic solutions are, in general different since the perturbed system is no longer invariant under the rotation* R_π. *At infinity, these solutions behave like* $R_{\Omega_K t \pm \delta_K} X_K$ *or* $R_\pi R_{\Omega_K t \pm \delta_K} X_K$.

Remark. We used a method ad hoc to look for reversible homoclinic orbits. Rather than P_1 we may consider the hyperplane F spanned by $L\mathscr{H}_K(0)$ and the vector space $\mathbf{P}_1$. Let us write $X \in F$ as

$$X = uL\mathscr{H}_K(0) + v, \qquad \text{where } v \text{ lies in } \mathbf{P}_1.$$

When $\varepsilon = 0$, $W_s \cap F$ is a curve tangent to the $L\mathscr{H}_K(0)$ axis which can be written as:

$$v = f_0(u), \qquad \text{satisfying } f_0(0) = 0, \qquad \frac{df_0}{du}(0), \qquad f_0(u) = f_0(-u).$$

Now, for the perturbed system, we obtain a perturbed intersection

$$W_s(K, \varepsilon) \cap F = \{(u, v)/v = f(u, \varepsilon)\}$$

and in the same way

$$W_u(K, \varepsilon) \cap F = SW_s(K, \varepsilon) \cap F = \{(u, v)/v = f(-u, \varepsilon)\}$$

where we have by definition $f(u, 0) \equiv f_0(u)$. So (u, v) belongs to $W_s(K, \varepsilon) \cap W_u(K, \varepsilon)$ iff:

$$v = f(u, \varepsilon) = f(-u, \varepsilon).$$

This system always has the solution $(u, v) = (0, f(0, \varepsilon))$ (we already know that $f_0(u) = f_0(-u)$) corresponding to the reversible homoclinic solutions previously found, but other solutions of the form

$$(u_0, f(u_0, \varepsilon)) \text{ and } (-u_0, f(u_0, \varepsilon)) \qquad \text{where } u_0 \text{ satisfies } f(u_0, \varepsilon) = f(-u_0, \varepsilon)$$

might also occur. Such solutions, when they exist, correspond to homoclinic orbits which are not reversible but they are mapped onto each other by S. The difficulty is then to compute f up to a sufficiently high order in (u, ε) (order 4).

IV. The Reversible 1:1 Resonance Vector Field Case

Our original problem is of the following form in $\mathbb{R}^4$ or $\mathbb{C}^2$,

$$\frac{dX}{dt} = F(\mu, X) + \mathscr{R}(\mu, X), \qquad \mathscr{R}(\mu, X) = O(\|X\|^{N+1}), \tag{54}$$

where F is the vector field (4) and $\mathscr{R}$ represents the higher order terms which are not in normal form. As all the orbits of the normal form we are interested in (periodic or homoclinic orbits) are of order $\sqrt{|\mu|}$, let us perform the rescaling

$$X = \sqrt{|\mu|}\,\hat{X}, \tag{55}$$

and define $\hat{F}$ and $\hat{\mathscr{R}}$ as follows:

$$\hat{F}(\mu, \hat{X}) = \frac{1}{\sqrt{|\mu|}} F(\mu, \sqrt{|\mu|}\,\hat{X}), \qquad \hat{\mathscr{R}}(\mu, \hat{X}) = \frac{1}{\sqrt{|\mu|}^{N}} \mathscr{R}(\mu, \sqrt{|\mu|}\,\hat{X}).$$

Now, by making $\varepsilon =: \sqrt{|\mu|}^{N-1}$ in the following new system

$$\frac{d\hat{X}}{dt} = \hat{F}(\mu, \hat{X}) + \varepsilon\hat{\mathscr{R}}(\mu, \hat{X}), \tag{56}$$

one recovers system (54). Moreover (56) satisfies the hypothesis required on system (47).

IV.1. *Periodic Orbits*

Let us look for a solution of (56) of the form

$$\hat{X}(t) = R_{\Omega_K t}(\hat{X}_K + \hat{Z}(t)) \qquad \text{where} \quad X_K =: \sqrt{|\mu|}\,\hat{X}_K.$$

Then $\hat{Z}$ is a solution of

$$\frac{d\hat{Z}}{dt} - \mathscr{L}_K \hat{Z} = \mathscr{N}(\hat{Z}, t, \mu) + \varepsilon\tilde{\mathscr{R}}(\hat{Z}, t, \mu), \tag{57}$$

where

$$\mathscr{N}(\hat{Z}, t, \mu) = \hat{F}(\mu, \hat{X}_K + \hat{Z}) - \hat{F}(\mu, \hat{X}_K) - D\hat{F}(\hat{X}_K)\,\hat{Z}$$
$$\tilde{\mathscr{R}}(\hat{Z}, t, \mu) = R_{-\Omega_K t}\hat{\mathscr{R}}(\mu, R_{\Omega_K t}(\hat{X}_K + \hat{Z})).$$

Notice that $\mathcal{N}$ involves terms coming from F of order at least $O(\sqrt{|\mu|}\,\|\hat{Z}\|^2)$ and that $\tilde{\mathcal{R}}$ involves terms coming from $\mathcal{R}$ of order at least $O(\sqrt{|\mu|})$. Under these conditions, we can apply the result of part III.1 to derive an equation similar to equation $h(x, y, \varepsilon) = 0$ which yields

$$y \sinh\left(\frac{\lambda\pi}{\Omega_K}\right) + O(\sqrt{|\mu|}\,(\|x\|^2 + \|y\|^2 + \varepsilon)) = 0,$$

$$\text{with} \quad \frac{\lambda\pi}{\Omega_K} = O(\sqrt{|\mu|}) \text{ since } \lambda = \mu^{1/2}\lambda_K$$

(see (44)). So, dividing by $\sqrt{|\mu|}$, we finally have to solve an equation which first order writes $y + O(\|x\|^2 + \|y\|^2 + \varepsilon) = 0$, This equation can be solved with respect to y by means of the implicit fuction theorem provided that ε is small enough. Now, coming back to the relationship between ε and μ, we obtain the persistence of the periodic solutions for $N \geqslant 2$.

Remark for Periodic Solutions of Elliptic Structure. In that case, corresponding to the curves OE, OE′, OG, OG′, FF′ of Figs. 3, 4, 5 the same estimates on λ, which is now purely imaginary, gives $|\mu|^{1/2}$ in factor in $\sin(\lambda\pi/\Omega_K)$ and the same proof holds.

Bibliographical Remark. In [10], the periodic solutions are found via Lyapounov–Schmidt reduction. Figures 1 and 2 of this paper are strictly included our Figs. 3 to 5, but notice that we use a different classification for our periodic orbits (hyperbolic and elliptic).

IV.2. *Homoclinic Orbits to Periodic Solutions in the Supercritical Case*

Let us now look for a solution of (56) of the form

$$\hat{X}(t) = R_{\Omega_K t + \varphi}(\hat{\mathcal{H}}_K(t) - R_{\delta_K} X_K + \hat{Z}(t)) + \hat{X}_K(t + \tau, \varepsilon, \mu),$$

where $\mathcal{H}_K =: \sqrt{|\mu|}\,\hat{\mathcal{H}}_K$, $\hat{X}_K(t, \varepsilon, \mu)$ is the periodic solution of (56) such that $\hat{X}_K(t, 0, \mu) = R_{\Omega_K t}\hat{X}_K$, and $\tau = (\delta_K + \varphi)/\Omega_K$.

Then $\hat{Z}$ is a solution of

$$\frac{d\hat{Z}}{dt} - \mathcal{L}_K(t)\,\hat{Z} = \mathcal{N}'(\hat{Z}, t, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathcal{R}}'(\hat{Z}, t, \varphi, \varepsilon, \mu), \tag{58}$$

where $\mathcal{L}_k(t) = DF(\mu, \mathcal{H}_K(t)) - \Omega_K L$ and we notice that the equation $d\hat{Z}/dt - \mathcal{L}_k(t)\,\hat{Z} = 0$ is the linearized equation around the homoclinic orbit. We have by construction

$$\mathcal{N}'(\hat{Z}, t, \varphi, \varepsilon, \mu) = \hat{F}[\mu, \hat{\mathcal{H}}_K - R_{\delta_K}\hat{X}_K + \hat{Z} + R_{(\Omega_K t + \varphi)}\hat{X}_K(t+\tau, \varepsilon, \mu)]$$
$$- \hat{F}(\mu, \hat{\mathcal{H}}_K) + \hat{F}(\mu, R_{\delta_K}\hat{X}_K)$$
$$- \hat{F}[\mu, R_{(\Omega_K t + \varphi)}\hat{X}_K(t+\tau, \varepsilon, \mu)] - D\hat{F}(\mu, \hat{\mathcal{H}}_K)\,\hat{Z},$$
$$\hat{\mathcal{R}}'(\hat{Z}, t, \varphi, \varepsilon, \mu) = R_{-(\Omega_K t + \varphi)}\hat{\mathcal{R}}[\mu, R_{\Omega_K t + \varphi}(\hat{\mathcal{H}}_K - R_{\delta_K}\hat{X}_K + \hat{Z})$$
$$+ \hat{X}_K(t+\tau, \varepsilon, \mu)] - R_{-(\Omega_K t + \varphi)}\hat{\mathcal{R}}[\mu, \hat{X}_K(t+\tau, \varepsilon, \mu)].$$

Notice that if $\hat{Z}$ is a solution of (58) going to 0 as t goes to ∞, the corresponding function $\hat{X}$ is a solution of (56) starting when $t=0$ near the point $R_\varphi \hat{\mathcal{H}}_K(0)$ and approaching the periodic orbit $\hat{X}_K(t, \varepsilon, \mu)$ as t goes to ∞. So, in order to study the stable manifold of the perturbed periodic orbit, we shall now concentrate our attention in looking for solutions of (58) decreasing exponentially as $t \to \infty$. Let $0 < \nu < \lambda$ and

$$E_\nu =: \{\hat{Z}: \mathbb{R}^+ \to \mathbb{R}^4 / \hat{Z} \text{ is continuous and } \operatorname*{Sup}_{t \in \mathbb{R}+} \|\hat{Z}\|\, e^{\nu t} < \infty\};$$

E_ν is a Banach space for the norm $\|\hat{Z}\|_\nu =: \operatorname{Sup}_{t \in \mathbb{R}+} \|\hat{Z}\|\, e^{\nu t}$. Let us first note that both $\mathcal{N}'$ and $\hat{\mathcal{R}}'$ map E_ν into itself. More precisely, notice that the difference $\hat{X}_K(t, \varepsilon, \mu) - R_{\Omega_K t}\hat{X}_K$ is of order at least ε, and that there exists a constant M such that for each $\hat{Z}$

$$\|\mathcal{N}'(\hat{Z}(t), t, \varphi, \varepsilon, \mu)\| < M\sqrt{|\mu|}\,(\varepsilon e^{-\lambda t} + \varepsilon\,\|\hat{Z}(t)\| + \|\hat{Z}(t)\|^2),$$

$$\|\hat{\mathcal{R}}'(\hat{Z}(t), t, \varphi, \varepsilon, \mu)\| < M\sqrt{|\mu|}\,(e^{-\lambda t} + \|\hat{Z}(t)\|),$$

$$\|\mathcal{N}'(\hat{Z}(t), t, \varphi, \varepsilon, \mu) - \mathcal{N}'(\hat{Z}(t), t, \varphi, \varepsilon, \mu)\|$$
$$< M\sqrt{|\mu|}\,(\varepsilon + \|\hat{Z}(t)\| + \|\hat{Z}'(t)\|)\|(\hat{Z} - \hat{Z}')(t)\|,$$

$$\|\hat{\mathcal{R}}'(\hat{Z}'(t), t, \varphi, \varepsilon, \mu) - \hat{\mathcal{R}}'(\hat{Z}(t), t, \varphi, \varepsilon, \mu)\|$$
$$< M\sqrt{|\mu|}\,\|(\hat{Z} - \hat{Z}')(t)\|,$$

where we assume that $\varepsilon \ll \sqrt{|\mu|}$.

As we found the solutions of the linearized equation around the periodic orbits, one can see that $\partial\mathcal{H}_K/\partial t$, $L\mathcal{H}_K$, $(\partial\mathcal{H}_K/\partial K) + t(d\Omega_K/dK)\,L\mathcal{H}_K$ are solutions of $d\hat{Z}/dt - \mathcal{L}_k(t)\,\hat{Z} = 0$. Let us denote $p(t) =: (\partial\mathcal{H}_K/\partial t)\,(t)$, $r(t) =: L\mathcal{H}_K(t)$, $s(t) =: (\partial\mathcal{H}_K/\partial K)(t)$, and let us choose a vector $q(0)$ such that $Sq(0) = q(0)$ and $\{p(0), q(0), r(0), s(0)\}$ is a basis of $\mathbb{R}^4$ and let us denote by $q(t)$ the solution of $d\hat{Z}/dt - \mathcal{L}_k(t)\,\hat{Z} = 0$ starting at $q(0)$ at time $t = 0$. Then, one can see (for example, looking at the wronskian of $\{p(t), q(t), r(t), s(t)\}$) that $q(t)$ is a reversible solution growing exponentially as

$e^{\lambda t}$ as t goes to $+\infty$, tangentially to the eigenspace belonging to the eigenvalue λ. Let us define $P(t)$, the projection onto the direction $p(t)$, along the directions $\{q(t), r(t), s(t)\}$, $Q(t) =: \mathrm{Id} - P(t)$, and let us denote by $X(t, s)$ the fundamental matrix of $d\hat{Z}/dt - \mathscr{L}_k(t)\hat{Z} = 0$.

One can see that if $\hat{Z}$ is such that $\int_t^{\infty} X(t,s)\, Q(s)(\mathscr{N}'(\hat{Z})(s), s, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathscr{R}}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu))\, ds$ converges for all $t \geq 0$ and $\hat{Z}$ satifies the functional equation (which characterizes the stable manifold)

$$\hat{Z}(t) = X(t,0)\, P(0)\, \hat{Z}(0) + \int_0^t X(t,s)\, P(s)$$
$$\times (\mathscr{N}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathscr{R}}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu))\, ds$$
$$- \int_t^{\infty} X(t,s)\, Q(s)(\mathscr{N}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathscr{R}}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu)\, ds, \tag{59}$$

then $\hat{Z}$ is solution of our problem (58). Moreover, if we denote by $\{\hat{p}(t), \hat{q}(t), \hat{r}(t), \hat{s}(t)\}$ the adjoint basis of $\{p(t), q(t), r(t), s(t)\}$, then $\{\hat{p}(t), \hat{q}(t), \hat{r}(t) - (d\Omega_K/dK)\, t\hat{s}(t), \hat{s}(t)\}$ is a family of solutions of the adjoint equation:

$$\frac{d\hat{Z}}{dt} + ({}^{t}DF(\mu, \mathscr{H}_K)(t) + \Omega_K L)\, \hat{Z} = 0 \qquad \text{(observe that } {}^{t}L = -L\text{)}.$$

It is easy to check that (59) writes

$$\hat{Z}(t) = \xi p(t) + \int_0^t [\mathscr{N}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathscr{R}}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) \mid \hat{p}(s)]\, ds\, p(t)$$
$$- \int_t^{\infty} [\mathscr{N}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathscr{R}}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) \mid \hat{q}(s)]\, ds\, q(t)$$
$$- \int_t^{\infty} \Big[\mathscr{N}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathscr{R}}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) \mid \hat{r}(s)$$
$$- \frac{d\Omega_K}{dK}(s - t)\, \hat{s}(s)\Big]\, ds\, r(t)$$
$$- \int_t^{\infty} [\mathscr{N}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) + \varepsilon\tilde{\mathscr{R}}'(\hat{Z}(s), s, \varphi, \varepsilon, \mu) \mid \hat{s}(s)]\, ds\, s(t). \tag{60}$$

where $[\cdot \mid \cdot]$ is the usual scalar product in $\mathbb{R}^4$, and $\xi = [\hat{Z}(0) \mid \hat{p}(0)] \in \mathbb{R}$. Let us denote by $\Theta_{\xi}(\hat{Z})(t)$ the right-hand side of (60).

Then $\Theta_\xi : E_\nu \to E_\nu$ is a continuous map and there exists a constant M' such that:

$$\|\Theta_\xi(\hat{Z})\|_\nu \leqslant |\xi| + M'\left[\|\hat{Z}\|_\nu^2 + \varepsilon\left(1 + \frac{1}{1-\nu/\lambda}\right)\right],$$

$$\|\Theta_\xi(\hat{Z}) - \Theta_\xi(\hat{Z}')\|_\nu \leqslant M'\left[\varepsilon\left(1 + \frac{1}{1-\nu/\lambda}\right) + \|\hat{Z}\|_\nu + \|\hat{Z}'\|_\nu\right]\|\hat{Z} - \hat{Z}'\|_\nu.$$

These estimates only use the facts that:

(i) all the functions $r(t)$, $s(t)$, $\hat{r}(t)$, $\hat{s}(t)$ are bounded;

(ii) $p(t)$, $\hat{q}(t)$ decrease as $e^{-\lambda t}$ as t goes to ∞; $q(t)$, $\hat{p}(t)$ increase as $e^{\lambda t}$ as t goes to ∞;

(iii) $te^{(\nu-\lambda)t} \leqslant 1/e(\lambda-\nu)$, $\int_t^\infty (s-t)\, e^{-\nu(s-t)}\, ds = 1/\nu^2$,

(iv) λ is order $\sqrt{|\mu|}$, $d\Omega_K/dK$ is of order $1/|\mu|$, $s(t)$ is of order $1/|\mu|$ and $r(t)$ is of order $\sqrt{|\mu|}$.

Let $E_d = \{\hat{Z} \in E_\nu / \|\hat{Z}\|_\nu \leqslant d\}$. A standard argument of existence of a fixed point for a contraction shows that, in choosing $\varepsilon = d/4M'(1-\nu/\lambda) \ll \sqrt{|\mu|}$, for each ε, μ, φ and ξ such that $|\xi| \leqslant d/2$, then Θ_ξ admits a unique fixed point in E_d which is a solution of (60).

Let us denote the set of symmetric points $P =: \{\hat{Z}/S\hat{Z} = \hat{Z}\}$ and let P' be the tangent space to the stable manifold of the periodic orbit $\{R_{\Omega_K t}\hat{X}_K\}$ at the point $\hat{\mathscr{H}}_K(0)$; then P intersects orthogonally P' at $\hat{\mathscr{H}}_K(0)$ (it is a consequence of the fact that S is a self-adjoint operator). It follows that there exists a point on the stable manifold of the periodic orbit $\hat{X}_K(t, \varepsilon, \mu)$ such that the tangent plane based at this point is nearly parallel to P' and intersects transversally P at a distance of order ε of $\hat{\mathscr{H}}_K(0)$ (provided that ν is not too close to λ). So, we can conclude that, provided that μ is small enough and $\varepsilon \ll \sqrt{|\mu|}$, the sable manifold of the orbit $\hat{X}_K(t, \varepsilon, \mu)$ intersects P. This proves, as in Section III.2, the persistence of two reversible homoclinic solutions for system (56).

Now coming back to the link between ε and μ, we have the persistence of these homoclinic orbits for $N \geqslant 3$ in (54) (it is then sufficient to take cubic terms for the normal form; the remaining terms are such that $N = 4$).

Remark. If we wish an exponential tendency of the perturbed orbit $\hat{X}_K(t)$ with an exponent ν very close to λ, and if we wish an estimate of the same order on d (the distance between this orbit and $R_{\Omega_K t+\delta}\hat{\mathscr{H}}_K(t)$), we need to increase the value of N in (54) to improve ε.

IV.3. *Homoclinic Orbit in the Subcritical Case*

We recall that in this situation ($q_2 < 0$ and $\mu < 0$), the origin is a hyperbolic fixed point of (54) and the corresponding eigenvalues of the operator $DF(\mu, 0) - \Omega'_0 L$ are $\pm\lambda'$ where $\lambda' = \sqrt{Q(\mu, 0, 0)}$ is of order $\sqrt{-\mu}$. Moreover, there exists a solution homoclinic to the origin of the form (46) for the normal form (36).

We are faced with the problem of the persistence of a homoclinic orbit to a hyperbolic fixed point for the perturbed vector field (56). Such a problem was studied by many authors (see [11], for example). In the frame of reversible vector fields, special cases were studied by Kirchgässner in [2, 12] and the case when the principal eigenvalues are real is treated in [13]. In the present case, we have two pairs of symmetric complex eigenvalues which leads to a different study.

Let us look for a solution of (56) of the form:

$$\hat{X}(t) = R_{\Omega'_0 t + \varphi}(\hat{\mathscr{H}}'_0(t) + \hat{Z}(t)), \qquad \text{where} \quad \mathscr{H}'_0 =: \sqrt{|\mu|}\, \hat{\mathscr{H}}'_0.$$

Then $\hat{Z}$ is a solution of

$$\frac{d\hat{Z}}{dt} - \mathscr{L}'_0(t)\hat{Z} = \mathscr{N}''(\hat{Z}, t, \mu) + \varepsilon\tilde{\mathscr{R}}''(\hat{Z}, t, \varphi, \mu), \tag{61}$$

where $\mathscr{L}'_0(t) = DF(\mu, \mathscr{H}'_0(t)) - \Omega'_0 L$ (notice that the equation $d\hat{Z}/dt - \mathscr{L}'_0(t)\hat{Z} = 0$ is the linearized equation around the homoclinic orbit), and where we have defined:

$$\mathscr{N}''(\hat{Z}, t, \mu) = \hat{F}(\mu, \hat{\mathscr{H}}'_0(t) + \hat{Z}) - \hat{F}(\mu, \hat{\mathscr{H}}'_0(t)) - D\hat{F}(\mu, \hat{\mathscr{H}}'_0(t))\hat{Z},$$

$$\mathscr{R}''(\hat{Z}, t, \varphi, \mu) = R_{-(\Omega'_0 t + \varphi)}\hat{\mathscr{R}}(\mu, R_{\Omega'_0 t + \varphi}(\hat{\mathscr{H}}'_0(t) + \hat{Z})).$$

If $\hat{Z}$ is a small, bounded solution of (61), as we are in a hyperbolic situation, $\hat{Z}$ in fact tends to 0 as t goes to ∞, and the corresponding $\hat{X}$ is an orbit starting, at $t = 0$, near the point $R_\varphi \hat{\mathscr{H}}'_0(0)$, and spiraling towards the origin as $|t|$ goes to ∞. So, let

$$E =: \{\hat{Z} : \mathbb{R}^+ \to \mathbb{R}^4 / \hat{Z} \text{ is continuous and } \sup_{t \in \mathbb{R}_+} \|\hat{Z}\| < \infty\};$$

E is a Banach space for the norm $\|\hat{Z}\|_{\infty} =: \operatorname{Sup}_{t \in \mathbb{R}_+} \|\hat{Z}\|$. In order to show that (61) admits small solutions in E we need some information about the linearized equation around the homoclinic orbit; more precisely, we need the following:

PROPOSITION. *The equation*

$$\frac{d\hat{Z}}{dt} - \mathscr{L}'_0(t)\hat{Z} = 0 \tag{62}$$

admits an exponential dichotomy on $\mathbb{R}^+$; *i.e., there exists a continuous family of projection matrices* $P'(t)$ *and* $K > 0$ *such that, if we denote by* $X'(t, s)$ *the fundamental matrix of* (62):

$$\begin{aligned} &X'(t, s)\, P'(s) = P(t)\, X'(t, s), \\ &\quad \|X'(t, s)\, P'(s)\| \leqslant K e^{\lambda'(t-s)} \qquad \text{for} \quad 0 \leqslant s \leqslant t, \\ &\quad \|X'(t, s)(\mathrm{Id} - P'(s))\| \leqslant K e^{\lambda'(s-t)} \qquad \text{for} \quad 0 \leqslant t \leqslant s. \end{aligned}$$

For a proof of the existence of such a solution of (62) see [14]. In fact, $P'(t)$ appears as the unique solution of

$$\frac{dP'}{dt} = \mathscr{L}'_0(t)\, P' - P'\mathscr{L}'_0(t), \tag{63}$$

such that when t goes to ∞, $P'(t)$ tends towards the matrix P'_∞ of the projection onto the eigenspace of the operator $DF(\mu, 0) - \Omega'_0 L$ belonging to the double eigenvalues $-\lambda'$, along the eigenspace belonging to the double eigenvalues λ'. Now, one can check that if $\hat{Z}$ is such that

$$\int_t^\infty X'(t, s)(\mathrm{Id} - P'(s))[\mathscr{N}''(\hat{Z}(s), s, \mu) + \varepsilon\tilde{\mathscr{R}}''(\hat{Z}(s), s, \varphi, \mu)]\, ds$$

converges for each t, and $\hat{Z}$ satisfies the functional equation

$$\begin{aligned} \hat{Z}(t) = X'(t, 0)\, P'(0)\, \eta &+ \int_0^t X'(t, s)\, P'(s) \\ &\times [\mathscr{N}''(\hat{Z}(s), s, \mu) + \varepsilon\tilde{\mathscr{R}}''(\hat{Z}(s), s, \varphi, \mu)]\, ds \\ &- \int_t^\infty X'(t, s)(\mathrm{Id} - P'(s)) \\ &\times [\mathscr{N}''(\hat{Z}(s), s, \mu) + \varepsilon\tilde{\mathscr{R}}''(\hat{Z}(s), s, \varphi, \mu)]\, ds, \end{aligned} \tag{64}$$

for some η in $\mathbb{R}^4$, then $\hat{Z}$ is a solution of (61). Let us denote by $\Theta'_\eta(t)$ the right-hand side of (64), then Θ'_η maps $E \to E$, and notice that $\mathscr{N}''$ is of order at least $\sqrt{-\mu}\|\hat{Z}\|_{\infty}^2$, $\mathscr{R}''$ is of order at least $\sqrt{-\mu}$, and λ is of order $\sqrt{-\mu}$; there exists a constant K' such that:

$$\|\Theta'_\eta \hat{Z}\|_{\infty} \leqslant K\|\eta\| + K'(\|\hat{Z}\|_{\infty}^2 + \varepsilon),$$
$$\|\Theta'_\eta \hat{Z} - \Theta'_\eta \hat{Z}'\|_{\infty} \leqslant K'(\varepsilon + \|\hat{Z}\|_{\infty} + \|\hat{Z}'\|_{\infty})\|\hat{Z} - \hat{Z}'\|_{\infty}.$$

So, in choosing $\varepsilon = d/4K'$, for each ε, μ, and η such that $\|\eta\| \leqslant d/2$, Θ'_η admits a unique fixed point in the ball of radius d of E. When in addition we have $\varepsilon \ll \sqrt{|\mu|}$, it is easy to show, as in Section IV.2, that this solution tends towards 0 as $t \to \infty$ (hyperbolicity of the origin is conserved). Now, observing that we have two reversible homoclinic orbits for the normal form (see (46)), we can conclude as in Section IV.2 that we have the *persistence of two reversible orbits homoclinic to the origin* for $N \geqslant 3$.

Remark. In the subcritical case, we do not consider the question of the existence of a one-parameter family of reversible periodic orbits with a large period, and tending to the homoclinic orbit as the period goes to infinity. The existence of such a family is clear on the normal form (see Fig. 5, h, κ near 0). It remains to prove a persistence result for such orbits. We thank the referee for indicating to us that this is a general result proved in a Devaney paper [15] (see also a recent work of Fiedler and Vanderbauwhede).

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