Existence of orthogonal domain walls in Bénard-Rayleigh convection

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Abstract

In Bénard-Rayleigh convection we consider the pattern defect in orthogonal domain walls connecting a set of convective rolls with another set of rolls orthogonal to the first set. This is understood as an heteroclinic orbit of a reversible system where the x - coordinate plays the role of time. This appears as a perturbation of the heteroclinic orbit proved to exist in a reduced 6-dimensional system studied by a variational method in [2], and analytically in [8]. We then prove the existence of a one-parameter family of heteroclinic connections between orthogonal sets of rolls, completing the result of [2].

Key words: Reversible dynamical systems, Bifurcations, Heteroclinic connection, Domain walls in convection

1 Introduction

The Bénard-Rayleigh convection problem is a classical problem in fluid mechanics. It concerns the flow of a three-dimensional viscous fluid layer situated between two horizontal parallel plates and heated from below. Upon increasing the difference of temperature between the two plates, the simple conduction state looses stability at a critical value of the temperature difference corresponding to a critical value $\mathcal{R}_c$ of the Rayleigh number. Beyond the instability threshold, a convective regime develops in which patterns are formed, such as convective rolls, hexagons, or squares [9]. Observed patterns are often accompanied by defects as for instance domain walls which occur between rolls with different orientations. We refer to the works [1, 10, 11], and the references therein, for experimental and analytical results, and detailed descriptions of these patterns and defects.

Mathematically, the governing equations are the Navier-Stokes equations coupled with an equation for the temperature, and completed by boundary conditions at the two plates. Observed patterns are then found as particular steady solutions of these equations. In [4] and [5] Haragus and Iooss handled the full governing equations and proved, for various boundary conditions, the existence of symmetric domain walls in convection (however not yet observed experimentally).

The existence of orthogonal domain walls (effectively observed experimentally) has been studied formally by Manneville and Pomeau in [11], and more recently by Buffoni et al [2], where the authors handle the full governing equations, showing that the study leads to a small perturbation of the reduced system of amplitude equations in $\mathbb{R}^6$, the same system as the one predicted in [11]:

$$\begin{aligned} A^{(4)} &= A(1 - A^2 - gB^2) \\ B'' &= \varepsilon^2 B(-1 + gA^2 + B^2). \end{aligned} \tag{1}$$

By a variational argument Boris Buffoni et al [2] prove the existence of an heteroclinic orbit, for any $g > 1$, and ε small enough, such that

$$\begin{aligned} A_*(x), B_*(x) &> 0, \\ (A_*(x), B_*(x)) &\to \begin{cases} M_- = (1,0) \text{ as } x \to -\infty \\ M_+ = (0,1) \text{ as } x \to +\infty \end{cases}. \end{aligned}$$

This orbit is expected to represent the connection between a set of convecting rolls parallel to the x direction, with a set of orthogonal rolls. Unfortunately, this type of elegant proof does not allow to prove the persistence of such heteroclinic curve under reversible perturbations of the vector field. Our purpose here is to use the analytic results of [8] for proving its persistence applied to orthogonal domain walls in Bénard-Rayleigh convection. It should be noticed that even though the present analysis looks similar to the one made in [4] and [5], it really needs serious adaptation since, here we loose the symmetry of the wall defect, which plays an important role in [4] and [5].

Starting from a formulation of the steady governing equations as an infinite-dimensional dynamical system in which the horizontal coordinate x plays the role of evolutionary variable (spatial dynamics), and looking for solutions periodic in y, a center manifold reduction is performed, which leads to a 12-dimensional reduced reversible dynamical system, reducing to 8-dimensional ($\mathbb{R}^4 \times \mathbb{C}^2$), after restricting to solutions with reflection symmetry $y \to -y$. A normal form up to cubic order for this reduced system is obtained in [2], which, after some calculations and rescaling (see Appendix A.1) becomes

$$\begin{aligned} A_0^{(4)} &= k_- A_0'' + A_0\Big(1 - \frac{k_-^2}{4} - A_0^2 - g|B_0|^2\Big) + \widehat{f}, \\ B_0'' &= \varepsilon^2 B_0(-1 + gA_0^2 + |B_0|^2) + \widehat{g}, \end{aligned} \tag{2}$$

with additional "cubic" terms of the form

$$\begin{aligned} \widehat{f}_0 = {} & id_1\varepsilon A_0(B_0\overline{B_0}' - \overline{B_0}B_0') + \varepsilon^2[\sigma_0 k_- A_0^3 + d_3 A_0'' + d_4 A_0^2 A_0'' + d_2 A_0 A_0'^2 + d_6 A_0 |B_0'|^2 \\ & + d_7 A_0'(B_0\overline{B_0}' + \overline{B_0}B_0') + d_5 A_0''|B_0|^2] + id_8\varepsilon^3 A_0''(B_0\overline{B_0}' - \overline{B_0}B_0') + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{3}$$

$$
\begin{aligned}
\widehat{g_0} &= \varepsilon^3[ic_0B_0' + ic_1B_0'|A_0|^2 + ic_2B_0'|B_0|^2 + ic_3B_0^2\overline{B_0}' + ic_9B_0A_0A_0'] \\
&+\varepsilon^4[c_4B_0'(B_0\overline{B_0}' - \overline{B_0}B_0') + c_5B_0A_0A_0'' + c_6B_0A_0'^2 + c_7B_0'A_0A_0'] \\
&+\varepsilon^5[ic_8B_0A_0A_0''' + ic_7B_0'A_0A_0'' + ic_{10}B_0'A_0'^2 + ic_{11}B_0A_0'A_0'' + \mathcal{O}(\varepsilon^6),
\end{aligned} \tag{4}
$$

where parameters are defined as (see Appendix A.1)

$$
\begin{aligned}
\varepsilon^4 &\sim \mathcal{R}^{1/2} - \mathcal{R}_c^{1/2},\ \mathcal{R} \text{ Rayleigh number}, \\
& \quad k_c(1+\varepsilon^2 k_-) \text{ wave number in } y \text{ direction},
\end{aligned}
$$

coefficients c_j, d_j are real and

$$
\begin{aligned}
\widehat{f} &= \widehat{f_0} + \widehat{f_1} \\
\widehat{g} &= \widehat{g_0} + \widehat{g_1},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{f_1} &= \varepsilon^4\mathcal{O}[|X|(|X|^2 + |Y|^2 + \varepsilon^4)^2], \\
\widehat{g_1} &= \varepsilon^6\mathcal{O}[(|X|^2 + |Y|)(|X|^2 + |Y|^2 + \varepsilon^4)^2],
\end{aligned}
$$

with

$$
\begin{aligned}
X &= (A_0, A_0', A_0'', A_0''')^t \in \mathbb{R}^4, \\
Y &= (B_0, B_0')^t \in \mathbb{C}^2.
\end{aligned}
$$

Moreover the system (2) commutes with the reversibility symmetry S_1 :

$$
(A_0, A_0', A_0'', A_0''', B_0, B_0') \mapsto (A_0, -A_0', A_0'', -A_0''', \overline{B_0}, -\overline{B_0'}),
$$

and

$$
\begin{aligned}
&A_0^{(4)} \text{ is odd in } X, \\
&B_0'' \text{ is even in } X,
\end{aligned}
$$

which results from the equivariance of the original system under the shift by half of a wave length in the y direction (fixing the symmetry $y \mapsto -y$). The estimates for $\widehat{f_1}$ and $\widehat{g_1}$ result from the property of the normal form which does not contain terms of degree 4 in (X, Y), and from the inequality

$$
(a+b)^4 \leq 4(a^2+b^2)^2 \text{ for } a, b \in \mathbb{R}.
$$

Remark 1 *Notice that the above reduction is valid for the three classical boundary conditions for the Bénard-Rayleigh convection problem: rigid-rigid, free-free, free-rigid. However in the case of rigid-rigid or free-free boundary conditions,* $Y = 0$ *is an invariant subspace, which simplifies the estimate for* $\widehat{g_1}$.

Remark 2 *Notice that the system (2) becomes just the system (1) for* $k_- = \widehat{f} = \widehat{g} = 0$, *and* B_0 *real.*

Remark 3 *Notice also that the high order terms of size* $\mathcal{O}(\varepsilon^4)$ *for* $A_0^{(4)}$ *and* $\mathcal{O}(\varepsilon^6)$ *for* B_0'' *are functions of* $e^{\pm i\frac{x}{2\varepsilon}}$. *This is due to the fact that* $B_0 e^{i\frac{x}{2\varepsilon}}$ *is the original amplitude of the* Y *mode (see (60)).*

Let us give here the results obtained in [8] for the system (1):

Theorem 4 *Let us choose* $0 < \delta_0 < 1/3$, *then for* $\delta_0 \leq \delta \leq 1$, η_0 *such that for* ε *small enough with* $\alpha = \varepsilon^{1/3}$, *where* $0 < \alpha = [(1+\delta^2)\eta_0^2 - 1]^{1/2}$, *the 3-dim unstable manifold of* M_- *intersects transversally the 3-dim stable manifold of* M_+, *except for a finite umber of values of* δ. *The connecting curve* $(A_*, B_*)(x)$ *which is obtained is locally unique (it is the only curve for this intersection). Moreover its dependency in parameters* (ε, δ) *is analytic. In addition we have* $B_*(x)$ *and* $B_*'(x) > 0$ *on* $(-\infty, +\infty)$, *the principal part of* $B_*(x)$ *being given*

i) for $x \in (-\infty, 0]$, *by*

$$\begin{aligned} B_*(x) &= \frac{1}{(1+\frac{\delta^2}{2})^{1/2}\cosh(x_0 - \varepsilon\delta x)}, \\ \cosh x_0 &= \frac{1}{B_{00}(1+\frac{\delta^2}{2})^{1/2}}, \\ B_{00} &= B_*(0) = (1-\eta_0^2\delta^2)^{1/2}, \end{aligned}$$

ii) for $x \in [0, +\infty)$, *by*

$$B_*(x) = \frac{\tanh(\varepsilon x/\sqrt{2}) + B_{00}}{1 + B_{00}\tanh(\varepsilon x/\sqrt{2})}.$$

For $x \to -\infty$ *we have* $(A_* - 1, A_*', A_*'', A_*''', B_*, B_*') \to 0$ *at least as* $e^{\varepsilon\delta x}$, *while for* $x \to +\infty$, $(A_*, A_*', A_*'', A_*''') \to 0$ *at least as* $e^{-\sqrt{\frac{\delta}{2}}x}$, *and* $(B_* - 1, B_*') \to 0$ *at least as* $e^{-\sqrt{2}\varepsilon x}$.

Moreover, choosing $\delta_* < \delta$ we have the following estimates

Corollary 5 *For* $x \in (-\infty, 0]$ *there exists* $c > 0$ *independent of* ε *such that for the heteroclinic curve*

$$\begin{aligned} |A_*(x) - 1| &\leq ce^{2\varepsilon\delta_* x}, \\ |A_*'(x)| + |A_*''(x)| + |A_*'''(x)| &\leq c\sqrt{\varepsilon}e^{2\varepsilon\delta_* x}, \\ 0 &< B_*(x) \leq ce^{\varepsilon\delta_* x}, \\ 0 &< B_*'(x) \leq c\varepsilon e^{\varepsilon\delta_* x}. \end{aligned}$$

Corollary 6 *For* $x \in [0, +\infty)$ *there exists* $c > 0$ *independent of* ε *such that for the heteroclinic curve*

$$\begin{aligned} |A_*^{(m)}(x)| &\leq c\sqrt{\varepsilon}e^{-\frac{\delta_*^{1/2}x}{\sqrt{2}}}, \ m = 0,1,2,3, \\ |B_*(x) - 1| &\leq ce^{-\sqrt{2}\varepsilon x}, \ |B_*'(x)| \leq c\varepsilon e^{-\sqrt{2}\varepsilon x}. \end{aligned}$$

We intend to prove the following for the system (2):

Proposition 7 *Except for a finite number of values of $g = 1 + \delta^2$ and for ε such that Theorem 4 applies, the heteroclinic solution connecting an equilibrium at $-\infty$ (representing convective rolls parallel to x - axis) and a periodic solution at $+\infty$ (representing convective rolls orthogonal to the previous ones), persists as a one-parameter family of orthogonal domain walls, (see results in (56), (57)). The wave numbers of limiting periodic convective rolls are linked by a one parameter family of relationships depending on ε (the amplitude of rolls being of order ε^2) (see Remarks 17 and 18).*

Remark 8 *The wave numbers of the sets of rolls at $-\infty$ and at $+\infty$ differ in general. This is a major difference with the symmetric case (of non orthogonal walls) treated in [4] and [5].*

Remark 9 *In the result above, the size of the perturbed wave numbers depend on bounds proved for some coefficients, specially the term $a_6\varepsilon^{5/2}$ in the last bifurcation equation (55). Notice that all such coefficients depend on the Prandtl number.*

Remark 10 *Values of δ such that $0.476 \leq \delta$ include values obtained for δ in the Bénard-Rayleigh convection problem where g is function of the Prandtl number $\mathcal{P}$ (see [4]). With rigid-rigid, rigid-free, or free-free boundaries the minimum values of g are respectively ($g_{\min} = 1.227$, 1.332, 1.423) corresponding to $\delta_{\min} = 0.476$, 0.576, 0.650. The restriction in Theorem 4 corresponds to $1 < g \leq 2$. The eligible values for the Prandtl number are respectively $\mathcal{P} > 0.5308, > 0.6222, > 0.8078$.*

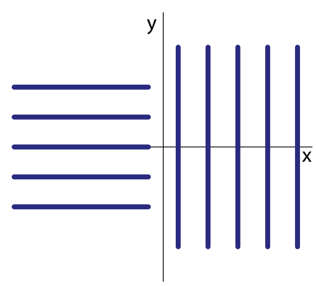

Figure 1: Orthogonal domain wall

2 Setting of the perturbed system

Since we leave now some freedom to the wave numbers, as well in the y direction, as in the x direction, the "end points" of the expected heteroclinic are no longer $(1, 0)$ at $-\infty$, and the circle $(0, e^{i\phi})$ at $+\infty$. In fact the classical study of steady convective rolls, shows that these should be respectively

$(A_0^{(-\infty)}(k_-), B_0^{(-\infty)}(k_-))$ and $(0, B_0^{(+\infty)}(\omega, x))$ (see [3] section 4.3.3, or [4] sections 2 and 6.2, and Appendix A.2, A.3). We have indeed

$$\begin{aligned}
(A_0^{(-\infty)})^2 &= 1 - \frac{k_-^2}{4} + \sigma_0\varepsilon^2 k_- + \mathcal{O}(\varepsilon^2|k_-|^3 + \varepsilon^4), \\
1 - (A_0^{(-\infty)}) \overset{def}{=} -\frac{\widetilde{\omega}_-^2}{2},\ \widetilde{\omega}_-^2 &= \frac{k_-^2}{4} - \sigma_0\varepsilon^2 k_- + \mathcal{O}[k_-^4 + \varepsilon^2|k_-|^3 + \varepsilon^4], \\
B_0^{(-\infty)} &= \mathcal{O}(\varepsilon^6),
\end{aligned}$$

$$\begin{aligned}
e^{i\frac{x}{2\varepsilon}} B_0^{(+\infty)}(\omega, x) &= r_0 e^{i\omega x} + \mathcal{O}(\varepsilon^6),\ A_0^{(+\infty)} = 0, \\
\omega \overset{def}{=} \frac{1}{2\varepsilon} + \varepsilon\widetilde{\omega}_+ &= \frac{1 + \varepsilon^2 k_+}{2\varepsilon} + \mathcal{O}(\varepsilon^7),
\end{aligned}$$

$$B_0^{(+\infty)} e^{-i\varepsilon\widetilde{\omega}_+ x} = C_0^{(+\infty)} + iD_0^{(+\infty)}$$

$$\begin{aligned}
r_0^2 &= 1 - \frac{k_+^2}{4} + \mathcal{O}(\varepsilon^2|k_+| + \varepsilon^4) = 1 - \mathcal{O}[(|\widetilde{\omega}_+| + \varepsilon^2)^2], \\
C_0^{(+\infty)} &= r_0 + \mathcal{O}(\varepsilon^6),\ \text{oscil. part}(C_0^{(+\infty)}) = \mathcal{O}(\varepsilon^6), \\
D_0^{(+\infty)} &= \mathcal{O}(\varepsilon^6).
\end{aligned}$$

Remark 11 *The coefficient σ_0 introduced in the expression of $(A_0^{(-\infty)})^2$ depends on the Prandtl number.*

Remark 12 *We may notice that in case the system has the symmetry S_0 representing $z \mapsto 1 - z$ (OK for rigid-rigid, or free-free boundary conditions), then $B_0^{(-\infty)} = 0$, which simplifies computations.*

Let us set

$$B_0 e^{-i\varepsilon\widetilde{\omega}_+ x} = C_0 + iD_0,$$

then (2) becomes

$$A_0^{(4)} = k_- A_0'' + A_0[1 - \frac{k_-^2}{4} - A_0^2 - g(C_0^2 + D_0^2)] + f \tag{5}$$

$$\begin{aligned}
C_0'' &= 2\varepsilon\widetilde{\omega}_+ D_0' + \varepsilon^2 C_0(-1 + \widetilde{\omega}_+^2 + gA_0^2 + C_0^2 + D_0^2) + g_r \\
D_0'' &= -2\varepsilon\widetilde{\omega}_+ C_0' + \varepsilon^2 D_0(-1 + \widetilde{\omega}_+^2 + gA_0^2 + C_0^2 + D_0^2) + g_i
\end{aligned} \tag{6}$$

with

$$f = \widehat{f},\quad g_r + ig_i = \widehat{g}e^{-i\varepsilon\widetilde{\omega}_+ x},$$

and where the exponential factor disappears in the cubic part when we replace B_0 by $(C_0 + iD_0)e^{i\varepsilon\widetilde{\omega}_+ x}$. Let us define

$$\begin{aligned}
f &= f_0(\varepsilon, k_-, X, Y, \overline{Y}) + f_1(\omega x, \varepsilon, k_-, X, Y, \overline{Y}) \\
g_r &= g_{r0}(\varepsilon, X, Y, \overline{Y}) + g_{r1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y}) \\
g_i &= g_{i0}(\varepsilon, X, Y, \overline{Y}) + g_{i1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y}),
\end{aligned}$$

where f_0, g_{r0}, g_{i0} come only from cubic terms of the normal form in (2), and where f_1, g_{r1}, g_{i1} are $2\pi-$periodic in ωx, smooth in their arguments, and satisfy estimates

$$\begin{aligned}
|f_1(\omega x, \varepsilon, k_-, X, Y, \overline{Y})| &\leq c\varepsilon^4 |X|(|X|^2 + |Y|^2)^2 \\
|g_{r1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y})| + |g_{i1}(\omega x, \varepsilon, k_-, X, Y, \overline{Y})| &\leq c\varepsilon^6 (|X|^2 + |Y|)(|X|^2 + |Y|^2)^2,
\end{aligned}$$

with

$$\begin{aligned}
X &= (A_0, A_0', A_0'', A_0''') \\
Y &= (C_0 + iD_0, C_0' + iD_0').
\end{aligned}$$

Then we have from (3), (4):

$$\begin{aligned}
f_0 = {} & d_1\varepsilon A_0(C_0D_0' - D_0C_0') + \sigma_0\varepsilon^2 k_- A_0^3 + d_2\varepsilon^2 A_0 A_0'^2 + d_3\varepsilon^2 A_0'' \\
& + d_4\varepsilon^2 A_0^2 A_0'' + d_5\varepsilon^2 A_0''(C_0^2 + D_0^2) + d_6\varepsilon^2 A_0(C_0'^2 + D_0'^2) + \\
& + d_7\varepsilon^2 A_0'(C_0C_0' + D_0D_0') + d_8\varepsilon^3 A_0''(C_0D_0' - D_0C_0') + \mathcal{O}(\varepsilon^4),
\end{aligned} \tag{7}$$

$$\begin{aligned}
g_{r0} + ig_{i0} = {} & i\varepsilon^3(C_0' + iD_0')[c_0 + c_1A_0^2 + c_2(C_0^2 + D_0^2)] \\
& + \varepsilon^3 c_3(C_0 + iD_0)(C_0D_0' - D_0C_0') + i\varepsilon^3 c_9(C_0 + iD_0)A_0A_0' \\
& + \varepsilon^4 c_4(C_0' + iD_0')(C_0D_0' - D_0C_0') + c_5\varepsilon^4 A_0A_0''(C_0 + iD_0) \\
& + \varepsilon^4[c_6A_0'^2(C_0 + iD_0) + c_7A_0A_0'(C_0' + iD_0')] \\
& + i\varepsilon^5(C_0' + iD_0')(c_7A_0A_0'' + c_{10}A_0'^2) \\
& + i\varepsilon^5(C_0 + iD_0)(c_8A_0A_0''' + c_{11}A_0'A_0'') + \mathcal{O}(\varepsilon^6).
\end{aligned} \tag{8}$$

Now, let us set

$$\begin{aligned}
A_0 &= A_* + \widetilde{A_0} \\
C_0 &= B_* + \widetilde{C_0} \\
D_0 &= \widetilde{D_0}
\end{aligned}$$

where we observe that we expect

$$\begin{aligned}
\widetilde{A_0} \underset{x=-\infty}{\rightarrow} A_0^{(-\infty)} - 1 &= -\frac{\widetilde{\omega}_-^2}{2}, \\
C_0 + iD_0 \underset{x=-\infty}{\rightarrow} C_0^{(-\infty)} &= B_0^{(-\infty)} = \mathcal{O}(\varepsilon^6), \\
\widetilde{C_0} + i\widetilde{D_0} \underset{x=+\infty}{\rightarrow} C_0^{(+\infty)} + iD_0^{(+\infty)} - 1 &\sim -\frac{(\widetilde{\omega}_+ + \mathcal{O}(\varepsilon^2))^2}{2}.
\end{aligned}$$

Then (5,6) becomes the "perturbed system"

$$\mathcal{M}_g(\widetilde{A_0}, \widetilde{C_0}) = \begin{pmatrix} -k_-(A_*'' + \widetilde{A_0}'') + \frac{k_-^2}{4}(A_* + \widetilde{A_0}) + \widetilde{\phi_0} \\ \frac{2\widetilde{\omega}_+}{\varepsilon}\widetilde{D_0}' + \widetilde{\omega}_+^2(B_* + \widetilde{C_0}) + \widetilde{\psi_{0r}} \end{pmatrix}, \tag{9}$$

$$\mathcal{L}_g\widetilde{D_0} = -\frac{2\widetilde{\omega}_+}{\varepsilon}(B_*' + \widetilde{C_0}') + \widetilde{\omega}_+^2\widetilde{D_0} + \widetilde{\psi_{0i}}, \tag{10}$$

where linear operators $\mathcal{M}_g$ and $\mathcal{L}_g$ are defined as

$$\mathcal{M}_g\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A \end{pmatrix}, \tag{11}$$

$$\mathcal{L}_gD = \frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D, \tag{12}$$

and where $\widetilde{\phi_0}, \widetilde{\psi_{0r}}, \widetilde{\psi_{0i}}$ are smooth functions of $(\omega x, \varepsilon, k_-, \widetilde{\omega}_+, \widetilde{X}, \widetilde{Y})$ where

$$\begin{aligned} \widetilde{X} &= (\widetilde{A_0}, \widetilde{A_0}', \widetilde{A_0}'', \widetilde{A_0}''') \\ \widetilde{Y} &= (\widetilde{C_0}, \widetilde{D_0}, \widetilde{C_0}', \widetilde{D_0}') \end{aligned}$$

$$\begin{aligned} \widetilde{\phi_0} &= \widetilde{\phi_{00}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) + \widetilde{\phi_{01}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) \\ \widetilde{\psi_{0r}} &= \widetilde{\psi_{0r0}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) + \widetilde{\psi_{0r1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) \\ \widetilde{\psi_{0i}} &= \widetilde{\psi_{0i0}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) + \widetilde{\psi_{0i1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) \end{aligned}$$

$$\begin{aligned} |\widetilde{\phi_{01}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y})| &\leq c\varepsilon^4 \\ |\widetilde{\psi_{0r1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y})| + |\widetilde{\psi_{0i1}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y})| &\leq c\varepsilon^4. \end{aligned}$$

More precisely, we have

$$\begin{aligned} \widetilde{\phi_{00}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) &= 3A_*\widetilde{A_0}^2 + \widetilde{A_0}^3 + 2gB_*\widetilde{A_0}\widetilde{C_0} \\ &\quad + g(A_* + \widetilde{A_0})(\widetilde{C_0}^2 + \widetilde{D_0}^2) + f_{00}, \end{aligned} \tag{13}$$

$$\begin{aligned} \widetilde{\psi_{0r0}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) &= 2gA_*\widetilde{A_0}\widetilde{C_0} + gB_*\widetilde{A_0}^2 + 2B_*\widetilde{C_0}^2 + g\widetilde{A_0}^2\widetilde{C_0} \\ &\quad + (B_* + \widetilde{C_0})(\widetilde{C_0}^2 + \widetilde{D_0}^2) + g_{00r}, \end{aligned} \tag{14}$$

$$\begin{aligned} \widetilde{\psi_{0i0}}(\varepsilon, k_-, \widetilde{X}, \widetilde{Y}) &= 2gA_*\widetilde{A_0}\widetilde{D_0} + 2B_*\widetilde{C_0}\widetilde{D_0} + g\widetilde{A_0}^2\widetilde{D_0} \\ &\quad + \widetilde{D_0}(\widetilde{C_0}^2 + \widetilde{D_0}^2) + g_{00i}, \end{aligned} \tag{15}$$

and in using Theorem 4, Corollaries 5 and 6,

$$
\begin{aligned}
f_{00} &= \sigma_0\varepsilon^2 k_- A_*^3 + \mathcal{O}[\varepsilon^{5/2}(e^{\varepsilon\delta x}\chi_{(-\infty,0)} + e^{-\frac{\delta' x}{\sqrt{2}}}\chi_{(0,\infty)}) + \varepsilon^2(|\widetilde{X}| + |\widetilde{Y}|) \\
&\quad + \varepsilon|\widetilde{D_0}'|(\chi_{(-\infty,0)} + e^{-\frac{\delta' x}{\sqrt{2}}}\chi_{(0,\infty)})], \\
g_{00r} &= \mathcal{O}[\varepsilon^{5/2}(e^{\varepsilon\delta x}\chi_{(-\infty,0)} + e^{-\frac{\delta' x}{\sqrt{2}}}\chi_{(0,\infty)}) + \varepsilon^2(|\widetilde{X}| + |\widetilde{Y}|) \\
&\quad + \varepsilon(|\widetilde{C_0}'| + |\widetilde{D_0}'|)], \\
g_{00i} &= \mathcal{O}[\varepsilon^{3/2}(e^{\varepsilon\delta x}\chi_{(-\infty,0)} + e^{-\frac{\delta' x}{\sqrt{2}}}\chi_{(0,\infty)}) + \varepsilon|\widetilde{X}| \\
&\quad + \varepsilon(|\widetilde{C_0}'| + |\widetilde{D_0}'| + \varepsilon|\widetilde{D_0}|)].
\end{aligned}
$$

where f_{00} and $g_{00r} + ig_{00i}$ are smooth functions which come from the rest of the cubic normal form written in (7,8)) and $\chi_{(-\infty,0)}$ and $\chi_{(0,\infty)}$ are the characteristic functions on the corresponding intervals.

Remark 13 *We notice that the estimates for the main terms independent of* $\widetilde{X}, \widetilde{Y}$ *come from*

$$
\begin{aligned}
\text{for } f_{00} &: \quad \sigma_0\varepsilon^2 k_- A_*^3 + d_2\varepsilon^2 A_* A_*'^2 + d_3\varepsilon^2 A_*'' + d_4\varepsilon^2 A_*^2 A_*'' + d_5\varepsilon^2 A_*'' B_*^2, \\
\text{for } g_{00r} &: \quad c_5\varepsilon^2 A_* A_*'' B_* + c_6\varepsilon^2 A_*'^2 B_* + c_7 A_* A_*' B_*' \\
\text{for } g_{00i} &: \quad \varepsilon B_*'(c_0 + c_1 A_*^2 + c_2 B_*^2) + \varepsilon c_9 B_* A_* A_*'.
\end{aligned}
$$

Moreover, notice that, below, we need to compute $\int f_{00} A_*' dx$, $\int g_{00r} B_*' dx$, $\int g_{00i} B_* dx$, *which, for terms independent of* $\widetilde{X}, \widetilde{Y}$ *leads to*

$$
\begin{aligned}
\text{for } \int f_{00} A_*' dx &= -\frac{\sigma_0\varepsilon^2 k_-}{4} + \varepsilon^2 \int (d_2 A_* A_*'^3 + d_4 A_*^2 A_*' A_*'') dx + \mathcal{O}(\varepsilon^3) \\
&= -\frac{\sigma_0\varepsilon^2 k_-}{4} + \mathcal{O}(\varepsilon^{5/2}), \\
\text{for } \int g_{00r} B_*' dx &\sim \varepsilon^2 \int_{\mathbb{R}} c_5 A_* A_*'' B_* B_*' dx \\
&= -\varepsilon^2 c_5 \int_{\mathbb{R}} [A_*'^2 B_* B_*' + A_* A_*' (B_* B_*')'] dx = \mathcal{O}(\varepsilon^3), \\
\text{for } \int g_{00i} B_* dx &= \varepsilon(\frac{c_0}{2} + \frac{c_2}{4}) + \varepsilon(c_1 - c_9) \int_{\mathbb{R}} A_*^2 B_* B_*' = \mathcal{O}(\varepsilon),
\end{aligned}
$$

where we notice

$$
\begin{aligned}
\int A_*'' A_*' dx &= 0, \ \varepsilon^2 \int A_*'' B_*^2 A_*' dx = -\varepsilon^2 \int A_*'^2 B_* B_*' dx = \mathcal{O}(\varepsilon^3), \\
\int (d_2 A_* A_*'^3 + d_4 A_*^2 A_*' A_*'') dx &= (d_2 - d_4) \int A_* A_*'^3 dx = \mathcal{O}(\varepsilon^{1/2}),
\end{aligned}
$$

and taking care of the convergence in $e^{\varepsilon\delta x}$ *at* $-\infty$, *which implies a division by* ε *in the integral on* $(-\infty, 0)$.

Before solving the system we need to change variables so that the variables and the right hand side of (9,10) tend towards 0 at infinity. Let us denote

$$\begin{aligned}
\widetilde{X}^{(-\infty)} &= (A_0^{(-\infty)}-1,0,0,0) = (\mathcal{O}(\widetilde{\omega}_-^2),0,0,0)\\
\widetilde{Y}^{(-\infty)} &= (C_0^{(-\infty)},0,0,0) = (\mathcal{O}(\varepsilon^6),0,0,0),\\
\widetilde{X}^{(+\infty)} &= 0\\
\widetilde{Y}^{(+\infty)} &= (C_0^{(+\infty)}-1,D_0^{(+\infty)},C_0^{(+\infty)\prime},D_0^{(+\infty)\prime}) = [\mathcal{O}((\widetilde{\omega}_+ +\varepsilon^2)^2),\mathcal{O}(\varepsilon^6),\mathcal{O}(\varepsilon^5),\mathcal{O}(\varepsilon^5)],
\end{aligned}$$

then, taking care, in (5,6), of the forms of f, g_r, g_i, we notice that the limit terms in the right hand side of (9,10) as $x \to -\infty$ are

$$\begin{aligned}
&\frac{k_-^2}{4}A_0^{(-\infty)} + \widetilde{\phi_0}(\omega x,\varepsilon,k_-,\widetilde{X}^{(-\infty)},\widetilde{Y}^{(-\infty)}) \text{ exp limit as } e^{\varepsilon\delta x} \text{ (see } f_{00}),\\
&\widetilde{\omega}_+^2 C_0^{(-\infty)} + \widetilde{\psi_{0r}}(\omega x,\varepsilon,k_-,\widetilde{X}^{(-\infty)},\widetilde{Y}^{(-\infty)}) \text{ exp limit as } e^{\varepsilon\delta x} \text{ (see } g_{00r})\\
&\widetilde{\psi_{0i}}(\omega x,\varepsilon,k_-,\widetilde{X}^{(-\infty)},\widetilde{Y}^{(-\infty)}) \text{ exp limit as } e^{\varepsilon\delta x} \text{ (as } B_*' \text{ and see } g_{00i}).
\end{aligned}$$

The limit terms of the right hand side of (9,10) as $x \to +\infty$ is

$$\begin{aligned}
&0 \text{ exp limit as } e^{-\frac{\sqrt{\delta}}{\sqrt{2}}x} \text{ (as } A_*)\\
&\frac{2\widetilde{\omega}_+}{\varepsilon}(D_0^{(+\infty)})' + \widetilde{\omega}_+^2 C_0^{(+\infty)} + \widetilde{\psi_{0r}}(\omega x,\varepsilon,k_-,0,\widetilde{Y}^{(+\infty)}) \text{ exp limit as } e^{-\sqrt{\frac{\delta}{2}}x} \text{ (see } g_{00r}),\\
&-\frac{2\widetilde{\omega}_+}{\varepsilon}(C_0^{(+\infty)})' + \widetilde{\omega}_+^2 D_0^{(+\infty)} + \widetilde{\psi_{0i}}(\omega x,\varepsilon,k_-,0,\widetilde{Y}^{(+\infty)}) \text{ exp limit as } e^{-\varepsilon\sqrt{2}x} \text{ (see } g_{00i}).
\end{aligned}$$

Let us change variables as

$$\begin{aligned}
\widetilde{A_0} &= \alpha_-\chi_- + \widehat{A_0}\\
\widetilde{C_0} &= \beta_-\chi_- + \beta_+\chi_+ + \widehat{C_0},\\
\widetilde{D_0} &= \gamma_+\chi_+ + \widehat{D_0},
\end{aligned}$$

with (in using Appendix A.2 and (66) in Appendix A.3)

$$\begin{aligned}
\alpha_- &= (A_0^{(-\infty)}-1) = -\widetilde{\omega}_-^2/2,\ \beta_- = B_0^{(-\infty)},\\
\beta_+ &= (C_0^{(+\infty)}(\omega x)-1),\ \gamma_+ = D_0^{(+\infty)}(\omega x),
\end{aligned}$$

$$\text{const part of } \beta_+ \overset{def}{=} \beta_+^{(c)} = -\frac{\widetilde{\omega}_+^2}{2} + \frac{\sigma_1\varepsilon^2\widetilde{\omega}_+}{2} + \frac{\sigma_2\varepsilon^4}{2} + \mathcal{O}[(|\widetilde{\omega}_+| + \varepsilon^2)^4], \quad (16)$$

and where χ_- and χ_+ are smooth functions, such that

$$\begin{aligned}
\chi_- &= 1 \text{ for } x \in (-\infty,-1),\\
&= 0 \text{ for } x > 0\\
0 &< \chi_- < 1 \text{ for } x \in (-1,0),
\end{aligned}$$

$$
\begin{aligned}
\chi_+ &= 1 \text{ for } x \in (1,\infty),\\
&= 0 \text{ for } x < 0\\
0 &< \chi_+ < 1 \text{ for } x \in (0,1),
\end{aligned}
$$

such that

$$(\widehat{A_0}, \widehat{C_0}, \widehat{D_0}) \to 0 \text{ as } |x| \to \infty.$$

2.1 Properties of linear operators $\mathcal{M}_g$ and $\mathcal{L}_g$

Let us define the Hilbert spaces

$$L^2_\eta = \{u; u(x)e^{\eta|x|} \in L^2(\mathbb{R})\},$$

$$
\begin{aligned}
\mathcal{D}_0 &= \{(A,C) \in H^4_\eta \times H^2_\eta; A \in H^4_\eta, C \in \mathcal{D}_1\}\\
\mathcal{D}_1 &= \{C \in H^2_\eta; \varepsilon^{-2}||C''||_{L^2_\eta} + \varepsilon^{-1}||C'||_{L^2_\eta} + ||C||_{L^2_\eta} \overset{def}{=} ||C||_{\mathcal{D}_1} < \infty\}
\end{aligned}
$$

equiped with natural scalar products. Then we have the following result (proved in [8]):

Lemma 14 *Except maybe for a set of isolated values of g, the kernel of $\mathcal{M}_g$ in L^2_η is one dimensional, spanned by (A'_*, B'_*), and its range has codimension 1, L^2- orthogonal to (A'_*, B'_*). $\mathcal{M}_g$ has a pseudo-inverse acting from L^2_η to $\mathcal{D}_0$ for any $\eta > 0$ small enough, with bound independent of ε.*

The operator $\mathcal{L}_g$ has a trivial kernel, and its range which has codimension 1, is L^2- orthogonal to B_* ($B_* \notin L^2$). $\mathcal{L}_g$ has a pseudo-inverse acting respectively from L^2_η to $\mathcal{D}_1$ for $\eta > 0$ small enough, with bound independent of ε.

3 Calculation and estimates for $\mathcal{M}_g(\widehat{A_0}, \widehat{C_0})$ and $\mathcal{L}_g\widehat{D_0}$

3.1 First component of $\mathcal{M}_g(\widehat{A_0}, \widehat{C_0})$

The first component is now the sum of small terms linear in $(\widehat{A_0}, \widehat{C_0})$ plus quadratic terms and terms independent of $(\widehat{A_0}, \widehat{C_0})$ which tend exponentially to 0 as $e^{\varepsilon\delta x}$ for $x \to -\infty$ and $e^{-\sqrt{2}\varepsilon x}$ for $x \to +\infty$:

$$\mathcal{M}_g(\widehat{A_0}, \widehat{C_0})|_1 = -k_-\widehat{A_0}'' + \frac{k_-^2}{4}\widehat{A_0} + \widehat{\phi_0} + \varphi_1(k_-) \tag{17}$$

with

$$
\begin{aligned}
\varphi_1(k_-) &= -k_-(A''_* + \alpha_-\chi''_-) + \frac{k_-^2}{4}(A_* - \chi_-) + \alpha_-\chi_-^{(4)} \\
&\quad -3(1-A_*^2)\alpha_-\chi_- + gB_*^2\alpha_-\chi_- + 2gA_*B_*(\beta_-\chi_- + \beta_+\chi_+),\\
\widehat{\phi_0} &= \widetilde{\phi_0}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) - \chi_-\widetilde{\phi_0}(\omega x, \varepsilon, k_-, \widetilde{X}^{(-\infty)}, \widetilde{Y}^{(-\infty)}).
\end{aligned}
\tag{18}
$$

More precisely we have, from (13)

$$\begin{aligned}
\widehat{\phi_0} &= 3[\alpha_-^2(A_*\chi_-^2-\chi_-)+2\alpha_-A_*\chi_-\widehat{A_0}+A_*\widehat{A_0}^2]+\alpha_-^3(\chi_-^3-\chi_-) \\
&\quad +3\alpha_-^2\chi_-^2\widehat{A_0}+3\alpha_-\chi_-\widehat{A_0}^2+\widehat{A_0}^3+2gB_*[\alpha_-\chi_-\widehat{C_0}+(\beta_-\chi_-+\beta_+\chi_+)\widehat{A_0}+\widehat{A_0}\widehat{C_0}] \\
&\quad +g(A_*+\alpha_-\chi_-+\widehat{A_0})[(\beta_-\chi_-+\beta_+\chi_++\widehat{C_0})^2+(\gamma_+\chi_++\widehat{D_0})^2] \\
&\quad -\chi_-g(1+\alpha_-)\beta_-^2+\widehat{f_{00}}, \\
\widehat{f_{00}} &= \sigma_0\varepsilon^2k_-(A_*^3-\chi_-)+\mathcal{O}[\varepsilon^{5/2}(e^{\varepsilon\delta x}\chi_{(-\infty,0)}+e^{-\frac{\delta' x}{\sqrt{2}}}\chi_{(0,\infty)})+\varepsilon(|\widehat{D_0}'|+\varepsilon|\widehat{D_0}|)+\varepsilon^2|\widehat{X}|].
\end{aligned} \tag{19}$$

We notice that for $\eta=\varepsilon\delta/2$, and due to Corollary 6,

$$\begin{aligned}
\frac{1}{\varepsilon^2}\beta_+' &= \mathcal{O}(\varepsilon^3),\ \frac{1}{\varepsilon^2}\gamma_+'=\mathcal{O}(\varepsilon^3), \\
||A_*'||_{L_\eta^2} &= \mathcal{O}(1),\ ||B_*'||_{L_\eta^2}=\mathcal{O}(\varepsilon^{1/2}), \\
||A_*'^2||_{L_\eta^2} &= \mathcal{O}(\varepsilon^{1/2}),\ ||B_*'^2||_{L_\eta^2}=\mathcal{O}(\varepsilon^{3/2}), \\
||A_*''||_{L_\eta^2} &= \mathcal{O}(1),\ ||B_*''||_{L_\eta^2}=\mathcal{O}(\varepsilon^{3/2}).
\end{aligned}$$

Then we have the estimates (we use extensively $2|ab|\leq a^2+b^2$)

$$\begin{aligned}
||\varphi_1(k_-)||_{L_\eta^2} &\leq c\left(\sqrt{\varepsilon}|k_-|+\frac{k_-^2+\varepsilon^4}{\sqrt{\varepsilon}}+\widetilde{\omega}_+^2+\varepsilon^2|\widetilde{\omega}_+|)\right), \qquad (20) \\
\int_{\mathbb{R}}\varphi_1(k_-)A_*'dx &= \mathcal{O}[(|k_-|+|\widetilde{\omega}_+|+\varepsilon^2)^2],
\end{aligned}$$

where we use $\eta=\varepsilon\delta/2$ ($\eta<\varepsilon\delta$ is necessary), integration by parts and

$$\begin{aligned}
\int_{\mathbb{R}}A_*'A_*''dx &= 0, \\
\int_{\mathbb{R}}(A_*-\chi_-)A_*'dx &= \mathcal{O}(1) \\
\int_{\mathbb{R}}(1-A_*^2)A_*'\chi_-dx &= \mathcal{O}(1).
\end{aligned}$$

Let us use the following little Lemma (adapted from a simple Sobolev inequality)

Lemma 15 *For any $u\in H_\eta^1$ and ε sufficiently small, we have*

$$|u(x)|\leq c(||u||_{L_\eta^2}+\frac{1}{\varepsilon}||u'||_{L_\eta^2})$$

where c is independent of ε.

Then in the next estimates we use extensively

$$\begin{aligned}
|\widehat{A_0}^{(m)}(x)| &\leq \frac{c}{\varepsilon}||\widehat{A_0}||_{H_\eta^4},\ m=0,1,2,3 \\
|\widehat{C_0}^{(m)}(x)| &\leq c\varepsilon^m||\widehat{C_0}||_{\mathcal{D}_1},\ m=0,1.
\end{aligned}$$

We obtain, for sufficiently small $\varepsilon, k_-, \widetilde{\omega}_+, \widehat{A_0}, \widehat{C_0}, \widehat{D_0}$ in $\mathbb{R}^3 \times \mathcal{D}_0 \times \mathcal{D}_1$

$$||\widehat{\phi_0}||_{L^2_\eta} \le c\left(\varepsilon^2 + \varepsilon^{3/2}|k_-| + \frac{\widetilde{\omega}_-^4}{\sqrt{\varepsilon}} + \widetilde{\omega}_+^4 + \frac{1}{\varepsilon}||(\widehat{A_0}, \widehat{C_0})||^2_{\mathcal{D}_0} + \frac{1}{\varepsilon^2}||(\widehat{A_0}||^3_{H^4_\eta} + ||\widehat{D_0}||^2_{\mathcal{D}_1}\right). \tag{21}$$

3.2 Second component of $\mathcal{M}_g(\widehat{A_0}, \widehat{C_0})$

For the second component we have

$$\mathcal{M}_g(\widehat{A_0}, \widehat{C_0})|_2 = \frac{2\widetilde{\omega}_+}{\varepsilon}\widehat{D_0}' + \widetilde{\omega}_+^2\widehat{C_0} + \widehat{\psi_{0r}} + \varphi_2(k_-), \tag{22}$$

with

$$\begin{aligned}
\varphi_2(k_-) &= \widetilde{\omega}_+^2(B_* - \chi_+) - \frac{1}{\varepsilon^2}\beta_-\chi_-'' - \frac{2}{\varepsilon^2}\beta_+'\chi_+' - \frac{1}{\varepsilon^2}\beta_+\chi_+'' + \frac{2\widetilde{\omega}_+}{\varepsilon}\gamma_+\chi_+' \qquad (23)\\
&\quad -(3 - gA_*^2 - 3B_*^2)\beta_+\chi_+ + [1 - \chi_- - g(A_*^2 - \chi_-)]\beta_-\chi_- + 2gA_*B_*\alpha_-\chi_-,\\
\widehat{\psi_{0r}} &= \widetilde{\psi_{0r}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) - \chi_+\widetilde{\psi_{0r}}(\omega x, \varepsilon, k_-, 0, \widetilde{Y}^{(+\infty)})\\
&\quad -\chi_-\widetilde{\psi_{0r}}(\omega x, \varepsilon, k_-, \widetilde{X}^{(-\infty)}, \widetilde{Y}^{(-\infty)}),
\end{aligned}$$

where $\gamma_+ = D_0^{(+\infty)}$. For $\widehat{\psi_{0r}}$ we have

$$\begin{aligned}
\widehat{\psi_{0r}} &= 2gA_*(\alpha_-\chi_-\widehat{C_0} + (\beta_-\chi_- + \beta_+\chi_+)\widehat{A_0} + \widehat{A_0}\widehat{C_0}) \qquad (24)\\
&\quad +g(B_* + \beta_+\chi_+ + \widehat{C_0})(\alpha_-^2\chi_-^2 + 2\alpha_-\chi_-\widehat{A_0} + \widehat{A_0}^2)\\
&\quad +g\beta_-\chi_-[(\alpha_-^2(\chi_-^2 - 1) + 2\alpha_-\chi_-\widehat{A_0} + \widehat{A_0}^2]\\
&\quad +[B_*(\beta_-\chi_- + \beta_+\chi_+)^2 - \chi_+\beta_+^2] + [B_*(\gamma_+\chi_+)^2 - \chi_+\gamma_+^2]\\
&\quad +\beta_+\chi_+(\chi_+^2 - 1)(\beta_+^2 + \gamma_+^2) + \beta_-^3\chi_-(\chi_-^2 - 1)\\
&\quad +\widehat{C_0}[(\beta_-\chi_- + \beta_+\chi_+ + \widehat{C_0})^2 + (\gamma_+\chi_+ + \widehat{D_0})^2]\\
&\quad +2(B_* + \beta_+\chi_+)(\beta_-\chi_- + \beta_+\chi_+\widehat{C_0} + \gamma_+\chi_+\widehat{D_0})\\
&\quad +(B_* + \beta_-\chi_- + \beta_+\chi_+)(\widehat{C_0}^2 + \widehat{D_0}^2) + \widehat{g_{00r}},
\end{aligned}$$

$$\widehat{g_{00r}} = \mathcal{O}(\varepsilon^{5/2}(e^{\varepsilon\delta x}\chi_{(-\infty,0)} + e^{-\frac{\delta' x}{\sqrt{2}}}\chi_{(0,\infty)}) + \varepsilon^2(|\widehat{X}| + |\widehat{Y}|) + \varepsilon(|\widehat{C_0}'| + |\widehat{D_0}'|).$$

Now we obtain for sufficiently small $\varepsilon, k_-, \widetilde{\omega}_+, \widehat{A_0}, \widehat{C_0}, \widehat{D_0}$ in $\mathbb{R}^3 \times \mathcal{D}_0 \times \mathcal{D}_1$

$$\begin{aligned}
||\widehat{\psi_{0r}}||_{L^2_\eta} &\le c\left(\varepsilon^2 + \frac{\widetilde{\omega}_-^4 + \widetilde{\omega}_+^4}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon}||(\widehat{A_0}, \widehat{C_0})||^2_{\mathcal{D}_0} + ||\widehat{D_0}||^2_{\mathcal{D}_1}\right) \qquad (25)\\
&\quad +c\left((\widetilde{\omega}_-^2 + \widetilde{\omega}_+^2)||(\widehat{A_0}, \widehat{C_0})||_{\mathcal{D}_0} + \varepsilon^2||\widehat{D_0}||_{\mathcal{D}_1}\right),
\end{aligned}$$

$$\begin{aligned}
||\varphi_2(k_-)||_{L^2_\eta} &\le c(\frac{\widetilde{\omega}_-^2}{\sqrt{\varepsilon}} + \frac{(|\widetilde{\omega}_+| + \varepsilon^2)^2}{\varepsilon^2}), \qquad (26)\\
\int_{\mathbb{R}} \varphi_2(k_-)B_*' dx &= \mathcal{O}[(\widetilde{\omega}_-^2 + \widetilde{\omega}_+^2 + \varepsilon^4)],
\end{aligned}$$

where the last estimates use

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_0^1 \beta'_+\chi'_+B'_*dx &= \mathcal{O}(\varepsilon^4)\\ \frac{1}{\varepsilon^2}\int_0^1 \beta_+\chi''_+B'_*dx &= \mathcal{O}(|\widetilde{\omega}_+|+\varepsilon^2)^2\end{aligned}$$

obtained, for the first integral in integrating by parts, and for the second one in separating the oscillating part of order ε^6 from the constant part $\beta_+^{(c)}$ of β_+, for which we make an integration by parts, in using $B''_* = \mathcal{O}(\varepsilon^2 B_*)$. More precisely we have

$$\begin{aligned}\int_{\mathbb{R}}\varphi_1(k_-)A'_*dx + \int_{\mathbb{R}}\varphi_2(k_-)B'_*dx &= a_2\frac{k_-^2}{4} + a_3\sigma_0\varepsilon^2 k_- \\ &\quad +\mathcal{O}(|k_-^3| + \varepsilon^2 k_-^2 + \widetilde{\omega}_+^2 + \varepsilon^4),\end{aligned} \tag{27}$$

with

$$\begin{aligned}a_2 &= \int_{\mathbb{R}}(A_* - \chi_-)A'_*dx - a_3,\\ a_3 &= \int_{-1}^0 \chi_-^{(4)}A'_* - 3\int_{\mathbb{R}}(1-A_*^2)A'_*\chi_- dx + g\int_{\mathbb{R}}(A_*B_*^2)'\chi_- dx,\end{aligned}$$

We observe that (see Corollay 5)

$$\begin{aligned}\int_{\mathbb{R}}(A_* - \chi_-)A'_*dx &= \frac{1}{2} + \mathcal{O}(\varepsilon^{1/2})\\ \int_{-1}^0 \chi_-^{(4)}A'_*dx &= \mathcal{O}(\varepsilon^{1/2})\\ g\int_{-\infty}^0 (A_*B_*^2)'\chi_- dx &= -g\int_{-1}^0 (A_*B_*^2)\chi'_- dx = \mathcal{O}(\varepsilon^{1/3})\\ -3\int_{-\infty}^0 (1-A_*^2)\chi_- A'_* dx &= 3\int_{-1}^0 (A_* - \frac{A_*^3}{3} - \frac{2}{3})\chi'_- dx = 2 + \mathcal{O}(\varepsilon^{1/3})\\ \varepsilon^6 b_0 &= -\frac{1}{\varepsilon^2}\int_{-1}^0 \beta_-\chi''_-B'_*dx.\end{aligned} \tag{28}$$

so that

$$a_2 = -3/2 + \mathcal{O}(\varepsilon^{1/3}), \tag{29}$$

$$a_3 = 2 + \mathcal{O}(\varepsilon^{1/3}). \tag{30}$$

3.3 Component $\mathcal{L}_g\widehat{D_0}$

For the third component we obtain

$$\mathcal{L}_g\widehat{D_0} = -\frac{2\widetilde{\omega}_+}{\varepsilon}\widehat{C_0}' + \widetilde{\omega}_+^2\widehat{D_0} + \widehat{\psi_{0i}} + \varphi_3(k_-), \tag{31}$$

$$
\begin{aligned}
\varphi_3(\widetilde{\omega}, k_-, \omega x) &= -\frac{2\widetilde{\omega}_+}{\varepsilon}[B_*' + \beta_- \chi_-' + \beta_+ \chi_+'] - \frac{2}{\varepsilon^2}\gamma_+' \chi_+' \\
&\quad -\frac{1}{\varepsilon^2}\gamma_+ \chi_+'' - (1 - gA_*^2 - B_*^2)\gamma_+ \chi_+,
\end{aligned}
$$

and for

$$
\begin{aligned}
\widehat{\psi_{0i}} &= \widetilde{\psi_{0i}}(\omega x, \varepsilon, k_-, \widetilde{X}, \widetilde{Y}) - \chi_+ \widetilde{\psi_{0i}}(\omega x, \varepsilon, k_-, 0, \widetilde{Y}^{(+\infty)}) \\
&\quad -\chi_- \widetilde{\psi_{0i}}(\omega x, \varepsilon, k_-, \widetilde{X}^{(-\infty)}, \widetilde{Y}^{(-\infty)}),
\end{aligned}
$$

and for sufficiently small $\varepsilon, k_-, \widetilde{\omega}_+, \widehat{A_0}, \widehat{C_0}, \widehat{D_0}$ in $\mathbb{R}^3 \times \mathcal{D}_0 \times \mathcal{D}_1$, the estimate

$$
\begin{aligned}
||\widehat{\psi_{0i}}||_{L^2_\eta} &\leq c\{\varepsilon + (\widetilde{\omega}_-^2 + \widetilde{\omega}_+^2)||\widehat{D_0}||_{\mathcal{D}_1} + ||\widehat{A_0}\widehat{D_0}||_{L^2_\eta} \\
&\quad +||(\widehat{C_0}\widehat{D_0})||_{L^2_\eta} + ||\widehat{D_0}||^2_{\mathcal{D}_1}\}, \qquad (32)
\end{aligned}
$$

$$
||\varphi_3||_{L^2_\eta} \leq c(\varepsilon^3 + \frac{|\widetilde{\omega}_+|}{\sqrt{\varepsilon}} + \frac{|\widetilde{\omega}_+^3|}{\varepsilon}), \qquad (33)
$$

4 Bifurcation equation

Let us use an adapted Lyapunov-Schmidt method. Since

$$
\mathcal{M}_g(A_*', B_*') = 0,
$$

we now decompose $(\widehat{A_0}, \widehat{C_0}, \widehat{D_0})$ as

$$
\begin{aligned}
\widehat{A_0} &= zA_*' + u, \qquad (34) \\
\widehat{C_0} &= zB_*' + v, \\
\widehat{D_0} &= w,
\end{aligned}
$$

then equations (17,22) give (Q_0 is the projection in L^2 on the range of $\mathcal{M}_g$)

$$
\mathcal{M}_g(u, v) = Q_0 \begin{pmatrix} -k_-(zA_*' + u)'' + \frac{k_-^2}{4}(zA_*' + u) + \widehat{\phi_0} + \varphi_1(k_-) \\ \frac{2\widetilde{\omega}_+}{\varepsilon} w' + \widetilde{\omega}_+^2(zB_*' + v) + \widehat{\psi_{0r}} + \varphi_2(k_-) \end{pmatrix}. \qquad (35)
$$

4.1 Resolution in $\widetilde{\omega}_+$ and w

We observe that (u, v) and w appear non symmetrically, so, we first solve equation (31), where the kernel of $\mathcal{L}_g$ is empty, and its range of codimension 1 (see Lemma 14). This has the advantage to give w and $\widetilde{\omega}_+$ in function of $(u, v, z, k_-, \varepsilon)$.

Since

$$
\int_0^1 \frac{1}{\varepsilon^2}\gamma_+' \chi_+' B_* dx = -\int_0^1 \frac{1}{\varepsilon^2}\gamma_+ (\chi_+' B_*)' dx = \mathcal{O}(\varepsilon^4),
$$

we obtain, in using Remark 13

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_3 B_* dx &= -\frac{\widetilde{\omega}_+}{\varepsilon}[1+\mathcal{O}(|\widetilde{\omega}_+|+\varepsilon^2)^2]+\mathcal{O}(\varepsilon^4),\\
\int_{\mathbb{R}} \widehat{\psi_{0i}} B_* dx &= \mathcal{O}[\varepsilon+(\widetilde{\omega}_-^2+\widetilde{\omega}_+^2)||\widehat{D_0}||_{\mathcal{D}_1}+||\widehat{D_0}||^2_{\mathcal{D}_1}+||\widehat{A_0}\widehat{D_0}||_{L^2_\eta}+||\widehat{C_0}\widehat{D_0}||_{L^2_\eta}]\\
&= \mathcal{O}[\varepsilon+\varepsilon^{1/2}|z|||w||_{\mathcal{D}_1}+||(u,v)||^2_{\mathcal{D}_0}+||w||^2_{\mathcal{D}_1}+(\widetilde{\omega}_-^2+\widetilde{\omega}_+^2)||w||_{\mathcal{D}_1})].
\end{aligned}$$

The compatibility condition for equation (31) leads to

$$\frac{2\widetilde{\omega}_+}{\varepsilon}\int_{\mathbb{R}} B_*' B_* dx = \int_{\mathbb{R}} \left[-\frac{2\widetilde{\omega}_+}{\varepsilon}(zB_*''+v')+\widetilde{\omega}_+^2 w+\widehat{\psi_{0i}}+\varphi_3\right] B_* dx,$$

which gives

$$\begin{aligned}
\widetilde{\omega}_+ &= \int_{\mathbb{R}} \left[-2\widetilde{\omega}_+(zB_*''+v')+\varepsilon\widetilde{\omega}_+^2 w\right] B_* dx\\
&\quad +\mathcal{O}[\varepsilon^2+|\widetilde{\omega}_+|(|\widetilde{\omega}_+|+\varepsilon^2)^2+\varepsilon^{3/2}|z|||w||_{\mathcal{D}_1}]\\
&\quad +\varepsilon\mathcal{O}(||(u,v)||^2_{\mathcal{D}_0}+||w||^2_{\mathcal{D}_1}+(\widetilde{\omega}_-^2+\widetilde{\omega}_+^2)||w||_{\mathcal{D}_1}).
\end{aligned}$$

which is a smooth function of its arguments and may be solved with respect to $\widetilde{\omega}_+$ (or equivalently with respect to k_+ since $\widetilde{\omega}_+ = \frac{k_+}{2}+\mathcal{O}(\varepsilon^6)$) by implicit function theorem in the neighborhood of 0 for

$$\begin{aligned}
(u,v) &\in \mathcal{D}_0,\ w\in\mathcal{D}_1,\ (\varepsilon,\widetilde{\omega}_-,z)\in\mathbb{R}^3,\\
\widetilde{\omega}_+ &= \mathfrak{k}_+(\varepsilon,\widetilde{\omega}_-,z,(u,v),w)\in C^1(\mathbb{R}^3\times\mathcal{D}_0\times\mathcal{D}_1).
\end{aligned}$$

Indeed we have the estimate

$$|\mathfrak{k}_+|\le c[\varepsilon^2+\varepsilon^{3/2}|z|||w||_{\mathcal{D}_1}+\varepsilon\widetilde{\omega}_-^2||w||_{\mathcal{D}_1}+\varepsilon(||(u,v)||^2_{\mathcal{D}_0}+||w||^2_{\mathcal{D}_1})]. \tag{36}$$

For solving equation (31) we now have

$$w=\mathcal{L}_g^{-1}[-\frac{2\mathfrak{k}_+}{\varepsilon}(zB_*''+v')+\mathfrak{k}_+^2 w+\varphi_3+\widehat{\psi_{0i}}]$$

which may be solved by implicit function theorem with respect to w in $\mathcal{D}_1$ for

$$(\varepsilon,\widetilde{\omega}_-,z,(u,v))\in\mathbb{R}^3\times\mathcal{D}_0.$$

We obtain, using (33), (32), (36) and

$$||\frac{B_*''}{\varepsilon}||_{L^2_\eta}=\mathcal{O}(\varepsilon^{1/2}),\quad ||\frac{v'}{\varepsilon}||_{L^2_\eta}\le||v||_{\mathcal{D}_1},$$

then

$$w=\mathfrak{w}(\varepsilon,\widetilde{\omega}_-,z,u,v)$$

with

$$||\mathfrak{w}||_{\mathcal{D}_1}\le c(\varepsilon+\varepsilon^{1/2}||(u,v)||^2_{\mathcal{D}_0}), \tag{37}$$

and we obtain

$$|\mathfrak{k}_+|\le c(\varepsilon^2+\varepsilon||(u,v)||^2_{\mathcal{D}_0}). \tag{38}$$

Remark 16 *The term of order ε in $\mathfrak{w}$ is $\varepsilon w_1 + \mathcal{O}(\varepsilon^{3/2})$ with*

$$w_1 = c_9 \mathcal{L}_g^{-1}[B_* A_* A_*' - 2B_*' \int_{\mathbb{R}} B_*^2 A_* A_*' dx], \ ||w_1||_{\mathcal{D}_1} = \mathcal{O}(1), \tag{39}$$

and the compatibility condition is satisfied with

$$||2B_*' \int_{\mathbb{R}} B_*^2 A_* A_*' dx||_{L^2_\eta} = \mathcal{O}(\varepsilon^{1/2}).$$

4.2 Resolution in (u, v)

Now, we replace w and $\widetilde{\omega}_+$ by their expressions $\mathfrak{w}$ and $\mathfrak{k}_+$, and consider (35) which may be solved by implicit function theorem with respect to (u, v) in a neighborhood of 0 in $\mathcal{D}_0$ for (ε, k_-, z) close to 0 in $\mathbb{R}^3$. Indeed, the right hand side of (35) is smooth in its arguments and assuming

$$|k_-| << \varepsilon^{3/4}, \text{ i.e. } |\widetilde{\omega}_-| << \varepsilon^{3/4}, \tag{40}$$

$$|z| << \varepsilon^{3/4}, \tag{41}$$

$$||(u, v)||_{\mathcal{D}_0} << \varepsilon, \tag{42}$$

using (34) and collecting results of (17,20,21) for the first component, and (22,26,25) for the second component, estimates in L^2_η of the right hand side are as follows

$$\begin{aligned} \text{1st comp.} &= \mathcal{O}(\frac{(k_-^2 + z^2)}{\sqrt{\varepsilon}} + \varepsilon^{1/2}|k_-| + \varepsilon^2 + \frac{1}{\varepsilon}||(u, v)||^2_{\mathcal{D}_0} + (1/\varepsilon^2)||u||^3_{\mathcal{D}_0}) \\ \text{2nd comp.} &= \mathcal{O}(\varepsilon^2 + \frac{(k_-^2 + z^2)}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon}||(u, v)||^2_{\mathcal{D}_0}). \end{aligned}$$

Applying implicit function theorem for (ε, k_-, z) satisfying (40,41) in $\mathbb{R}^3$, leads to

$$(u, v) = (\mathfrak{u}, \mathfrak{v})(\varepsilon, k_-, z) \in \mathcal{D}_0$$

with

$$||(\mathfrak{u}, \mathfrak{v})||_{\mathcal{D}_0} \leq c(\varepsilon^2 + \frac{(k_-^2 + z^2)}{\sqrt{\varepsilon}} + \varepsilon^{1/2}|k_-|), \tag{43}$$

which satisfies the a priori estimate (42). Now using (37), (38), (40), (41) and (43)

$$\begin{aligned} ||\mathfrak{w}||_{\mathcal{D}_1} &\leq c\varepsilon, &(44) \\ |\mathfrak{k}_+| &\leq c\varepsilon^2, &(45) \end{aligned}$$

where (40) and (41) need to be checked at the end.

4.3 Final bifurcation equation

The orthogonality in L^2 of the right hand side of $\mathcal{M}_g(\widehat{A_0}, \widehat{C_0})$ with (A'_*, B'_*) (see Lemma 14), gives one relationship, expressed as the cancelling of a function of (z, k_-, ε), from which we extract the family of bifurcating solutions. It gives

$$\begin{aligned} 0 &= \int_{\mathbb{R}} [-k_-(zA'''_* + u'') + \frac{k_-^2}{4}(zA'_* + u)]A'_* dx + \int_{\mathbb{R}} (\widehat{\phi_0} + \varphi_1)A'_* dx \\ &\quad + \int_{\mathbb{R}} [\frac{2\widetilde{\omega}_+}{\varepsilon} w' + \widetilde{\omega}_+^2(zB'_* + v)]B'_* dx + \int_{\mathbb{R}} (\widehat{\psi_{0r}} + \varphi_2)B'_* dx. \end{aligned} \tag{46}$$

We define

$$a_1 = -\int_{\mathbb{R}} A'''_* A'_* dx = \int_{\mathbb{R}} A''^2_* dx > 0, \ (= \mathcal{O}(1)) \tag{47}$$

and we have, from (19), (24), (42), (44), (45), (40), (41) and Remark 13

$$\int_{\mathbb{R}} \widehat{\phi_0} A'_* dx = z^2[a'_0 + \mathcal{O}(\varepsilon^{3/2})] + \sigma'_0 \varepsilon^2 k_- + \mathcal{O}[\varepsilon^{5/2} + |z|(\varepsilon^{3/2} + k_-^2)],$$

with

$$\begin{aligned} a'_0 &= \int_{\mathbb{R}} (3A_* A'^3_* + 2gB_* B'_* A'^2_* + gA_* A'_* B'^2_*) dx = \mathcal{O}(\varepsilon^{1/2}), \\ \sigma'_0 &= \sigma_0 \int_{\mathbb{R}} A'_*(A_*^3 - \chi_-) dx = \sigma_0[\frac{3}{4} + \mathcal{O}(\varepsilon^{1/2})], \end{aligned} \tag{48}$$

where (for example) the estimated term in $\varepsilon^{3/2}|z|$ comes from

$$|\int_{\mathbb{R}} zA'^2_* u dx| \leq c|z| ||A'^2_*||_{L^2_\eta} ||u||_{L^2_\eta} \leq c' \varepsilon^{1/2} |z| \varepsilon,$$

occuring in

$$3 \int_{\mathbb{R}} A_* A'_* (A'_* z + u)^2 dx.$$

We also obtain

$$\int_{\mathbb{R}} \widehat{\psi_{0r}} B'_* dx = z^2 a''_0 + \mathcal{O}(\varepsilon^{5/2} + \varepsilon^2 |z|)$$

with

$$a''_0 = \int_{\mathbb{R}} (gB_* B'_* A'^2_* + 2gA_* A'_* B'^2_* + B_* B'^3_*) dx = \mathcal{O}(\varepsilon).$$

Hence

$$\int_{\mathbb{R}} \widehat{\phi_0} A'_* dx + \int_{\mathbb{R}} \widehat{\psi_{0r}} B'_* dx = z^2[a_0 + \mathcal{O}(\varepsilon^{3/2})] + \sigma'_0 \varepsilon^2 k_- + \mathcal{O}(\varepsilon^{5/2} + \varepsilon^{3/2}|z|), \tag{49}$$

where we define

$$a_0 = 3 \int_{\mathbb{R}} (A_* A'^3_* + gB_* B'_* A'^2_* + gA_* A'_* B'^2_* + (1/3) B_* B'^3_*) dx = \mathcal{O}(\varepsilon^{1/2}). \tag{50}$$

Using Corollaries 5 and 6, we notice that the main contribution of this integral is on $(-\infty, 0)$ and precisely

$$\int_{-\infty}^{0} 3A_* A_*'^3 dx = \mathcal{O}(\varepsilon^{1/2}), \quad \int_{0}^{+\infty} 3A_* A_*'^3 dx = \mathcal{O}(\varepsilon^2).$$

Now collecting the expressions (47), (27), (49) in (46) we obtain the bifurcation equation

$$\widetilde{a_0} z^2 + a_1 k_- z + a_2' \frac{k_-^2}{4} + a_3'' \varepsilon k_- + a_5 \varepsilon^{3/2} z + a_4' \varepsilon^{5/2} = \mathcal{O}(|k_-|^3 + \varepsilon^3), \qquad (51)$$

where

$$\begin{aligned} \widetilde{a_0} &= a_0 + \mathcal{O}(\varepsilon^{3/2}), \quad a_0 = \varepsilon^{1/2}\overline{a_0} = \mathcal{O}(\varepsilon^{1/2}) \text{ (see (50))} \\ a_1 &= \int_{\mathbb{R}} A_*'^2 dx + \mathcal{O}(\varepsilon^{1/2}) = \mathcal{O}(1) \\ a_2' &= a_2 + \mathcal{O}(\varepsilon^{3/2}) = -3/2 + \mathcal{O}(\varepsilon^{1/3}) \\ a_3'' &= a_3' + \varepsilon(\sigma_0' + a_3\sigma_0) = o(1), \\ a_4' &= a_4 + o(1), \ a_4 = \mathcal{O}(1), \quad a_5 = o(1), \end{aligned} \qquad (52)$$

with (see (40,41,43))

$$-k_- \int_{\mathbb{R}} A_*' u'' dx = \varepsilon k_- a_3', \quad a_3' = \mathcal{O}(\varepsilon^{-1} ||u||_{\mathcal{D}_0}) = o(1),$$

$$a_5 = \mathcal{O}(\varepsilon^{-1} ||(u,v)||_{\mathcal{D}_0}) = o(1),$$

and (see Remark 13 and (39))

$$(d_2 - d_4) \int_{\mathbb{R}} A_* A_*'^3 dx + \int_{\mathbb{R}} A_*' w_1^2 dx = a_4 \varepsilon^{1/2}, \qquad (53)$$

the term of order $o(1)$ in a_4' comes from the estimate of terms of order $\mathcal{O}(\varepsilon^{5/2})$ in

$$\sqrt{\varepsilon}(||u|| ||v|| + ||u||^2 + ||v||^2) + \frac{1}{\sqrt{\varepsilon}} ||u||^3$$

occuring in the estimate of

$$\int_{\mathbb{R}} \widehat{\phi_0} A_*' dx,$$

where we notice that (using Lemma 61))

$$| \int_{\mathbb{R}} A_*' u^3 dx| \leq c\sqrt{\varepsilon} ||u||_{C^0} ||u||_{\mathcal{D}_0}^2 \leq c \frac{1}{\sqrt{\varepsilon}} ||u||_{\mathcal{D}_0}^3.$$

The discriminant of the principal part of the quadratic form in (z, k_-) of the left hand side of (51) is

$$\Delta = a_1^2 - \widetilde{a_0} a_2' \sim a_1^2 \qquad (54)$$

which *it is positive*. The bifurcation equation (51) may be written as

$$\begin{aligned}&\left(\frac{a_2'k_-}{2}+a_1z+a_3''\varepsilon\right)^2-\Delta\left(z+\frac{a_1a_3''\varepsilon}{\Delta}-\frac{a_2'a_5\varepsilon^{3/2}}{2\Delta}\right)^2+a_6\varepsilon^{5/2} \\ =\ &\mathcal{O}\left(|k_-|^3+\varepsilon^3\right),\end{aligned} \tag{55}$$

where

$$\begin{aligned}a_6\varepsilon^{5/2} &= a_2'a_4'\varepsilon^{5/2}+\frac{a_2'a_3''^2\widetilde{a}_0\varepsilon^2-a_1a_2'a_3''a_5\varepsilon^{5/2}}{\Delta}, \\ a_6 &\sim -\frac{3}{2}a_4.\end{aligned}$$

Then, using the implicit function theorem, we obtain a family of solutions such that z and k_- are at least of order ε, with leads to

$$||(\mathfrak{u},\mathfrak{v})||_{\mathcal{D}_0}=\mathcal{O}(\varepsilon^{3/2}),$$

hence $a_3''=\mathcal{O}(\varepsilon^{1/2})$ and

i) if $a_4<0$

$$\begin{aligned}z &= \frac{1}{a_1}\sqrt{\frac{-3a_4}{2}}\varepsilon^{5/4}\sinh\phi+\mathcal{O}(\varepsilon^{3/2}). \\ k_- &= 2\sqrt{\frac{2a_4}{3}}\varepsilon^{5/4}\exp(-\phi)+\mathcal{O}(\varepsilon^{3/2}). \\ \phi &\in \mathbb{R};\end{aligned} \tag{56}$$

ii) if $a_4>0$

$$\begin{aligned}z &= \frac{1}{a_1}\sqrt{\frac{3a_4}{2}}\varepsilon^{5/4}\cosh\phi+\mathcal{O}(\varepsilon^{3/2}) \\ k_- &= -2\sqrt{\frac{2a_4}{3}}\varepsilon^{5/4}\exp(-\phi)+\mathcal{O}(\varepsilon^{3/2}) \\ \phi &\in \mathbb{R}.\end{aligned} \tag{57}$$

Checking that the conditions (40), (41) are satisfied for $\exp\phi << \varepsilon^{-1/2}$, Proposition 7 is proved.

Remark 17 *It should be noted that the one parameter family of solutions which are obtained, correspond to convective rolls at* $-\infty$ *with wave numbers*

$$k_c(1+\varepsilon^2k_-)$$

connected to convective rolls at $+\infty$ *with wave numbers*

$$k_c(1+2\varepsilon^2\widetilde{\omega}_+).$$

The calculations made above, show that we obtain $\widetilde{\omega}_+$ *and* k_- *as functions of* ε,ϕ *where* $\phi\in\mathbb{R}$ *such that* $\exp\phi << \varepsilon^{-1/2}$*. This is a one parameter family of relationships between wave numbers at each infinity, depending on the amplitude* ε^2 *of rolls.*

Remark 18 *We might examine the limit size of k_-. For example, is it possible to obtain the case $k_- = k_+ = 2\widetilde{\omega}_+ = \mathcal{O}(\varepsilon^2)$?.Then, looking at the bifurcation equation we need to solve at main orders*

$$(\overline{a_0}z^2 + a_4'\varepsilon^2)\varepsilon^{1/2} = \mathcal{O}(\varepsilon^3).$$

This is only possible if

$$\overline{a_0}a_4' < 0,$$

which coefficient is a function of the Prandtl number.

A Appendix

A.1 Reduction of the normal form

We start with

$$\begin{aligned} X &= (A_0, A_1, A_2, A_3)^t \in \mathbb{R}^4, \\ Y &= (B_0, B_1)^t \in \mathbb{C}^2, \end{aligned}$$

and the system under normal form (see [2])

$$\begin{aligned} \frac{dX}{dx} &= LX + N(X, Y, \overline{Y}, \mu, \widetilde{k}) + F(X, Y, \overline{Y}, \mu, \widetilde{k}), \\ \frac{dY}{dx} &= L_{k_c}Y + M(X, Y, \overline{Y}, \mu) + G(X, Y, \overline{Y}, \mu), \end{aligned} \tag{58}$$

$$\begin{aligned} LX &= (A_1, A_2, A_3, 0)^t, \\ L_{k_c}Y &= (ik_cB_0 + B_1, ik_cB_1)^t, \end{aligned}$$

$$\begin{aligned} |F(X, Y, \overline{Y}, \mu, \widetilde{k})| &\leq c|X|(|X|^2 + |Y|^2 + |\widetilde{k}| + |\mu|)^2 \\ |G(X, Y, \overline{Y}, \mu)| &\leq c(|X|^2 + |Y|)(|X|^2 + |Y|^2 + |\mu|)^2, \end{aligned} \tag{59}$$

$$N(X, Y, \overline{Y}, \mu) = \begin{pmatrix} 0 \\ A_0P_1 \\ A_1P_1 + c_8u_8 + c_{13}u_{13} \\ A_2P_1 + A_0P_3 + +c_8v_8 + c_{13}v_{13} + d_{14}u_{14} \end{pmatrix},$$

$$M(X, Y, \overline{Y}, \mu) = \begin{pmatrix} iB_0Q_0 + \alpha_{10}u_{10} \\ iB_1Q_0 + B_0Q_1 + \alpha_{10}v_{10} + i\beta_{10}u_{10} + i\beta_{12}u_{12} \end{pmatrix},$$

$$\begin{aligned} P_1 &= b_0\mu + b_0'\widetilde{k} + b_1u_1 + b_3u_3 + b_5u_5 + b_6u_6, \\ P_3 &= d_0\mu + d_0''\widetilde{k}^2 + d_1u_1 + d_1'\widetilde{k}u_1 + d_3u_3 + d_5u_5 + d_6u_6, \end{aligned}$$

$$
\begin{aligned}
Q_0 &= \alpha_0\mu + \alpha_1 u_1 + \alpha_3 u_3 + \alpha_5 u_5 + \alpha_6 u_6 \\
Q_1 &= \beta_0\mu + \beta_1 u_1 + \beta_3 u_3 + \beta_5 u_5 + \beta_6 u_6,
\end{aligned}
$$

where

$$
\begin{aligned}
u_1 &= A_0^2, \quad v_1 = A_0 A_1, \quad w_1 = \frac{1}{2}A_1^2, \\
u_3 &= 2A_0A_2 - A_1^2, \quad v_3 = 3A_0A_3 - A_1A_2 \\
u_5 &= B_0\overline{B_0}, \quad v_5 = \frac{1}{2}(B_0\overline{B_1} + \overline{B_0}B_1), \quad w_5 = \frac{1}{2}B_1\overline{B_1} \\
u_6 &= i(B_0\overline{B_1} - \overline{B_0}B_1).
\end{aligned}
$$

$$
\begin{aligned}
u_8 &= A_0v_3 - A_1u_3, \quad v_8 = A_1v_3 - 2A_2u_3, \\
u_{13} &= A_0v_5 - A_1u_5, \quad v_{13} = A_0w_5 - A_2u_5, \\
u_{14} &= A_0w_5 + A_2u_5 - A_1v_5,
\end{aligned}
$$

$$
\begin{aligned}
u_{10} &= B_0v_1 - B_1u_1, \quad v_{10} = 2B_0w_1 - B_1v_1 \\
u_{12} &= B_0v_3 - B_1u_3.
\end{aligned}
$$

The (reversible) system anticomutes with the symmetry $\mathbf{S}_1$ (represents the reflection $x \mapsto -x$). and commutes with $\boldsymbol{\tau}_\pi$ (shift by half of one period in y direction):

$$
\begin{aligned}
(A_0, A_1, A_2, A_3, B_0, B_1) &\mapsto {}^{\mathbf{S}_1}(A_0, -A_1, A_2, -A_3, \overline{B_0}, -\overline{B_1}), \\
(A_0, A_1, A_2, A_3, B_0, B_1) &\mapsto {}^{\boldsymbol{\tau}_\pi}(-A_0, -A_1, -A_2, -A_3, B_0, B_1).
\end{aligned}
$$

Remark 19 *In the case of rigid-free boundary conditions, we have no symmetry $z \mapsto 1-z$. We don't use this symmetry here (valid only in rigid-rigid or free-free boundaries). The symmetry $\boldsymbol{\tau}_\pi$ implies that F is odd in X and G even in X. It can be shown that there is no term of degree 4 in $X, Y, \overline{Y}$ in the normal form.*

Then, the X part of the system (58) may be written as a 4th order real ODE, while the Y part becomes a 2nd order complex ODE as

$$
\begin{aligned}
A_0^{(4)} = {} & A_0[d_0\mu + (d_0'' - b_0'^2)\widetilde{k}^2 + d_1A_0^2 + d_1'\widetilde{k}A_0^2 + d_5\widetilde{B_0}\overline{\widetilde{B_0}} + d_1'\widetilde{k}A_0^2 \\
& + id_6(\widetilde{B_0}\overline{\widetilde{B_0}}' - \overline{\widetilde{B_0}}\widetilde{B_0}')] + (a_0\mu + 3b_0'\widetilde{k})A_0'' + a_1A_0^2A_0'' + a_2A_0A_0'^2 \\
& + a_3A_0\widetilde{B_0}'\overline{\widetilde{B_0}}' + a_4A_0'(\widetilde{B_0}\overline{\widetilde{B_0}}' + \overline{\widetilde{B_0}}\widetilde{B_0}') + a_5A_0''\widetilde{B_0}\overline{\widetilde{B_0}} \\
& + 3ib_6A_0''(\widetilde{B_0}\overline{\widetilde{B_0}}' - \overline{\widetilde{B_0}}\widetilde{B_0}') + a_6A_0A_0'A_0''' + a_7A_0A_0''^2 + a_8A_0'^2A_0'' + \mathcal{O}_X(5),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B_0}'' = {} & \widetilde{B_0}[\beta_0\mu + \beta_1A_0^2 + \beta_5\widetilde{B_0}\overline{\widetilde{B_0}}] + ic_1\widetilde{B_0}'A_0^2 + ic_2\widetilde{B_0}'|\widetilde{B_0}|^2 + ic_3\overline{\widetilde{B_0}}'\widetilde{B_0}^2 \\
& + 2i\alpha_0\mu\widetilde{B_0}' + ic_4\widetilde{B_0}A_0A_0' - 2\alpha_6\widetilde{B_0}'(\widetilde{B_0}\overline{\widetilde{B_0}}' - \overline{\widetilde{B_0}}\widetilde{B_0}') \\
& + c_5\widetilde{B_0}A_0A_0'' + c_6\widetilde{B_0}A_0'^2 + c_7\widetilde{B_0}'A_0A_0' + ic_8\widetilde{B_0}A_0A_0''' \\
& ic_9\widetilde{B_0}'A_0A_0'' + ic_{10}\widetilde{B_0}'A_0'^2 + ic_{11}\widetilde{B_0}A_0'A_0'' + \mathcal{O}_Y(5),
\end{aligned}
$$

with real coefficients $d_j, d_1' d_0'', a_j, b_j, b_0', c_j, \beta_j, \alpha_j$ and

$$\widetilde{B_0} = B_0 e^{-ik_c x},\ \widetilde{B_1} = B_1 e^{-ik_c x}, \tag{60}$$

$$\begin{aligned} d_0 &= -4k_c^2\beta_0 > 0,\ d_1 = -4k_c^2\beta_5 < 0, \\ \frac{\beta_1}{\beta_5} &= \frac{d_5}{d_1} := g > 0,\ b_0' = \frac{4k_c^2}{3}, d_0'' = -\frac{20}{9}k_c^4, \end{aligned}$$

$$\begin{aligned} \mathcal{O}_X(5) &= \mathcal{O}(|X|(|X|^2 + |Y|^2 + \widetilde{k}^2 + |\mu|)^2), \\ \mathcal{O}_Y(5) &= \mathcal{O}[(|X|^2 + |Y|)(|X|^2 + |Y|^2 + |\mu|)^2], \\ X &= (A_0, A_0', A_0'', A_0''')^t \\ Y &= (\widetilde{B_0}, \widetilde{B_0}'). \end{aligned}$$

Notice that the high order rests $\mathcal{O}_X(5)$ and $\mathcal{O}_Y(5)$ are no longer autonomous, since they are functions of $e^{\pm ik_c x}$.

Now, as indicated in [2] we make the following scaling

$$\begin{aligned} x &= \frac{1}{2\varepsilon k_c}\widetilde{x},\ \mu = \frac{4k_c^2}{-\beta_0}\varepsilon^4,\ \widetilde{k} = \varepsilon^2 k_- \\ A_0(x) &= \frac{2k_c}{\sqrt{\beta_5}}\varepsilon^2 \widetilde{A_0}(\widetilde{x}),\ \widetilde{B_0}(x) = \frac{2k_c}{\sqrt{\beta_5}}\varepsilon^2 \widetilde{\widetilde{B_0}}(\widetilde{x}), \end{aligned} \tag{61}$$

so that the system above becomes, after suppressing the tildes,

$$\begin{aligned} A_0^{(4)} &= k_- A_0'' + A_0(1 - \frac{k_-^2}{4} - A_0^2 - g|B_0|^2) + \widehat{f}, \\ B_0'' &= \varepsilon^2 B_0(-1 + gA_0^2 + |B_0|^2) + \widehat{g}, \end{aligned} \tag{62}$$

with additional cubic terms of the form

$$\begin{aligned} \widehat{f} &= id_1\varepsilon A_0(B_0\overline{B_0}' - \overline{B_0}B_0') + \sigma_0\varepsilon^2 k_- A_0^3 + \varepsilon^2[d_3A_0'' + d_4A_0^2A_0'' + d_2A_0A_0'^2 + d_6A_0|B_0'|^2 \\ &\quad + d_7A_0'(B_0\overline{B_0}' + \overline{B_0}B_0') + d_5A_0''|B_0|^2] + id_8\varepsilon^3 A_0''(B_0\overline{B_0}' - \overline{B_0}B_0') + \mathcal{O}(\varepsilon^4), \end{aligned}$$

$$\begin{aligned} \widehat{g} &= \varepsilon^3[ic_0B_0' + ic_1B_0'|A_0|^2 + ic_2B_0'|B_0|^2 + ic_3B_0^2\overline{B_0}' + ic_9B_0A_0A_0'] \\ &\quad + \varepsilon^4[c_4B_0'(B_0\overline{B_0}' - \overline{B_0}B_0') + c_5B_0A_0A_0'' + c_6B_0A_0'^2 + c_7B_0'A_0A_0'] \\ &\quad + \varepsilon^5[ic_8B_0A_0A_0''' + ic_7B_0'A_0A_0'' + ic_{10}B_0'A_0'^2 + ic_{11}B_0A_0'A_0'' + \mathcal{O}(\varepsilon^6). \end{aligned}$$

A.2 Equilibrium solution at $x = -\infty$

Let us look for equilibria of (2), which should correspond to the convective rolls at $x = -\infty$ parallel to x - axis. Cancelling all derivatives with respect to x, we obtain a system commuting with the symmetry $(A_0, B_0) \mapsto (A_0, \overline{B_0})$. It then results a system of 2 real equations for A_0, B_0 :

$$\begin{aligned} A_0(1 - \frac{k_-^2}{4} - A_0^2 + \sigma_0\varepsilon^2 k_- A_0^2 - gB_0^2) + \mathcal{O}(\varepsilon^4) &= 0 \\ B_0(-1 + gA_0^2 + B_0^2) + \mathcal{O}(\varepsilon^4) &= 0, \end{aligned}$$

where we may observe that the terms $\mathcal{O}(\varepsilon^4)$ in the second equation contain at least terms of degree 1 in B_0, since they come from terms of order 5 in $(A_0, B_0, \overline{B_0})$. The first terms not containing B_0 may be found at order 6 in A_0, which makes order ε^6 after the scaling (61) in the rest (12-6=6).

It then results that the equilibrium that we are looking for satisfies (by implicit function theorem)

$$\begin{aligned} A_0^2 &= 1 - \frac{k_-^2}{4} + \sigma_0\varepsilon^2 k_- + \mathcal{O}(\varepsilon^2|k_-|^3 + \varepsilon^4), \\ B_0 &= \mathcal{O}(\varepsilon^6). \end{aligned}$$

Remark 20 *In the cases where symmetry $z \mapsto 1-z$ applies, the additional symmetry S_0 changes the signs of A_0 and B_0, implying that $Y = 0$ is an invariant subspace, so that in such cases $B_0 = 0$ for equilibrium at $-\infty$.*

A.3 Periodic solution in M_+

Let us consider the 4-dimensional reversible vector field corresponding to the system (58) with $X = 0$ and rescaled. We intend to give precise estimates on the family of periodic bifurcating solutions $B_0^{(+\infty)}(k_+, x)$, here corresponding to the periodic convection rolls at infinity in M_+ with wave numbers close to k_c (becomes $1/2\varepsilon$ after the scaling (61)).

Since we use the normal form up to cubic order, and since there is no term of order 4, it takes the form (after the scaling used in [2], but before we incorporate $e^{\frac{ix}{2\varepsilon}}$ in B_0, so that the system is autonomous):

$$\begin{aligned} \frac{dB_0}{dx} &= \frac{i}{2\varepsilon}B_0 + B_1 + i\varepsilon^3 B_0 P + \varepsilon^7 g_0(\varepsilon, Y, \overline{Y}) \\ \frac{dB_1}{dx} &= \frac{i}{2\varepsilon}B_1 + \varepsilon^2 B_0 Q + i\varepsilon^3 B_1 P + \varepsilon^6 g_1(\varepsilon, Y, \overline{Y}), \end{aligned} \qquad (63)$$

with

$$\begin{aligned} Y &= (B_0, B_1) \\ P &= \alpha + \beta|B_0|^2 + \varepsilon\gamma K \\ Q &= -1 + |B_0|^2 + \varepsilon\delta K \\ K &= \frac{i}{2}(B_0\overline{B_1} - \overline{B_0}B_1) \end{aligned}$$

where we are looking for a periodic solution (B_0, B_1), with wave number ω close to $\frac{1+\varepsilon^2 k_+}{2\varepsilon}$.

A.3.1 Principal part

Let us first compute periodic solutions for $g_0 = g_1 \equiv 0$. Then these small terms will be perturbations treated by an adapted implicit function theorem.

Without g_0 and g_1, let us use polar coordinates (see [3] section 4.3.3)

$$\begin{aligned} B_0 &= r_0 e^{i\theta_0} \\ B_1 &= i r_1 e^{i\theta_1} \end{aligned}$$

then

$$\begin{aligned} K &= r_0 r_1 \cos(\theta_0 - \theta_1) = \text{const} \\ \frac{dr_0}{dx} &= r_1 \sin(\theta_0 - \theta_1) \\ \frac{dr_1}{dx} &= \varepsilon^2 r_0 \sin(\theta_0 - \theta_1) Q(\varepsilon, r_0^2, K) \\ r_0 \frac{d\theta_0}{dx} &= \frac{r_0}{2\varepsilon} + r_1 \cos(\theta_0 - \theta_1) + \varepsilon^3 r_0 P \\ r_1 \frac{d\theta_1}{dx} &= \frac{r_1}{2\varepsilon} - \varepsilon^2 r_0 \cos(\theta_0 - \theta_1) Q(\varepsilon, r_0^2, K) + \varepsilon^3 r_1 P. \end{aligned}$$

The required periodic solutions correspond to

$$\begin{aligned} &r_0 \text{ and } r_1 \text{ const} \\ \theta_0 &= \theta_1, \ \frac{d\theta_0}{dx} = \frac{1 + \varepsilon^2 k_+}{2\varepsilon} \\ K &= r_0 r_1, \end{aligned}$$

hence

$$\frac{\varepsilon k_+}{2} = \frac{r_1}{r_0} + \varepsilon^3 P \tag{64}$$

$$(\frac{r_1}{r_0})^2 = -\varepsilon^2 Q. \tag{65}$$

Solving (64) with respect to r_1 gives

$$\begin{aligned} r_1 &= \varepsilon r_0 \frac{k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)}{2(1 + \varepsilon^4 \gamma r_0^2)} \\ &= \frac{\varepsilon r_0}{2}[k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)](1 + \mathcal{O}(\varepsilon^4)), \end{aligned}$$

and (65) leads to

$$\frac{1}{4}[k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)]^2 + \frac{\varepsilon^2 \delta r_0^2}{2}[k_+ - 2\varepsilon^2(\alpha + \beta r_0^2)] = (1 - r_0^2)(1 + \gamma \varepsilon^4 r_0^2)^2$$

which is solved with respect to r_0^2, by implicit function theorem:

$$r_0^2 = 1 - \frac{k_+^2}{4} + \sigma_1 \varepsilon^2 k_+ + \sigma_2 \varepsilon^4 + \mathcal{O}[(|k_+| + \varepsilon^2)^4], \tag{66}$$

$$r_1 = \frac{\varepsilon r_0}{2} k_+ + \mathcal{O}(\varepsilon^3),$$

where we notice that coefficients σ_1 and σ_2 are functions of the Prandtl number. We obtain a one-parameter family of periodic solutions (parameter k_+), with only the Fourier modes $e^{\pm is}$.

A.3.2 Estimates of higher order terms

The proof below is new and self contained. There is a geometrical proof without estimates in Iooss-Pérouème [7], and a more precise proof by Horn in [6] section 3.5.

Let us define by ω the frequency of periodic solutions, where ω is close to

$$\omega_0 = \frac{1+\varepsilon^2 k_+}{2\varepsilon},$$

and set

$$\begin{aligned} s &= \omega x,\ \omega = \omega_0 + \widehat{\omega} \\ B_0(s) &= r_0 e^{is} + \widehat{B_0} \\ B_1(s) &= i r_1 e^{is} + i\widehat{B_1}, \end{aligned}$$

where B_0 and B_1 are $2\pi-$ periodic in s, and r_0, r_1 are solution of (64,65). Let us introduce the linear operator

$$L_0 = \begin{pmatrix} -(i\omega_0 \frac{d}{ds} + \frac{1}{2\varepsilon} + \varepsilon^3 P_0) & -1 \\ \varepsilon^2 Q_0 & -(i\omega_0 \frac{d}{ds} + \frac{1}{2\varepsilon} + \varepsilon^3 P_0) \end{pmatrix},$$

acting in the function space $H^1(\mathbb{R}/2\pi\mathbb{Z}) \times L^2(\mathbb{R}/2\pi\mathbb{Z})$. It appears that L_0 has a one-dimensional kernel

$$(r_0 e^{is}, r_1 e^{is}) \stackrel{def}{=} V_0 e^{is}$$

since (64,65) implies

$$\begin{aligned} [(\omega_0 - \frac{1}{2\varepsilon} - \varepsilon^3 P_0] r_0 - r_1 &= 0 \\ \varepsilon^2 Q_0 r_0 + [(\omega_0 - \frac{1}{2\varepsilon} - \varepsilon^3 P_0] r_1 &= 0, \end{aligned}$$

with

$$\begin{aligned} P_0 &= \alpha + \beta r_0^2 + \varepsilon\gamma r_0 r_1, \\ Q_0 &= -1 + r_0^2 + \varepsilon\delta r_0 r_1. \end{aligned}$$

Then the system (63), to be completed by its complex conjugate, becomes:

$$\begin{aligned} \widehat{\omega} V_0 e^{is} + L_0 \begin{pmatrix} \widehat{B_0} \\ \widehat{B_1} \end{pmatrix} &= i\widehat{\omega} \frac{d}{ds} \begin{pmatrix} \widehat{B_0} \\ \widehat{B_1} \end{pmatrix} + \begin{pmatrix} \varepsilon^3 r_0 P_{lin} \\ -\varepsilon^2 r_0 Q_{lin} + \varepsilon^3 r_1 P_{lin} \end{pmatrix} \\ &+ \begin{pmatrix} R_0(\widehat{Y}, \overline{\widehat{Y}}) \\ R_1(\widehat{Y}, \overline{\widehat{Y}}) \end{pmatrix}, \end{aligned} \tag{67}$$

where

$$
\begin{aligned}
P_{lin} &= e^{2is}[\beta r_0\overline{\widehat{B_0}} + \frac{\varepsilon\gamma}{2}(r_0\overline{\widehat{B_1}} + r_1\overline{\widehat{B_0}})] \\
&\quad +[\beta r_0\widehat{B_0} + \frac{\varepsilon\gamma}{2}(r_0\widehat{B_1} + r_1\widehat{B_0})] \\
Q_{lin} &= e^{2is}[-r_0\overline{\widehat{B_0}} + \frac{\varepsilon\delta}{2}(r_0\overline{\widehat{B_1}} + r_1\overline{\widehat{B_0}})] \\
&\quad +[-r_0\widehat{B_0} + \frac{\varepsilon\delta}{2}(r_0\widehat{B_1} + r_1\widehat{B_0})],
\end{aligned}
$$

$$
\begin{aligned}
R_0(\widehat{Y}, \overline{\widehat{Y}}) &= \varepsilon^3 r_0 e^{is} P_{quad} + \varepsilon^3\widehat{B_0}(e^{-is}P_{lin} + P_{quad}) - i\varepsilon^7 g_0, \\
R_1(\widehat{Y}, \overline{\widehat{Y}}) &= -\varepsilon^2 r_0 e^{is} Q_{quad} - \varepsilon^2\widehat{B_0}(e^{-is}Q_{lin} + Q_{quad}) \\
&\quad +\varepsilon^3 r_1 e^{is} P_{quad} + \varepsilon^3\widehat{B_1}(e^{-is}P_{lin} + P_{quad}) - \varepsilon^6 g_1,
\end{aligned}
$$

with

$$
\begin{aligned}
Q_{quad} &= \widehat{B_0}\overline{\widehat{B_0}} + \frac{\varepsilon\delta}{2}(\widehat{B_0}\overline{\widehat{B_1}} + \widehat{B_1}\overline{\widehat{B_0}}) \\
P_{quad} &= \beta\widehat{B_0}\overline{\widehat{B_0}} + \frac{\varepsilon\gamma}{2}(\widehat{B_0}\overline{\widehat{B_1}} + \widehat{B_1}\overline{\widehat{B_0}}).
\end{aligned}
$$

Let us decompose

$$
\begin{pmatrix} \widehat{B_0} \\ \widehat{B_1} \end{pmatrix} = \widehat{y}\begin{pmatrix} r_1 e^{is} \\ -r_0 e^{is} \end{pmatrix} + \begin{pmatrix} \widetilde{B_0} \\ \widetilde{B_1} \end{pmatrix}
$$

where $\widetilde{B_0}$ and $\widetilde{B_1}$ have no Fourier component in e^{is}, and we take the component in e^{is} orthogonal to $V_0 e^{is}$, since adding a component proportional to (r_0, r_1) is equivalent to adapt (r_0, r_1).

We first solve (67) with respect to $(\widetilde{B_0}, \widetilde{B_1})$ in using the implicit function theorem, since we observe (notice the term $n\omega_0 = \frac{n}{2\varepsilon}(1 + \varepsilon^2 k_+)$ in the operator for a Fourier component e^{nis}), that the pseudo-inverse of L_0 is bounded from $H^1(\mathbb{R}/2\pi\mathbb{Z}) \times L^2(\mathbb{R}/2\pi\mathbb{Z})$ to $H^2(\mathbb{R}/2\pi\mathbb{Z}) \times H^1(\mathbb{R}/2\pi\mathbb{Z})$. Let us notice that the difference with the classical Hopf bifurcation proof is that, norms in these spaces are chosen as, for example

$$
||u||_{H^2} = \frac{1}{\varepsilon^2}||u''||_{L^2} + \frac{1}{\varepsilon}||u'||_{L^2} + ||u||_{L^2},
$$

and notice that $H^1(\mathbb{R}/2\pi\mathbb{Z})$ is an algebra. It results that we obtain an estimate such that

$$
||(\widetilde{B_0}, \widetilde{B_1})||_{H^2 \times H^1} \le c(\varepsilon^2|\widehat{y}| + \varepsilon^6).
$$

It then remains to solve the 2-dimensional system in $(\widehat{\omega}, \widehat{y})$ which is a real system, due to the reversibility symmetry:

$$
\begin{aligned}
\widehat{\omega} r_0 + \widehat{y} r_1 &= -\widehat{\omega}\widehat{y} r_1 + \mathcal{O}(\varepsilon^4|\widehat{y}| + \varepsilon^3|\widehat{y}| + \varepsilon^7) \\
\widehat{\omega} r_1 - \widehat{y} r_0 &= \widehat{\omega}\widehat{y} r_0 + \mathcal{O}(\varepsilon^3|\widehat{y}| + \varepsilon^2|\widehat{y}| + \varepsilon^6),
\end{aligned}
$$

which gives

$$\begin{aligned}\widehat{\omega} &= \mathcal{O}(\varepsilon^7)\\ \widehat{y} &= \mathcal{O}(\varepsilon^6).\end{aligned}$$

It results finally that the family of periodic solutions at M_+ are such that

$$\begin{aligned}B_0 &= r_0 e^{i\omega x} + \mathcal{O}(\varepsilon^6),\\ B_1 &= ir_1 e^{i\omega x} + \mathcal{O}(\varepsilon^6),\\ \omega &= \frac{1}{2\varepsilon} + \frac{\varepsilon k_+}{2} + \mathcal{O}(\varepsilon^7).\end{aligned} \tag{68}$$

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