"Minimal surfaces in symmetric spaces and Labourie's conjecture"

List of talks

Higher rank Teichmüller theory studies special connected components of the character variety of a surface group into a non-compact semi-simple Lie group G. Those components, called higher Teichmüller spaces, share many properties with the classical Teichmüller space, corresponding to the case $G = PSL(2, \mathbb{R})$. Since their discovery by Hitchin in 1992, the study of higher Teichmüller spaces have seen an explosion of interest from the mathematical community.

In the last fifteen year, a central focus has been a conjecture of Labourie. Given a representation ρ in a higher Teichmüller space for G, the conjecture predicts the existence of a unique minimal surface in the symmetric space of G that is preserved by ρ. This conjecture contrasts with the case of quasi-Fuchsian representations (which are not higher Teichmüller spaces) where uniqueness is know to fail in general. In many aspects, this conjecture can be thought of an analogue of the uniformization theorem in higher rank, and such a result would have important applications to the geometry of higher Teichmüller spaces (such as the existence of a natural complex structure and Kähler metrics). Special cases of this conjecture have been proved, relying either on pseudo-Riemannian geometry or affine geometry of Higgs bundle theory, and the conjecture holds when the rank of G equals 2. However, a remarkable development came in April 2021 from a preprint of Marković in which he proved the existence of a counter-example when $G = PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$.

We plan to have 9 talks during the week, on the following topics (more detailed description and references will be given to the participants later):

1 Introduction to minimal surfaces and harmonic maps.

This introductory talk should define the basic notions of harmonic maps, its relation to minimal surfaces and the existence theorems of Eells-Sampson and Corlette-Donaldson. The talk will focus on the case of surfaces and (if time permits), Wolf's parametrization of the Teichmüller space can be sketched.

References

[1] B. LOUSTAU, *Harmonic maps from Kähler manifolds*. Arxiv:2010.03545.

[2] J. EELLS, J. H. SAMPSON, *Harmonic mappings of Riemannian manifolds.* American Journal of Mathematics (1964).

[3] M. WOLF, *The Teichmüller theory of harmonic maps*. Journal of Differential Geometry, Vol. 29, 449–479 (1989).

2 Higher Teichmüller spaces and Labourie's conjecture.

This talk will be the first encounter with Labourie's conjecture. After defining higher Teichmüller spaces (mostly Hitchin components and maximal representations), Hitchin's parametrization using holomorphic differentials will be described (from the point of view of harmonic maps). The proof of the existence part of Labourie's conjecture will be given (assuming the well-displacing property of Anosov representations).

This talk aims at defining higher Teichmüller spaces (and specially Hitchin's components), and describe Hitchin's parametrization. Then, Labourie's conjecture will be stated and the proof of the existence part will be given.

References

[1] F. LABOURIE, *Cross Ratios, Anosov Representations and the Energy Functional on Teichmüller Space* **(Sections 5,6 and 8)**. Annales Scientifiques de l'ENS, IV (2008).

3 Quasi-Fuchsian manifolds $\big(G = \mathrm{PSL}(2, \mathbb{C})\big)$.

This talk will introduce quasi-Fuchsian representations and will study of minimal surfaces in $\mathbb{H}^3$ that are preserved by a quasi-Fuchsian representation. The main goal will be to present the examples of non-uniqueness constructed by Huang-Wang.

References

[1] Z. HUANG AND B. WANG, *Counting minimal surfaces in quasi-Fuchsian three-manifolds.* Trans. Am. Math. Soc., 367(9):6063–6083, 2015.

4 Marković's counter-example $\big(G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})\big)$.

In this talk, two recent papers of Marković will be presented. On the one hand, the uniqueness result for minimal surfaces in the product of any two closed Riemannian surfaces; on the other, the examples of non-uniqueness of minimal surfaces in the product of three hyperbolic surfaces, providing a counterexample to Labourie's conjecture in rank greater than two.

References

[1] V. MARKOVIC, *Uniqueness of minimal diffeomorphisms between surfaces*, To appear in Bulletin of the LMS. `http://people.maths.ox.ac.uk/~markovic/M-minimal.pdf`

[2] V. MARKOVIC, *Non-uniqueness of minimal surfaces in a product of closed Riemann surfaces*, To appear in GAFA. `http://people.maths.ox.ac.uk/~markovic/M-minimalac.pdf`

5 Affine spheres $\big(G = \mathrm{PSL}(3, \mathbb{R})\big)$.

This talk will describe the notion of affine spheres in $\mathbb{R}^3$ and explain its relations with minimal surfaces in the symmetric space of $\mathrm{PSL}(3, \mathbb{R})$. This gives a proof of the conjecture for Hitchin representations in $\mathrm{PSL}(3, \mathbb{R})$.

References

[1] Y. BENOIST AND D. HULIN, *Cubic differentials and finite volume convex projective surfaces.* Geometry and Topology 17 (2013) 595–620

[2] F. LABOURIE, *Flat Projective Structures on Surfaces and Cubic Holomorphic Differentials.* Pure and Applied Mathematics Quaterly 3 № 4, 1057–1099 (2007).

[3] J. LOFTIN, *Applications of affine differential geometry to* $\mathbb{RP}^2$ *surfaces.* PhD thesis
`https://sites.rutgers.edu/john-loftin/wp-content/uploads/sites/229/2019/08/thesis.pdf`

[4] J. LOFTIN, *Survey on affine spheres.* Ji, Lizhen (ed.) et al., Handbook of geometric analysis. No. 2. Somerville, MA: International Press; Beijing: Higher Education Press (ISBN 978-1-57146-204-6/pbk). Advanced Lectures in Mathematics (ALM) 13, 161-191 (2010).

6 Constant curvature surfaces in Minkowski space $\big(G = \mathrm{SO}_0(2, 1) \ltimes \mathbb{R}^3\big)$.

This talk will be the first glimpse to the pseudo-Riemannian aspects of Labourie's conjecture. The notion of K-surface in the Minkowski 3-space will be introduced and the existence of a unique K-surface preserved by an affine deformation of a Fuchsian representation will be proved. In the second part, a generalization to the setting of affine differential geometry – namely the existence result for surfaces of constant affine curvature preserved by an affine deformation of a Hitchin representations in $\mathrm{PSL}(3, \mathbb{R})$, originally due to Labourie – will be presented following the work of Nie-Seppi.

References

[1] T. BARBOT, F. BÉGUIN, AND A. ZEGHIB, *Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes: application to the Minkowski problem in the Minkowski space.* Ann. Inst. Fourier (Grenoble), 61(2):511–591, 2011.

[2] F. BONSANTE, A. SEPPI AND P. SMILLIE, *Entire surfaces of constant curvature in Minkowski 3-space.*, Math. Annalen (2019) 374: 1261-1309.

[3] X. NIE AND A. SEPPI, *Affine deformations of quasi-divisible convex cones*, Arxiv:2009.12311.

7 Maximal surfaces in anti-de Sitter space $\big(G = (PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})\big)$.

In this talk, the duality between minimal Lagrangian surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$ and maximal surfaces in the anti-de Sitter space of dimension 3 will be explained. Then a proof of the conjecture for $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ will be obtained, as an application of the existence and uniqueness result for maximal surfaces in Anti-de Sitter manifolds.

References

[1] F. BONSANTE AND A. SEPPI, *Anti-de Sitter geometry and Teichmüller theory* **(Sections 3,6 and 7)**. In the tradition of Thurston (to appear, V. Alberge, K. Oshika and A. Papadopoulos ed.). Springer Verlag, 2020.

[2] F. BONSANTE AND J.M SCHLENKER, *Maximal surfaces and the universal Teichmüller space.* Invent. Math. 182, No. 2, 279-333 (2010).

8 Maximal representations in rank 2 $\big(G = SO_0(2,n)\big)$.

This talk will describe the proof of Labourie's conjecture for maximal representations in rank 2 Hermitian Lie groups. The pseudo-hyperbolic space will be introduced and the duality between minimal surfaces in the symmetric space of G and maximal surfaces in the pseudo-hyperbolic space will be explored. This talk should also give a (gentle) introduction to Higgs bundles.

References

[1] B. COLLIER, N. THOLOZAN AND J. TOULISSE, *The geometry of maximal representations in* $SO(2,n)$. Duke Math. J., 168(15):2873-2949.

[2] J. TOULISSE, *Higgs bundles, pseudo-hyperbolic geometry and maximal representations* - Actes du séminaire TSG, Tome 34 arXiv:1709.06197.

9 Cyclic surfaces $\big(G = PSL(3,\mathbb{R}),\ PSp(4,\mathbb{R}),\ G_2\big)$.

In this talk, the notion of cyclic surface will be introduced, and Labourie's proof of his conjecture for Hitchin representations in rank 2 Lie groups will be explained (the talk can focus on the cases $G = PSp(4,\mathbb{R}),\ G_2)$ that greatly simplify the proofs).

References

[1] F. LABOURIE, *Cyclic surfaces and Hitchin components in rank 2.* Ann. of Math. (2) 185 , № 1, 1–58 (2017).