Self duality equations :

the continuity method

We want to show that in the context of $G = sl_n(\mathbb{C})$

$$\{\text{harmonic bundles}\}_{\text{irreducible}} \longrightarrow \{\text{Stable Higgs bundles}\}$$

injective surjective.

Given $\bar\partial, \phi$ with $\bar\partial\phi = 0$, we are looking for ∇, g so that

(i) $\bar\partial^{\nabla} = \bar\partial$

(ii) $\nabla g = 0$

(iii) $R^{\nabla} + [\phi, \phi^*] = 0$

To simplify, we shall assume that $d^\circ \mathcal{E} = 0$

I Preliminaries

proposition A

Assume there exists a solution for $(\bar\partial, \phi)$ of Hitchin self duality equations. Then $\mathcal{E}$ is polystable

◀ Same proof as in the case of Narasimhan – Seshadri ▶

proposition B

let Ψ be non zero from $(E_0, \partial_0, \phi_0)$ to $(E_1, \partial_1, \phi_1)$ assume that $(E_0, \partial_0, \phi_0)$ is stable and $(E_1, \partial_1, \phi_1)$ is polystable, with the same rk and degree 0. Then Ψ is an isomorphisms

◀ Assume first that $\ker\Psi = L_0$ and $\mathrm{Im}\Psi = L_1$ are bundles

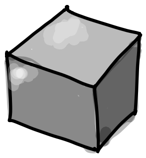
If Ψ is not an isomorphism then L_0 is a non trivial subbundle and thus

$\deg(L_0) < 0$, similarly $\deg(L_1) \leq 0$,

Thus φ is an isomorphism from

$F = E/L_0$ to L_1 which have degrees of different sign

$\rightsquigarrow$ a contradiction : they cannot be isomorphic.



In general the same ideas work replacing bundles by sheaves, free $\mathcal{O}_X$ modules, notions which are equivalent for curves ▶

proposition C [Actually a characterization of stability]

let $(\partial_n, \phi_n) \to (\partial_0, \phi_0)$ on E_0, let $(\hat{\partial}_n, \hat{\phi}_n) \to (\partial_1, \phi_1)$ on E_1

Assume that there exists isomorphisms φ_n so that

(i) $\varphi_n^* \hat{\partial}_n = \partial_n$

(ii) $\hat{\phi}_n \varphi_n = \varphi_n \phi_n$

Assume that (∂_0, ϕ_0) is polystable, (∂_1, ϕ_1) is stable.

Then there exists an isomorphism φ so that

(i) $\varphi^* \partial_1 = \partial_0$

(ii) $\phi_1 \varphi = \varphi \phi_0$

◀ let us take some auxiliary metrics on E_1 and E_0 and normalize φ_n (by a multiplicative constant) so that

$$\int \|\varphi_n\|^2 d\mu \equiv 1$$

It follows that we can extract a subsequence $\varphi_n \to \varphi$, with $\varphi^* \partial_1 = \partial_0$

It is enough to know this property for holomorphic functions :

« If $f_n : D \to \mathbb{C}$ is bounded in L^2, there exists a subsequence which converges C^∞ on every compact of D [use Cauchy integral formula + weak convergence] »

It follows that $\phi_1 \circ \varphi = \varphi \circ \phi_0$ since φ is non zero, it follows that φ is an isomorphisms by the proposition above ▶

Exercise: translate the above lemma in a statement about the closure of the orbit of the group of automorphisms.

II Setting up the problem:

$$\mathcal{M} = \{D, g \,;\, D \text{ flat}\} \xrightarrow{\quad\Psi\quad} \{(\bar\partial, \phi),\ \phi \in \Omega^{0,1}(\mathrm{End}(E))\} = \mathcal{N}$$

D of class C^k, g of class C^{k+1} — Banach manifold

every thing of class C^k = Banach manifold

$$(D, g) \longmapsto (\bar\partial^{\nabla}, D_g^{\#}) \quad \text{where} \quad D_g = g(D_g^{\#}\cdot, \cdot)$$
$$\nabla = D - D_g^{\#}$$

$\cup$ $\cup$

$\mathcal{H}$ = {Harmonic bundles}   {stable holomorphic bundles} = $\mathcal{B}$

We want to use the continuity method.

Step 1: Ψ is open from $\mathcal{M}$ to $\mathcal{N}$
Step 2: Ψ is proper from $\mathcal{H}$ to $\mathcal{B}$
$\Rightarrow$ Ψ is surjective and a local homeo.

Step 3: $\mathcal{B}$ is connected and Ψ is injective (at a point)

We will only show step 1 and step 2, and admit the (non trivial) step 3

I Openness: Our first goal is to show that Ψ is open

1. Step linearizing the equations

Let $\nabla^0, g^0, \bar\partial_0, \phi^0$ be satisfying (i), (ii), (iii)

Let now $\nabla^t, g^t, \bar\partial_t, \phi^t$ also satisfying (i) (ii) (iii)

and $A = \frac{d}{dt}(\nabla^t - \nabla^0)$, $h = \frac{d}{dt} G_t$ where $g(G_t u, v) = g_t(u, v)$

$B = \frac{d}{dt}(\bar\partial_t - \bar\partial_0)$; $\varphi = \frac{d}{dt}\varphi_t$. We end up with the equations.

$$\text{(i)} \quad d^\nabla A + \overbrace{[\varphi, \phi^*] + [\phi, \varphi^*]}^{\text{hermitian}} - \overbrace{[\phi, [h, \phi^*]]} = 0$$

$$\text{(ii)} \quad A^{0,1} = B$$

$$\text{(iii)} \quad A + A^* = d^\nabla h$$

explanation for , If $g_t(Wu, v) = g_t(u, W_t^* v)$

$\Rightarrow g(G_t W u, v) = g_t(G_t u, W_t^* v)$, thus $W_t^* = (G_t W_u G_t^{-1})^* = G_t^{-1} W^* G_t$

We can write this as an equation in h

$$R(h) := d^\nabla (d^\nabla h)^{1,0} - [\phi, [h, \phi^*]] = -[\varphi, \phi^*] - [\phi, \varphi^*] - d^\nabla(B + B^*) = Z(\varphi, B)$$

◀ One just writes $A = A^{1,0} + B$, $A^{1,0} + B^* = d^\nabla h^{1,0}$

thus $A = d^\nabla h^{1,0} + B + B^*$ ▶

The openness of Ψ is a consequence of the implicit function theorem in Banach spaces and the following Lemma :

Proposition : [Existence of the inverse of $T\Psi$] .

Given φ, B then there exists a unique h so that $R(h) = Z(\varphi, B)$

Moreover $\|h\|_{k+1} \leqslant A(\|\varphi\|_{C^k}, \|B\|_{C^k})$

lemma 1 If $R(h)=0$, then $h=0$

◀ $d^\nabla (d^\nabla h)^{0,1} = [\Phi,[h,\Phi^*]]$.

Bochner technique

$$\langle d^\nabla (d^\nabla h)^{0,1} | h\rangle = + \langle [h,\phi^*], [h,\phi^*]\rangle$$

$\|$

$$d(\langle d^\nabla h^{1,0} | h\rangle) - \langle d^\nabla h^{1,0}, d^\nabla h^{0,1}\rangle$$

thus $d(\langle d^\nabla h^{0,1} | h\rangle) = \langle [h,\phi^*], [h,\phi^*]\rangle + \langle d^\nabla h^{0,1}, d^\nabla h^{0,1}\rangle$

integrating we get

(i) $(d^\nabla h)^{1,0} = 0$, thus h is anti - holomorphic

(ii) $[h,\phi]=0$

Since $h = h^*$, h is also holomorphic. Hence $\nabla h = 0$, the eigenspacs E_i of h are then holomorphic subbundles.

write $E = \oplus E_i$, where E_i are stable par ϕ. Since (E,ϕ) is stable, then all $E_i = E$, thus $h = \lambda \mathrm{Id}$, since $\mathrm{tr}(h) = 0$. We get that $h=0$ ▶

lemma 2 : R is formally self adjoint

$$\int \langle R(h) | w\rangle = \int \langle h | R(w)\rangle$$

◀ Exercise in Stokes formula ▶

lemma 3 . Assume R is formally self adjoint and R is injective (and of the forme $\Delta +$ (lower order) in coordinates)

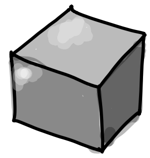
Then R has an inverse which is continuous

$$C^{k-1} \longrightarrow C^{k+1}$$

◀ True in finite dimensions, true for $\Delta : C^{k+1}(M) \to C^{k+1}(M)$

(for functions of zero integrals)

The result extends to this general case ▶



This is again a black box relying on Schauder estimates : that is

If indeed $R(h) = w$ then $\|h\|_{C^{k+2}} \leqslant K \|w\|_{C^k}$

II Properness

Goal : show that if

$(\partial_n, \phi_n) \longrightarrow (\partial_0, \phi_0)$

If (D_n, ∇_n, g_n) is a solution of H.S.D for (∂_n, ϕ_n)

then (D_n, ∇_n, g_n) subconverges to a solution of HSD for (∇_0, g_0)

We will denote by η_n the associated flat representations, and f_n the associated harmonic mappings. , ρ_n the transpose w.r to g_n.

Step 1 ; [a priori bounds]

there exists some bound B, only depending on ϕ so that

$$\int \mathrm{Tr}(\phi_n\, \rho_n(\phi_n))\, d\mu \leqslant B$$

[Actually B only depends on the eigenvalues of ϕ_n]

◀ We use the Böchner-Weitzenbock technique, let choose the hyperbolic metric on X

We first prove that $K\int \|\Phi\|^2 \geq \int \|[\Phi,\Phi^*]\|^2$, where K only depend on X

$$0 \leqslant \int_X \langle d^{\nabla^1}\phi \mid d^{\nabla^1}\phi\rangle \, d\mu \qquad \nabla^1 \text{ connection on } \Omega^1(X, \mathrm{End}E)$$

(Stokes formula) $$= -\int \langle R^{\nabla^1}\phi \mid \phi\rangle \, d\mu \quad (\text{use } d^{\nabla'} \circ d^{\nabla'} = R^{\nabla'})$$

$$= \int \langle -R^{\nabla}(\phi)\mid\phi\rangle + \int \langle \phi(R^{\nabla^0})\cdot\phi\rangle \, d\mu \quad \nabla = (g\text{-connection of } \mathrm{End}E)$$

$\nabla^0 =$ connection on X

(use HSD)
$$= \int \langle [[\phi,\phi^*],\phi] \mid \phi\rangle + K_1 \int \langle \phi \mid \phi\rangle \, d\mu \quad (\text{if I choose } R^{\nabla^0} \text{ of constant curvature} \cdot K_1]$$

Jacobi identity
$$= \int \langle [\phi,\phi^*] \mid [\phi,\phi^*]\rangle + K_1 \int \langle \phi\mid\phi\rangle \, d\mu$$

Thus we have obtained that

$$K_1 \|\phi\|^2 \geqslant \|[\phi,\phi^*]\|^2$$

But we have the following Inequality on matrices:

$$\|[A,A^*]\|^2 \geqslant K_2\|A\|^4 - \sum |\lambda_i(A)|^4$$

eigenvalus of A, only depends on the consjugacy class of A

■ the function $A \mapsto \|[A,A^*]\|^2 + \sum |\lambda_i(A)|^4$

is always >0 on $\|A\|=1$; (if $[A,A^*]=0$ then A is normal and $\sum|\lambda_i(A)|^4 = \|A^2\|$)

thus the result follows by homogeneity ■

Thus it follows that provided $\det(t-\phi(z))$ stays bounded

$$\int \|\phi\|^4 \leqslant K_3 \int \|\phi\|^2 + K_3$$

$$\mathrm{Area}(X) \int \|\phi\|^4 \geqslant \left(\int \|\phi\|^2\right)^2 \quad \text{by Cauchy-Schwarz}$$

Thus: $\int \|\phi\|^2 d\mu \leqslant K_6$ ▶

Step 2 for the corresponding harmonic mapping

$\exists B$ only depending on ϕ so that $E(f_n) \leqslant B_2$

◀ $Tf = \phi + \phi^*$

$e(f) = Tr(\phi^2) + Tr(\phi^{*2}) + Tr(\phi\phi^*)$

thus $E(f) \leqslant \int Tr(\phi\phi^*) d\mu + \int Re(Tr(\phi^2)) d\mu \leqslant B_2$ ▶

Step 3 : If f is a harmonic mapping, $\exists K$ only depending on X so that $\|e(f)\|^2 \leqslant K E(f)$

Moreover, all $\| f \|_{C^k} \leqslant K E(f)$

◀ See corollary at the end of previous lecture

the second part follows from Schauder estimates ▶

Step 5 . There exists a bundle E, on which you have (D_n, g_n) converges

And isomorphisms Ψ_n , $F \to E$ so that $(\Psi_n)_* (D_n, g_n)$

are solutions of the self duality equations for ϕ_n

◀ let us consider the universal cover of X and

consider g_n as maps from $\tilde{X} \to Sym(g)$

they are uniformly lipschitz and thus

$$d(g_n(x), \rho_n(\gamma) g_n(x)) \leqslant d(g_n(x), g_n(\gamma x)) \leqslant K(\gamma).$$

let us now consider $E \times \tilde{X}$ the trivial bundle on X

Then the family of harmonic maps

$$H_n \circ g_n \xrightarrow[C^\infty]{} \text{ to } g_\infty$$

Where $H_n \in G$ is so that

$$H_n g_n(x_0) \text{ is constant} := z_0$$

then $H_n \circ g_n$ is equivariant w.r to $\rho'_n = H_n \rho_n H_n^{-1}$

and $d(\rho_n(\gamma) z_0, z_0) \leqslant K$

For any $z_0 \in \mathrm{Sym}(G)$, $h \mapsto d(h z_0, z_0)$ is proper on G

$\Rightarrow$ implies that given S a finite set in $\pi_1(S)$

may extract a subsequence of ρ'_n so that

$$\rho'_n(\gamma) \mapsto \gamma_\infty =: \rho_\infty(\gamma)$$

since $\pi_1(S)$ is finitely presented

$$\rho'_n \longrightarrow \rho_\infty, \text{ Hence } D'_n \to D_\infty$$

since $g'_n \xrightarrow{C^\infty} g_\infty$ (using the previous step)

the result follows ▶

Conclusion : we can finally conclude the proof of the properness by using Proposition C