Parabolic Anosov representations.

I Parabolic subgroup for $Sl_n(\mathbb{K})$ $[\mathbb{K} = \mathbb{R}$ or $\mathbb{C}]$

(i) a **flag** in V, $\mathbb{K}$ vector space of dim k is

$\mathcal{F} = (E_0, \dots, E_p)$ where ; $E_{i+1} \subsetneq E_i \subsetneq V$

up to the action of $Sl(V)$, a flag is complementary determined

by $(n_1, \dots, n_p)$ $n_1 + \dots + n_p = \dim E$ $n_i = \dim(E_{i-1}/E_i)$ (where $E_{-1} = V$)

(ii) A **full flag** corresponds to the partition $(1, 1, \dots, 1)$

The flag manifold $\mathcal{F}_{(n)}$, is the space of flags corresponding to the partition (n)

(iii) A **parabolic subgroup** is the stabilizer of a flag

proposition: If N normalizes a parabolic subgroup P, then $N \subset P$

(iv) The **minimal parabolic subgroup (or Borel)** is the stabilizer of a full flag.

Obviously $\mathcal{F} \approx G/P$ $(G = Sl_n(\mathbb{K})$; P parabolic$)$

(v) A flag $F_0 \supset \dots \supset F_p$ is **transverse** to $E_0 \supset \dots \supset E_p$ if

$$E_i \oplus F_{n-p} = V$$

Prop: Two transverse flags $F \pitchfork E$, equivalent to a decomposition.

$$V = V_0 \oplus \dots \oplus V_p \text{ where } \dim V_i = n_i$$

$$E_j = V_{p-j} \oplus \dots \oplus V_p ; \quad F_j = V_0 \oplus \dots \oplus V_j$$

Observe that two $\pitchfork$ flags may or may not be conjugated.

If $Q \pitchfork P$ and $(n_0, \dots, n_p)$ is the partition associated to P, the partition associated to Q is the opposite partition $(n_p, \dots, n_0)$. A parabolic with the opposite partition is called an opposite parabolic.

(v) We have a $P \xrightarrow{\pi} \prod_i GL(V_i / V_{i-1}) \quad (V_{-1} = V) \quad := L$ Levi part of P

The unipotent radical U of P is $U = \ker(\pi)$

let $Q \pitchfork P$, let $\mathcal{F}_P = G \cdot P$; let U be the unipotent radical of Q

then $u \longmapsto u \cdot P$ is a bijection between

U and $\mathcal{B}_Q = \{ P' \mid P' \pitchfork Q \} :=$ Big bruhat cell

Given P, Q opposite parabolic the set

prop $\quad \mathcal{A}_{PQ} = \{ E \in \mathcal{F}_P, F \in \mathcal{F}_Q \mid P' \pitchfork Q' \}$ is a G-orbit

and the stabilizer of $(E, F) \simeq L : \prod_i GL(V_i)$

$\mathcal{F}_P$ is compact, $P = \text{Normalizer}(U) = \text{Normalizer of}(E)$

(vi) let A be a diagonal matrix with real eigenvalues $\lambda_0 > \cdots > \lambda_p$

$\rightsquigarrow$ a decomposition $V_0 \oplus \cdots \oplus V_p$ where $V_i = V_{\lambda_i}$

$\Rightarrow \pitchfork$ flags as above

$\Rightarrow$

$$P^+ = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & & * \end{pmatrix} ; \quad U^+ = \begin{pmatrix} 1 & * & * & x \\ & \boxed{1} & * & x \\ & & \boxed{1} & x \\ & & & * \end{pmatrix} ; \quad L = \begin{pmatrix} * & & \\ & \boxed{*} & \\ & & \boxed{*} \end{pmatrix} = Z(A)$$

$$P^- = \begin{pmatrix} * & & & \\ * & * & & \\ * & * & * & \\ * & * & * & * & * \end{pmatrix} ; \quad U^- = \begin{pmatrix} 1 & & & & \\ * & \boxed{1} & & & \\ * & * & * & \ddots & \\ * & * & * & * & \boxed{1} \end{pmatrix}$$

At the lie algebra level

$$\underline{U}^+ = \{ u \mid \lim_{n \to \infty} Ad(A^{-n}) u = 0 \}$$

proposition

A acts on $\mathcal{F}_P$. Its action has a unique attractive fixed point P^+, the Basin of attraction A is the big bruhat cell $U^- \cdot P^+$

II P- Anosov representations

Let P be a parabolic subgroup (i.e a partition $n_0 + \cdots + n_p = \dim V$)

A representation $\pi_1(S) \longrightarrow GL(V)$ is P- Anosov if there exists a decomposition of the associated bundle

$$\mathcal{V} = \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_p \text{ where } n_i = \mathrm{rk}(\mathcal{V}_i)$$

and $\mathcal{V}_0 > \mathcal{V}_1 > \cdots > \mathcal{V}_p$

Let $\mathcal{F}^+$ be the flag bundle whose fibers correspond to the flag with the partition $(n_0, \dots, n_p)$; and $\mathcal{F}^-$ with the opposite partition.

Observe that φ_t lifts by $/\!/^{ism}$ to actions ϕ_t^+ and ϕ_t^- on $\mathcal{F}^+$ and $\mathcal{F}^-$

Then the above decompositions give sections σ^+ and σ^- of $\mathcal{F}^+$ and $\mathcal{F}^-$

Prop : σ^+ and σ^- are attractive (resp.) repulsive fixed sections of $\mathcal{F}^+$ and $\mathcal{F}^-$

$\sigma^+ \pitchfork \sigma^-$

From the definition, σ^- lift to a map $\xi^- : U\mathbb{H}^2 \longrightarrow F^-$, constant along the flow and stable leaves, and thus to a map

$\xi^- : \partial_\infty \pi_1(S) \longrightarrow F^-$, called the repulsive limit map

$\xi^+ : \partial_\infty \pi_1(S) \rightarrow F^+$ called the attractive limit map

The limit maps are both ρ- equivariant and furthermore

If $x \neq y$; $\xi^+(x) \pitchfork \xi^-(y)$

Theorem (structural stability)

$\{$P- Anosov representations$\}$ is an open subset of the space of all representations.

Theorem (discreteness) [later]

Every Anosov representation is discrete and faithful.

III First examples, and Borel Anosov representations

A Fuchsian representation is Anosov in $SL_2(\mathbb{R})$

let η : representation of $SL_2(\mathbb{R})$ in G.

Proposition : $\eta \circ \rho$ is P-anosov where P is the parabolic subgroup associated to $\eta\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$.

◀ This is a consequence of the Weyl Chamber flow corollary of the previous lecture ▶

Examples

Proposition . All Fuchsian representation in the Hitchin component are Borel Anosov.

let ρ be a Borel Anosov reprentation (ie the associated flag is a full flag). $\rho : \Gamma \to SL(V)$ The limit map is a map

$$\partial_\infty \Gamma : \longrightarrow \text{Full flag manifold}$$
$$x \longmapsto (\xi(x) = (\xi_1(x), \dots\dots, \xi_{p-1}(\xi)) \quad ; \; p = \dim V$$

where $\dim(\xi_i(x)) = 1$

Proposition : Assume ρ is a Borel Anosov representation then

(i) $\forall \gamma \in \Gamma$, $\rho(\gamma)$ is $\mathbb{R}$-split with distinct eigenvalues in absolute value.

(ii) ρ is discrete (Assuming $\overline{\Gamma}^{\text{Zariski}}$ is simple)

We will later show that all Anosov representations are discrete.

◀ let $\xi^{\pm} : \partial_\infty \Gamma \to \mathcal{F}_B$ be the two limit maps.

By construction $\xi^+(\gamma^+) \pitchfork \xi^-(\gamma^-)$ and we have a decomposition along the closed orbit of γ

$V = \Sigma_1^l L_i$; $\rho(\gamma) L_i = L_i$, thus $\rho(\gamma)$ is $\mathbb{R}$-split ; moreover

$i \geqslant j \Rightarrow L_i > L_j$ thus $\lim_{n\to\infty} \left(\frac{\rho(\gamma)^n v_j}{\rho(\gamma)^n v_i} \right) = 0$

$$\| \quad \lim_{n\to\infty} \left(\frac{\lambda_j}{\lambda_i} \right)^n \left(\frac{\|v_j\|}{\|v_i\|} \right)$$ thus $|\lambda_j| < |\lambda_i|$

Now every Γ, zariski dense in a simple group is either discrete or dense : indeed h = lie algebra of $\overline{\Gamma}$ is an ideal .

But Γ is not dense : a rotation cannot be approximate by matrics with real eigenvalues. Thus Γ is discrete ▶