Orbit equivalence, periods and reparametrisation

I Orbit equivalence and conjugacy

Let $\varphi_t \circlearrowright M$, and $\psi_t \circlearrowright M$. An orbit equivalence is a Homeomorphism H so that there exists a continuous increasing function $s : t \to s(t)$

$$H\varphi_t = \psi_{s(t)} H$$

Ex a Hölder parametrisation is so that the identity is an orbit equivalence.
An equivalence, – or conjugacy – is a Homeomorphism H so that

$$\psi_{s(t)} = H^{-1}\varphi_t H$$

Remark : both equivalences preserves closed orbit, a conjugacy preserve the period or length of the closed orbit

Proposition

(i) If Φ_t is a lift of φ_t. If H is an orbit equivalence between φ_t and ψ_s, $\psi_{s(t)} = \varphi_t$; then $\Psi_{s(t)} := H^*\Phi_t$ is a lift of φ_t

(ii) If $L \longrightarrow M$, equipped is contracting w.r to Φ_t, then H^*L is contracting w. respect to Ψ

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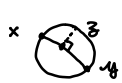
Proposition all hyperbolic geodesic flows on surfaces are orbit equivalent

◀ $UH^2 \xrightarrow{\theta_e} \{(x,y,z) \text{ cyclically ordered in } \partial_\infty H^2\} \xrightarrow{\psi_e} \{(x,y,z) \text{ cyclically } \partial_\infty \Gamma\}$



the orbit equivalence is $H\ \theta_{\rho_0}^{-1}\psi_{\rho_0}^{-1}\psi_{\rho_1}\theta_{\rho_1}$. ▶

Corollary : To be an Anosov representation is independant of the choice of the hyperbolic structure on S.

II Reparametrisation,

let $\{\varphi_t\}$ be a flow on M, and f a C^0 function > 0.

$\rightsquigarrow$ a new flow on M : ψ_s,

$$\psi_s(x) := \varphi_t(x) \quad \text{where} \quad s = \int_0^t f(\varphi_u(x))\,du$$

Proposition : ψ_s is a flow, if ℓ_γ is the length of a closed orbit γ of φ the length with respect to ψ is $\int_\gamma f\,dt$.

$$\blacktriangleleft \quad \psi_{s+\lambda}(x) = \varphi_T(x) \quad \text{with} \quad s+\lambda = \int^T f\circ\varphi_u(x)\,du$$

let t_1 so that $\int_0^{t_1} f\circ\varphi_u(x)\,dx = s$

then $\lambda = \int_{t_1}^T f\circ\varphi_u(x)dx = \int_0^{T-t_1} f\circ\varphi_u(\varphi_{t_1}(x))\,du$

Thus

$$\psi_s(x) = \varphi_{t_1}(x)$$

$$\psi_\lambda(\psi_s(x)) = \psi_\lambda(\varphi_{t_1}(x)) = \varphi_{T-t_1}(\varphi_{t_1}(x)) = \varphi_T(x) = \psi_{s+\lambda}(x).$$

Thus ψ is a flow : $\psi_{\lambda+s} = \psi_\lambda \circ \psi_s$

$\varphi_{\ell_\gamma}(x) = x$; thus $\psi_T(x) = x$ where $T = \int_0^{\ell_\gamma} f\circ\varphi_u\,du$ $\blacktriangleright$

Corollary :

(i) if ψ_s is a Hölder reparametrisation of φ ; then there exists $K > 0$ so that $\frac{1}{K}\ell_\gamma^\varphi \leqslant \ell_\gamma^\psi \leqslant K\ell_\gamma^\varphi$

(iii) φ is a reparametrisation of ψ using $g = \frac{1}{f}$.

Remark : the orbit equivalence for hyperbolic surface $\rightsquigarrow$ Hölder reparametrisation.

$\blacktriangleleft$ $K = \sup(f, \frac{1}{f})$;

(iii) ψ_s is a reparametrisation by f of φ if and only $\psi_s(x) = \varphi_{g(s,x)}(x)$; $\frac{d}{ds}\Big|_{s=0} g(s,x) = f(x)$

fixing x : $s \mapsto g(s,x) = \int_0^s f \circ \varphi_u \, du$ is a strictly monotone function, thus there exists h so that

$$g(h(s,x), x) = s$$

Thus $\psi_{h(s,x)}(x) = \varphi_s(x)$

by construction $1 = \frac{d}{ds'}(g) \cdot \frac{d}{ds} h$, $\hat{f} = \frac{1}{f}$ ▶

III Coboundary and conjugacy

A function f is a coboundary for the flow if there exists g

$$\forall t \quad \int_0^t f \circ \varphi_u(x) \, dx = g(\varphi_t(x)) - g(x)$$

Or equivalently

$$f(x) = \frac{d}{ds}\Big|_{s=0} g \circ \varphi_s(x)$$

Two functions are cohomologuous if they difer by a coboundary

Proposition :

If f is cohomologuous to 1 then

$\Leftrightarrow$ ψ and φ are conjugated : there exists H so that

$$H \circ \psi_t = \varphi_t \circ H$$

◀ Thus there exist so that

$$h(\varphi_t(x)) - h(x) = \int_0^t f \circ \varphi_u(x) \, du \; - t$$

Then $\psi_s = \varphi_t$ where , $s = \int_0^t f \circ \varphi_u(x) \, du$

Thus $s - t = h(\varphi_t(x)) - h(x)$ let $H(x) = \psi_{h(x)}(x)$

$\psi_t \circ H = \psi_{t+h(x)}(x)$; $H \circ \varphi_t(x) = \psi_{(h \circ \varphi_t(x))}(\varphi_t(x))$

$$= \psi_{h \circ \varphi_t(x) + s}(x)$$

this concludes the proof ▶

IV periods and conjugacy : Livšic theorem

The following is true for any Anosov flow. But we state it for the geodesic flow of a hyperbolic surface

$$\int_\gamma f \, d\mu_\gamma := \int^{\ell_\gamma} f \circ \varphi_t(x) dt \quad , \text{ where } \varphi_{\ell(\gamma)}(x) = x$$

Livšic theorem

Let h be a Hölder function. Assume that for all closed geodesic

$$\int_\gamma h \, d\mu_\gamma = 0$$

then there exists a function g so that $g(\varphi_t(x)) - g(x) = \int_0^t h \circ \varphi_u(x) du$

Observe that the condition is necessary :

Corollary . Let ψ_t be a Hölder reparametrisation of φ_t so that the closed orbits have the same length.
Then there exist an homeomorphism θ so that $\theta \circ \psi_t = \varphi_t \circ \theta$

Indication of the proof of the Livšic theorem

let f ; let $S_t^f(x) = \int_0^t f \circ \varphi_u(x) du$

Fact 1 There exists, $K, \alpha > 0$ so that given x_1, x_2 in $U\mathbb{H}^2$ and $\varepsilon > 0$ so to that $d(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon$, $\forall t \; |t| < T$
Then $d(\gamma_1(t), \gamma_1(t)) < \varepsilon \left(e^{-\alpha(T - |t|)}\right)$

« consequence of the fact that geodesics are exponentially diverging »

Corollary A : let f be a Hölder function, for all η, there exists $\varepsilon > 0$ so that

if $d(\varphi_t(x_1), \varphi_t(x_2)) \leqslant \varepsilon \quad \forall t \in [0,T]$

Then $\quad |S_T(f)(x_1) - S_T(f)(x_2)| \leqslant \eta$.

Fact 2 : the geodesic flow has a dense orbit (in the future). $t \mapsto \varphi_t(x_0)$

Fact 3: Closing lemma $\quad \exists\, \varepsilon, K, T_0 > 0$ so that if

$d(\varphi_T(z), z) \leqslant \varepsilon$, with $T \geqslant T_0$

Then there exists y, T' so that $\varphi_{T'}(y) = y$; $|T' - T| \leqslant K\varepsilon$

and $\forall\, t \in [0,T'] \quad d(\varphi_t(z), \varphi_t(y)) < K\varepsilon$

« A consequence of the density of (γ^+, γ^-) in $\partial_\infty \Gamma$ »

let t_n so that $\varphi_{t_n}(x_0) \longrightarrow x$; $t_n \to \infty$.

let q_n and p_n so that $|t_{q_n} - t_{p_n}| > T_0$, $p_n, q_n \to +\infty$

it follows from corollary A, the closing lemma and the hypothesis ($S^f_T(y) = 0$ if $\varphi_T(y) = y$) that

$$\left| S^f_{t_{p_n}}(x_0) - S^f_{t_{q_n}}(x_0) \right| \longrightarrow 0$$

It follows that

i) when $n \to \infty \quad S^f_{t_n}(x_0) \longrightarrow$

ii) let $g(x) = \lim_{t \to \infty} S^f_{t_n}(x_0)$

(iii) the same argument shows that g is continuous

Finally if $\quad g(x) = \lim_{n\to\infty} \left(S^t_{t_n}(x_0) \right) = \lim_{n \to \infty} \int_0^{t_n} f \circ \varphi_u(x_0)\, du$

$g(\varphi_s(x)) = \lim_{n\to\infty} \int_0^{t_n + s} f \circ \varphi_u(du)$

Thus $g(\varphi_s(x)) - g(x) = \int_0^s f \circ \varphi_u(x)\, du$. The Livšic theorem follows ▶

follows

Corollary: Two reparametrisation map have the same period $\Leftrightarrow$ they are cohomologuous.

V Reparametrisation and contracting line bundles

$\{$contracting lines$\}$ $\longleftrightarrow$ $\{$cohomology class of reparametrisation$\}$

① let f be a reparametrisation. let L be the trivial bundle over $M \circlearrowleft \varphi_t$

then $\Phi_t^f(x,u) = \left(\varphi_t(x), u \cdot \exp\left(-\int_0^t f\circ\varphi_u(x)\,dx\right)\right)$

is a contracting line bundle.

If f and g are cohomologuous, there exists a isomorphism H of L

so that $H\circ\Phi_t^f(x,u) = \Phi_t^g(x,u)\circ H$

◀ $\int_0^t f\circ\varphi_u(x)\,du - \int_0^t g\circ\varphi_u(x)\,dx = h(\varphi_t(x)) - h(x)$

Thus we take $H(x,u) = \left(x, u\cdot e^{h(x)}\right)$ ▶

② let Φ_t be a contradicting bundle map on the trivial bundle L, then there exists f so that $\Phi_t = \Phi_t^f$

Step 1 : there exists a metric on L so that $\forall\, t>0$ $\|\Phi_t^*(u)\| < \|u\|$

◀ let g be the trivial metric, since Φ_t is contracting it follows that $\Phi_t^* g \leqslant A e^{-t} g$

thus $g_t = \int_0^t \Phi_u^*(g)\,du \longrightarrow g_\infty = \int_0^\infty \Phi_u^*(g)\,du$

By definition $\Phi_t^* g_\infty = \int_t^\infty \Phi_u^*(g)\,du < g_\infty$. ▶

let us use such a metric on L to trivialize it. Then $\Phi_t(x,u) = (\varphi_t(x), u\exp(-H(t,x)))$

with $H(t,x) > 0$. Thus $f(x) = \frac{d}{dt}H(t,x)$ is a reparametrisation and $\Phi = \Phi^f$ ▶

Corollary of the proof, Let L be a contracting line bundle, then there exists a metric g on L, a reparametrisation of the flow so that

$$\Phi_t^*(g) = e^{-t} g.$$