non abelian hodge correspondance

I. Harmonic bundle :

let F be a vector space, $\mathrm{Met}(F) = \{$positive definite quadratic forms on $F\}$

is a Riemannian manifold : $T_g(\mathrm{Met}(F)) = \{A \in \mathrm{End}(F),$ symmetric w.r to $g\}$

$$\langle A, B \rangle_g := \mathrm{Tr}(AB)$$

A harmonic metric on a flat bundle F is a harmonic section of

$$\mathrm{Met}(F) \to M$$

A harmonic bundle (E, ∇, g) is a flat bundle (E, ∇) together with

a harmonic metric g

II. harmonic bundles and representations.

$$\{\text{Harmonic bundles}\} \to \mathrm{Rep}(\pi_1(S), GL_n(\mathbb{R}))$$

Theorem (Corlette - Donaldson) let M be a complete Riemannian

let $\rho : \pi_1(M) \to GL_n(\mathbb{R})$, assume $\overline{\rho(\pi_1(S))}^Z$ is reductive

then there exists an (essentially) unique harmonic metric on E_ρ

2 essentially unique $g_1 = g_2(A\cdot, A\cdot)$ with $D^e A = 0$

Next goal, relate harmonic bundles to Higgs bundles, and

extend the theory to other semi-simple G

↪ crash course on symmetric spaces.

III Harmonic bundles and Higgs bundles

A| From Harmonic Bundles to Higgs Bundles

Let (E, D, g) be a harmonic bundle over X; where E is complex and g hermitian, we consider g as a section of $Met(E)$

a) the holomorphic bundle $\bar{E}$ carries a flat connection D and a metric g

then we get a unitary metric ∇ charaterized by

$$\nabla = D - \frac{1}{2} D_g^{\#}$$

where $\quad Dg(u,v) = g(D_g^{\#}(u), v)$ (and $D_g^{\#}$ hermitian)

⚠ the holomorphic structure does not come from the flat connection

b) the Higgs field (ϕ)

$$g^*(T\,Met(E)) = \{ \text{hermitian endomorphism w.r to } g \} =$$
$$= \{ A \mid g(Au, v) = g(u, Av) \}$$
$$= \{ A \mid A = A^* \} \qquad (\text{for matrices } A^* = {}^t\bar{A})$$

<u>Fact</u>: let g_0 be a hermition metric on a complex vector space E

then $\{\text{Hermitian}\}_{\otimes \mathbb{C}} = End(E)$

◀ If A is hermitian $\Leftrightarrow$ JA is antihermitian ▶

Thus g harmonic $\Rightarrow$ $Tg^{(1,0)} \in \Omega^{1,0}(X, End(E))$ is holomorphic

The Higgs bundle $(\mathcal{E}, \phi)$ associated to the harmonic bundle (E, D, g) is $\mathcal{E} = (E, \bar{\partial}^{\nabla})$; $\phi = (T_g^{\vee})^{1,0}$

a last remark

(Actually $T_g^{\vee} = \frac{1}{2}(Dg)^{\#}$) ! thus $\frac{1}{2}D\underline{g}^{\#} = \phi + \phi^{\times}$

and $$D = \nabla + \phi + \phi^{*}$$

B| Hitchin self duality equations or the inverse construction

Starting from the holomorphic bundle we need to reconstruct the metric, the unitary connection is always obtained thanks to the holomorphic structure.

proposition let g be a hermitian metric on a holomorphic bundle, then there exists a unique hermitian connection ∇ so that $\bar{\partial}^{\nabla}$ gives the holomorphic structure

This unique hermitian connection is called the Chern connection of g.

◀ The space of $\bar{\partial}$ operator on E is an affine space:

$$\bar{\partial}_1 - \bar{\partial}_2 \in \Omega^{0,1}(X, \mathrm{End}(E))$$

Two unitary connections $\nabla_1 - \nabla_2 = B \in \Omega^1(X, \mathcal{A})$; $\mathcal{A}$ = antihermitian

$$\bar{\partial}^{\nabla_1} - \bar{\partial}^{\nabla_2} = B^{1,0} \rightsquigarrow \text{linear algebra} ▶$$

Let us start with a holomorphic bundle $(\mathcal{E}, \phi)$

Give a metric $g \rightsquigarrow$ its Chern connection ∇

Then the candidate for the flat connection is

$$D = \nabla + (\phi + \phi^*)$$

And the self duality or Hitchin equations is that D is flat

that is

$$\boxed{R^\nabla + [\phi \wedge \phi^*] = 0}$$

a second order equation on g

(the terms $\phi \wedge \phi$ and $\phi^* \wedge \phi^*$ vanish because we are on a curve, and $d^\nabla \phi$, $d^\nabla \phi^*$ because ϕ is holomorphic)

Theorem (Hitchin – Simpson)

Let $(\mathcal{E}, \phi)$ be a stable holomorphic Higgs bundle then there exists a unique hermitian metric g, so that

$$R^\nabla + [\phi \wedge \phi^*] = 0$$

where ∇ is the Chern connection of g

Remark, when $\phi = 0$, then ∇ is flat and we obtain Narasimhan – Seshadri theorem.