I The Hitchin section

Let us choose a spin structure, that is a line bundle S so that $S^2 = K$; by convention $K^{-1} = K^*$. Let us consider

$$\mathcal{E}_n = S^{-n} \oplus S^{-n+2} \oplus \cdots \oplus S^n$$

$$K \otimes \mathrm{End}(\mathcal{E}) = \left(\quad a_{ij} \quad \right), \qquad a_{ij} = K \otimes \left(S^{+n-2i+2}\right) \otimes S^{-n+2j-2} = K^{(j-i+1)}$$

consider $Q = (q_2, \dots, q_n)$, where $\omega_i \in H^0(K^i)$. Then the companion Higgs field is

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & q_n \\ 1 & \ddots & & & & \vdots \\ & \ddots & \ddots & & & \\ & & \ddots & \ddots & & q_2 \\ & & & & 1 & 0 \end{pmatrix} \quad \text{so that } \sigma^i(\Phi) = \omega_i$$

The Hitchin section is then

$$\sigma : Q \longmapsto (\mathcal{E}, \Phi_Q)$$

Observe that (as in Hodge theorem) $\mathcal{E}$ does not depend on Ω.

Theorem (Hitchin) [Hitchin section theorem]

(i) $(\mathcal{E}, \Phi_Q)$ is stable.

(ii) The representations associated to the Hitchin section are with values in $SL_n(\mathbb{R})$.

(iii) The representation associated to $\Phi_0 = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{pmatrix}$ factors through $\mathfrak{sl}_2(\mathbb{R})$.

II Proof of the Hitchin theorem.

A] Stability (we postpone that for later)

B] Reality of the representation

A little bit of Lie Theory

An involution of $\mathfrak{g}$ is $\sigma \in Aut(\mathfrak{g})$ so that $\sigma^2 = 1$

$\Rightarrow \mathfrak{h} = \{u \,|\, \sigma(u) = u\}$ is a lie sub algebra.

examples for $sl_n(\mathbb{C})$

(i) $A \to -{}^t\bar{A}$: the fixed pt $\approx su(n)$
(ii) $A \to \bar{A}$ " " " $\approx sl_n(\mathbb{R})$
(iii) the transpose w.r to a complex quadratic form $\approx SO(n, \mathbb{C})$

Case (i) and (ii) the involution is antilineaire then $\underline{sl_n(\mathbb{C})} = \mathfrak{h} + i\mathfrak{h}$, thus $sl_n(\mathbb{C}) = \mathfrak{h}_{\mathbb{C}}$: $\mathfrak{h}$ is a real form of $sl_n(\mathbb{C})$

Our goal, let $D = \nabla + \Phi + \Phi^*$, find an involution $\sigma \,|\, D\sigma = 0$, $Fix(\sigma) = sl_n(\mathbb{C})$

We move everything ∇, D to the adjoint bundle $\mathcal{G} = End_0(\mathcal{E})$

$$D = \nabla + B \text{ on } \mathcal{E} \longrightarrow D = \nabla + ad(B) \text{ on } \mathcal{E}.$$

(1). Our first involution coming from the metric g

$g \mapsto$ to the transpose $A \to A^*$ on $\mathcal{G}$

we define $\boxed{\rho : A \mapsto -A^*; \text{ then } \rho \text{ is (i) antilinear (ii) an involution}}$

$$\nabla \rho = 0$$
$$D = \nabla - \tfrac{1}{2}\rho \cdot D\rho$$

◀ We will make these computations much neatly later on. At this stage it is enough to check that if $\nabla = D + \frac{1}{2}\rho D\rho$, then $\nabla\rho = 0$

$$\nabla\rho = D\rho + \tfrac{1}{2}[\rho D\rho, \rho]$$
$$= D\rho + \tfrac{1}{2}\rho D\rho \cdot \rho - \tfrac{1}{2}D\rho = 0 \;\blacktriangleright$$

(2). A second involution

The vector bundle $\mathcal{E}$ carries a holomorphic quadratic form:

$$G_0(s_{-n}, s_{-n+2} \ldots, s_n) = \sum' s_i s_{-i}$$

The matrices symmetric for Q are «persymmetric» matrices: i.e the transpose is with respect to the other diagonal

ex Φ_0, or Φ_q $(n=2)$ are symmetric with respect to Q

There exists a quadratic form (non degenerate) G_Q for which Φ_Q is symmetric

$$G_Q = G_0 + \lambda_i(Q) G_i$$

where $\lambda_i(Q)$ is polynomial in Q ; $G_i(s) = s_n \cdot s_{n-2i}$

In particular G_Q is holomorphic

◀ let work on $\mathbb{R}^{n+1}$, with coordinates $(x_0, \ldots, x_n)$, with basis $e_0, \ldots, e_n$ and dual basis $e^0, \ldots, e^n$

let $G_0 = \sum' e^i \otimes e^{n-i}$; $\phi(e_n) = \sum_{i=1}^{n} a_i e_i$; $\phi(e_i) = e_{i+1}$

$0 = G_0(\phi(e_i), e_j) - G_0(\phi(e_i), e_j)$ if $i, j < n$ and $i = j = n$

$$G_0(\phi(e_n), e_j) - G_0(\phi(e_j), e_n) = a_{n-j}$$

let $G_i = e^n \otimes e^i + e^i \otimes e^n$; $i = 1, \ldots, n-1$

$$\begin{cases} G_j(\phi(e_i), e_k) = 0 \text{ if } i, k < n \\ G_i(\phi(e_j), e_n) = \delta_{j+1, i} \\ G_i(e_j, \phi(e_n)) = \delta_{i,j}\, a_n \end{cases}$$

Thus if $G_\lambda := G_0 - \sum' \lambda_i G_i$ we have

$$G_\lambda(\phi(e_j), e_n) - G_\lambda(e_n, \phi(e_j)) = a_{n-j} + \sum' \lambda_i [\delta_{j+1, i}] - \sum' \lambda_i \delta_{ij} a_n$$

We have to solve

$0 = a_{n-j} + \lambda_{j+1} - \lambda_j a_n \rightsquigarrow$ a recurrence relation $\rightsquigarrow$ the result ▶

$\boxed{\begin{array}{l}\text{The second involution is } I : (\text{opposite of transpose for } G_{\mathbb{C}}) \\ I \text{ is } \mathbb{C}\text{-linear, holomorphic and} \\ \qquad\qquad I(\Phi) = -\Phi\end{array}}$

It follows that J is an automorphism of $(\mathcal{G}, \Phi)$ to $(\mathcal{G}, -\Phi)$
Moreover the solution of HSD for $+\Phi$ is also a solution for $-\Phi$

Thus $\qquad I^*\rho = \rho \quad [\text{that is } I\rho I = \rho]$

$$I\,\phi^* = -\phi^*$$

(3) A third involution

$\boxed{\begin{array}{l}\text{let now } \sigma = \rho I \text{, then} \\ \qquad \text{(i) } \sigma \text{ is an antilinear involution} \\ \qquad \text{(ii) } D\sigma = 0\end{array}}$

◀ $\rho I = I\rho$, thus $\sigma^2 = 1$; let us consider I as a bundle automorphism $(\mathcal{G}$ ⟲, where $\mathcal{G} = \mathrm{End}_0(\mathcal{E}))$. We showed above that $I^*\nabla = \nabla$; but $D = \nabla - \mathrm{Ad}(B)$ where $B = \phi - \rho(\phi)$

then $I^*D = I^*\nabla - I^*(\mathrm{Ad}(B))$

However: $I^*\mathrm{Ad}(B) \underset{\text{def}}{\overset{\uparrow}{=}} I^{-1}\mathrm{Ad}(B)\, I = I\mathrm{Ad}(B) I = \mathrm{Ad}(IB)$
↑ I is an autom of $\mathcal{G}$

by $IB = -B$ (because $I\rho = \rho I$ and $I(\phi) = -\phi$)

thus $I^*D = \nabla + \mathrm{ad}(B)$

similarly $\rho^*\nabla = \nabla$ (because ∇ is metric), $\rho^*\mathrm{ad}(B) = \mathrm{ad}(\rho B) = -\mathrm{ad}\,B$

thus $\sigma^*D = \rho^* I^* (\nabla + \mathrm{ad}(B)) = \nabla + \mathrm{ad}(B)$

thus $\sigma^* D = D$, that is $D\sigma = 0$. ▶

(4) Conclusion

Proposition (Algebraic)

Let ρ be the involution associated to $K \approx SU(n) \subset SL(n,\mathbb{C})$

Let J " " " to $G \approx SO(n,\mathbb{C}) \subset SL(n,\mathbb{C})$

Assume $J\rho = \rho J$;

Then the fixed points of the involution $\sigma := \rho J$ is $\approx SL_n(\mathbb{R})$

◀ Since ρ and J commute. We can split $\underline{\mathfrak{sl}(n,\mathbb{C})}$ as

$$\mathfrak{sl}(n,\mathbb{C}) = \underset{(1,1)}{\mathfrak{k}} \oplus \underset{(-1,-1)}{\mathfrak{a}} \oplus \underset{(1,-1)}{\mathfrak{b}} \oplus \underset{(-1,1)}{\mathfrak{c}} \qquad = (\text{eig of } \rho \mid \text{eig of } J)$$

$\underline{\mathfrak{su}(n)} = \mathfrak{k} \oplus \mathfrak{b}$

$\underline{\mathfrak{so}(n,\mathbb{C})} = \mathfrak{k} \oplus \mathfrak{c}$, observe that $\mathfrak{c} = i\mathfrak{k}$, $\mathfrak{k}$ is compact

Thus $\mathfrak{k}$ is the maximal compact of $SO(n,\mathbb{C}) = SO(n,\mathbb{R})$

$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a}$, and $\mathfrak{a} = i\mathfrak{b}$

but $SO(n,\mathbb{R}) \circlearrowright \underline{\mathfrak{sl}(n,\mathbb{C})}$ decomposes in 4 irreducible representations

$\underline{\mathfrak{so}(n,\mathbb{R})} \oplus \mathrm{sym} \oplus i\,SO(n,\mathbb{R}) \oplus i\,\mathrm{Antisym}$

It follows $\mathfrak{a} \approx \mathrm{sym}$ as a representation of $\mathfrak{so}(n,\mathbb{R})$

$$\mathfrak{a} = e^{i\theta} \cdot \mathrm{Sym};$$

since $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{k}$, it follows that $e^{i\theta} = 1$, or $e^{i\theta} = i$

The sign of the Killing form implies that $e^{i\theta} = 1$.

And thus $\mathfrak{h} = \underline{\mathfrak{sl}(n,\mathbb{R})}$ ▶

III Proof of the stability :

Recall that $\mathcal{E} = S^{-n} \oplus \cdots\cdots \oplus S^{n}$; let $W_i = S^{-n} \oplus \cdots \oplus S^{n-2i}$

$$\phi_0 : W_i \to W_{i+1} .$$

It follows that $(\mathcal{E}, \phi_0)$ is stable : any ϕ_0- invariant bundle is of the form W_i ; but

$$\deg(W_j) = \left(\sum_{i=0}^{j} (-n+2i)\right) \deg(S) < 0$$

Let now $Q \in$ Hitchin basis, for $\lambda \in \mathbb{C}$, let

$$\lambda Q = (0, \lambda^2 q^2, \cdots\cdots, \lambda^n q^n), \text{ if } Q = (0, q_2, \cdots, q_n)$$

Let $\Delta_\lambda = \begin{pmatrix} \lambda^{+n} & & \\ & \ddots & \\ & & \lambda^{-n} \end{pmatrix}$ = Isomorphisms of $\mathcal{E}$.

- observe that

$$\Delta_\lambda \phi_Q \Delta_{\frac{1}{\lambda}} = \begin{pmatrix} 0 & & & \frac{1}{\lambda^{2(n-1)}} q_n \\ \lambda^2 & \ddots & & \vdots \\ & \ddots & \ddots & \frac{1}{\lambda^2} q_2 \\ & & \lambda^2 & 0 \end{pmatrix} = \lambda^2 \phi_{\frac{1}{\lambda^2} Q}$$

- Observe also that stability for $\phi \iff$ stability for $\mu\phi$

  Thus $(\mathcal{E}, \phi_Q)$ stable $\iff (\mathcal{E}, \phi_{\frac{Q}{\lambda^2}})$ stable.

- When $\lambda \to \infty$, $\qquad \phi_{\frac{Q}{\lambda^2}} \to \phi_0$

One line proof : « stability is an open condition » for experts only !

A) A baby case $\quad \hat{W}_i = S^{-n+2i} \oplus \cdots\cdots \oplus S^{n}$

Let $\mathcal{L}$ be stable par ϕ_Q and assume that

(1) $\mathcal{L} \not\subset \hat{W}_1 \qquad$ (2) $\operatorname{rk} \mathcal{L} = 1$

Recall that now $\mathcal{L}^\lambda := \Delta_\lambda \mathcal{L}$ is stable by $\Phi_{\frac{1}{\lambda^2}Q}$

However one now remarks that

(1) If x is so that $\mathcal{L}_x \pitchfork W_1$ then $\mathcal{L}^\lambda_x \xrightarrow{C^\infty} \bar{s}^{n}$ (largest eigenline of $\Delta_{\frac{1}{\lambda}}$)

(2) There are finitely many point x_i at which $\mathcal{L}_x \not\pitchfork W_1$

let us choose some auxiliary hermitian metric on $\mathcal{E}$ so that

In the neighborhood U_i of x_i, if ∇ is the Chern connection of g_0

- ∇ is flat
- s^i are // for ∇

let us consider the induced connection on $\mathcal{L}^\lambda$

then $$\deg(\mathcal{L}^\lambda) = \int R^\nabla = \underbrace{\int_{X-(U_1,\dots,U_n)} R^\nabla}_{(*)} + \underbrace{\sum_i \int_{U_i} R^\nabla}_{(**)}$$

$(*) \longrightarrow \deg(\bar{s}^n)$

$(**) \leqslant 0$ (we already computed that !)

Thus in the end $\deg(\mathcal{L}^\lambda) \leqslant \deg(\bar{s}^n) < 0$

B] The general case

(i) Assume now that $\mathcal{L} \subset W_i$, $\mathcal{L} \pitchfork W_{i+1}$, then outside finitely many pts $\mathcal{L}^\lambda \longrightarrow s^{-n+i}$ but this one is not stable by Φ_0

Thus for all rank 1 - cases.

(ii) If $\mathcal{F} \subset \mathcal{E}$ has rk k $\longrightarrow$ $\det(\mathcal{F}) \subset \Lambda^k \mathcal{E}$ is rank 1 and the same argument applies.

« A heuristic argument at the x_i : $\mathcal{L}^\lambda$ « bubbles off »

$\mathcal{L} : U \longrightarrow \mathbb{CP}(\mathcal{E})$

And $\deg(\mathcal{L}) = \deg(s^n) - \text{Area}(\text{Bubbles})$ »